Stellar Interiors - Hydrostatic Equilibrium and Ignition on the Main Sequence

http://apod.nasa.gov/apod/astropix.html

Outline of today's lecture

- Hydrostatic equilibrium: balancing gravity and pressure
- Nuclear fusion: the energy source for stars
- Energy transport: getting the energy out of a star

Review of 3 types of pressure in stars

Ideal gas pressure: collisions between gas particles

$$P_{\text{ideal}} \propto \rho T$$

$$P_{\text{ideal}} = 1.69 \rho N_A k T$$ (Special case for ionized gas)

Radiation pressure: collisions between photons and matter

$$P_{\text{rad}} \propto T^4$$

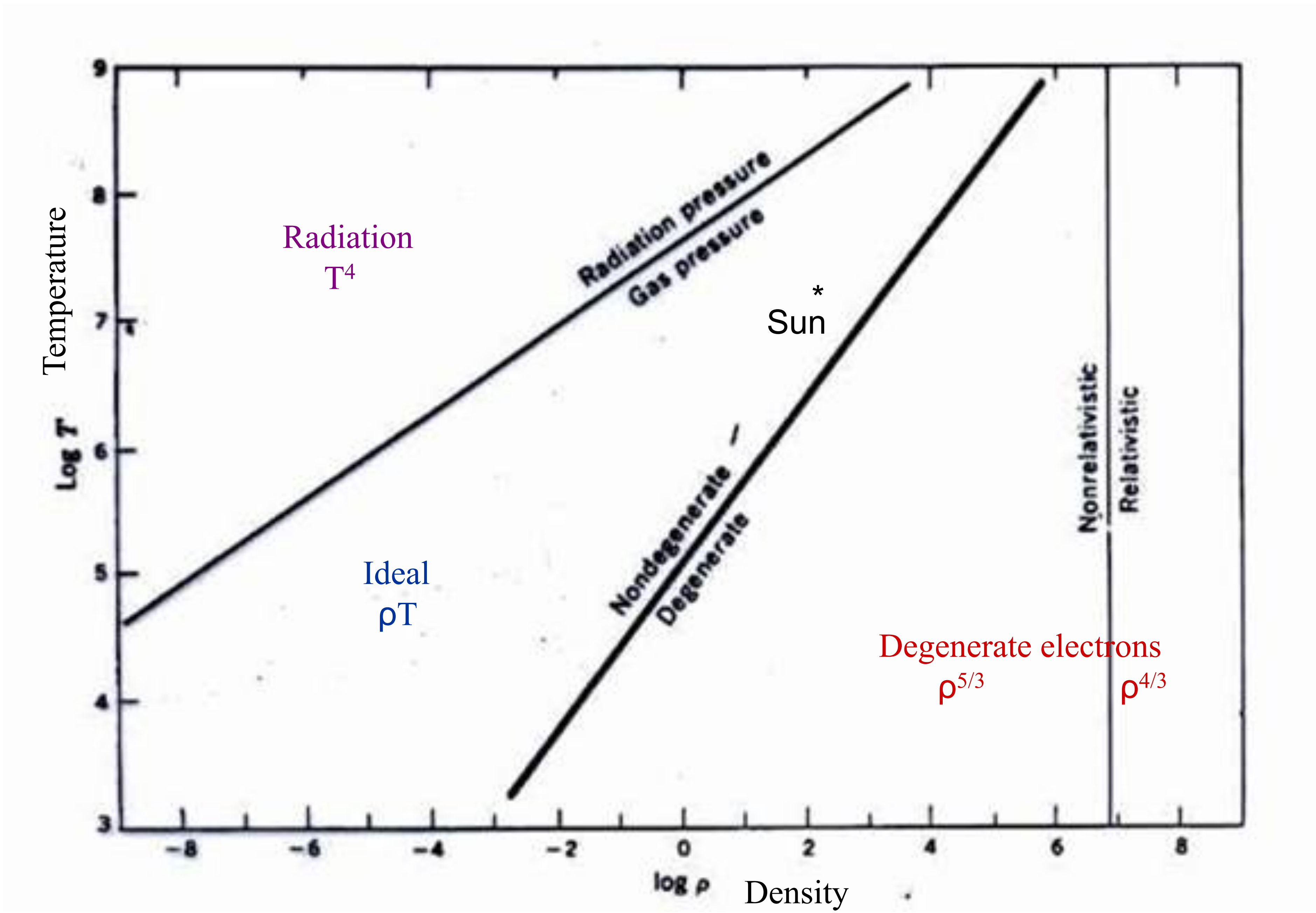
Degeneracy pressure: "resistance" of electrons (or neutrons) against compression into a smaller volume

$$P_{\text{deg}} \propto \rho^{5/3}$$

Non-relativistic speeds ($v < c$)

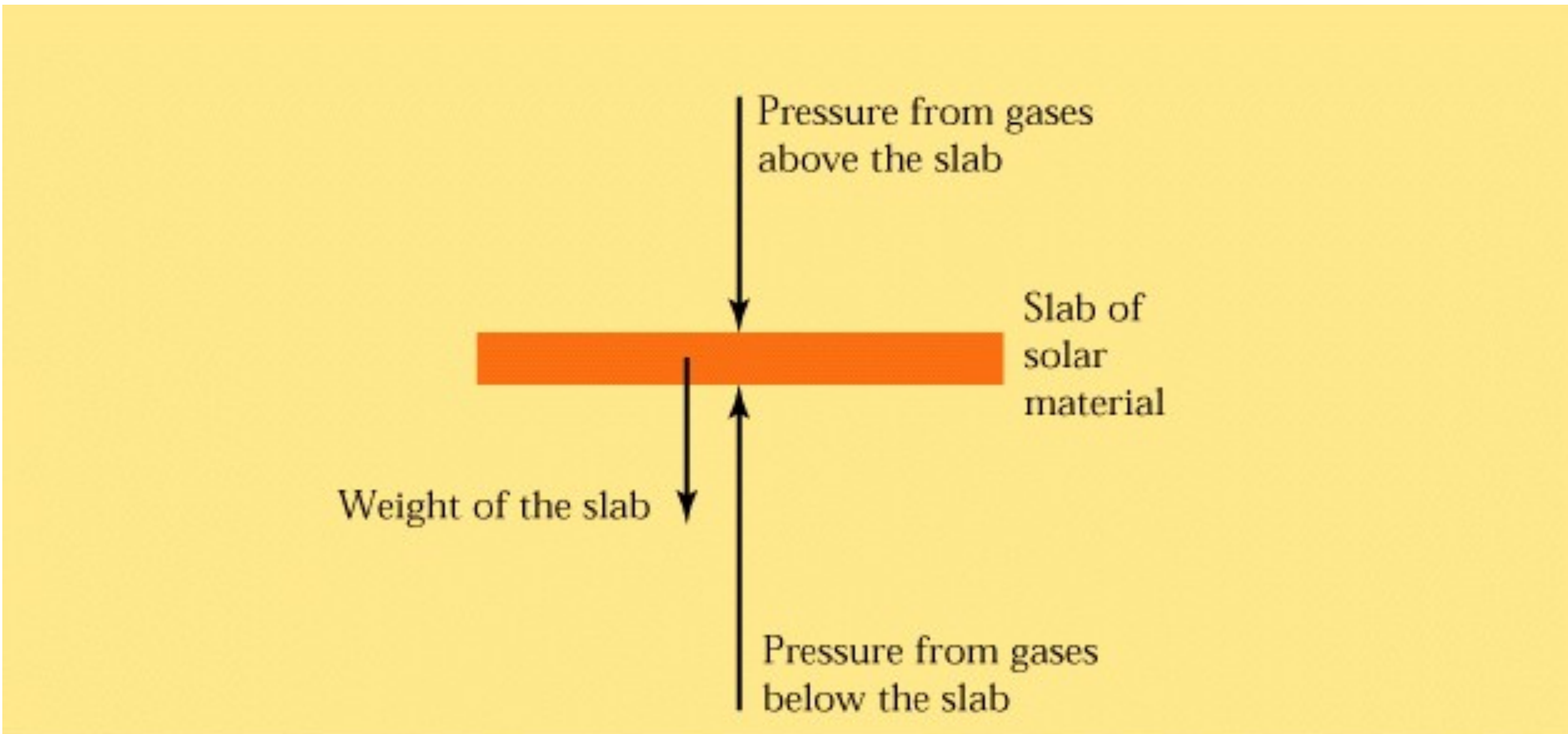
$$P_{\text{deg}} \propto \rho^{4/3}$$

Relativistic speeds ($v \sim c$)


Temperature
Log T
9
8
7
6
5
4
3
Radiation
T^4
Radiation pressure
Gas pressure
Sun
Ideal
ρT
Nondegenerate
Degenerate
Nonrelativistic
Relativistic
Degenerate electrons
$\rho^{5/3}$
$\rho^{4/3}$
−8
−6
−4
−2
0
2
4
6
8
log ρ
Density

HYDROSTATIC EQUILIBRIUM

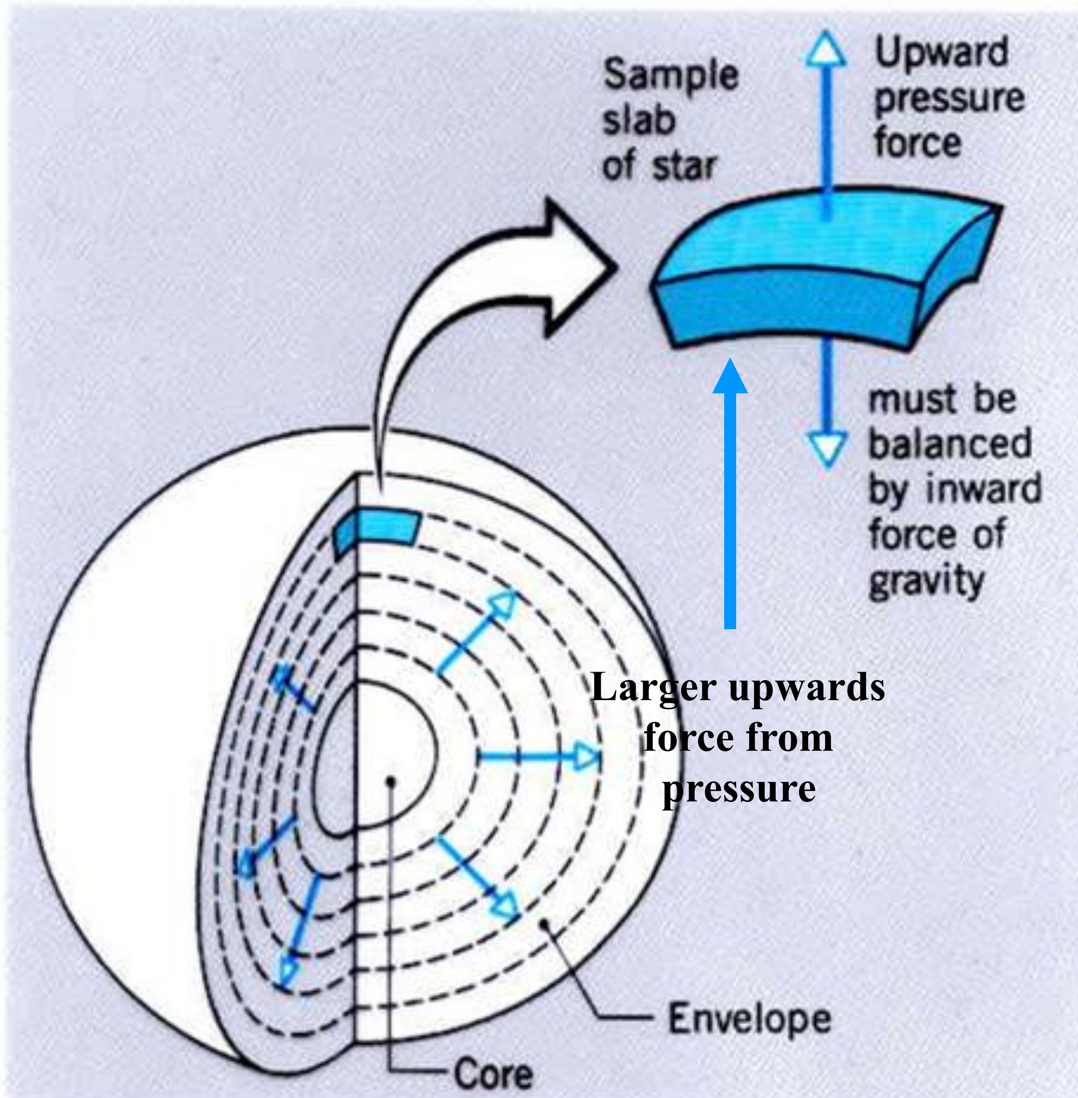
Is achieved when gravity pulling downward is balanced by pressure pushing upward



In hydrostatic equilibrium, forces balance

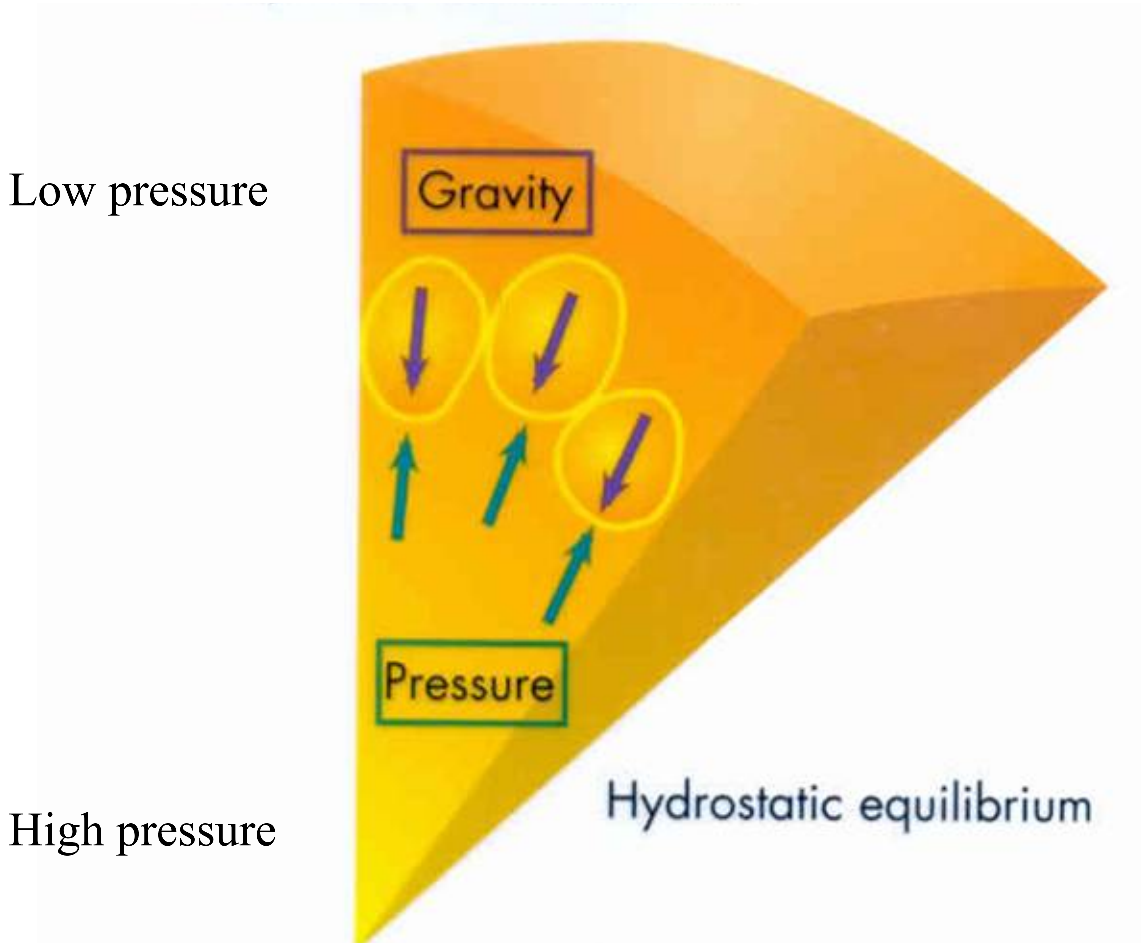
e.g. the Earth's atmosphere or a swimming pool

A star is "in hydrostatic equilibrium" when it is not collapsing or expanding



The balance of gas pressure and gravity in a star.

Inside a star the weight of the matter is supported by a *gradient* in the pressure. If the pressure on the top and bottom of a layer were exactly the same, the layer would fall because of its weight.

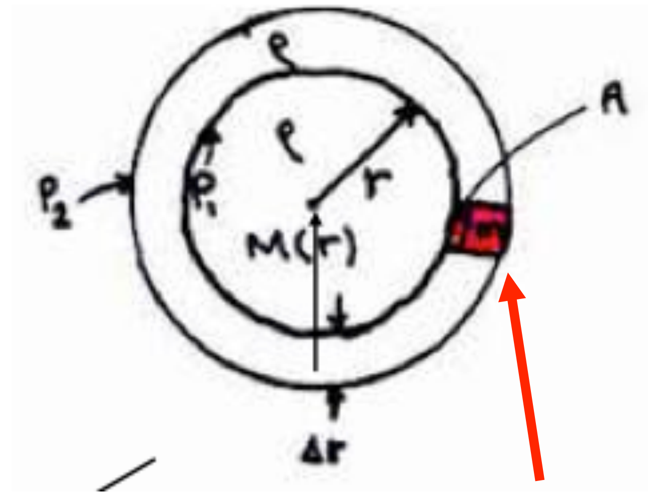


The difference between pressure times area on the top and the bottom balances the weight

Derivation of Equation of Hydrostatic Equilibrium

$$\text{Force up} = P_1 A - P_2 A = \Delta P \cdot A$$

$$\text{Force down} = -\frac{GM(r)m}{r^2}$$



$$m = \rho(A\Delta r)$$

To be in hydrostatic equilibrium, Force up = Force down

$$\Delta P \cdot A = -\frac{GM(r)}{r^2} m$$

$$\Delta P \cdot A = -\frac{GM(r)}{r^2} (\rho A \Delta r)$$

$$\Delta P = -\frac{GM(r)}{r^2} \rho \Delta r \quad \Rightarrow \quad \frac{\Delta P}{\Delta r} = -\frac{GM(r)\rho(r)}{r^2}$$

Or, more precisely…see next slide

For all kinds of gases – ideal, degenerate, whatever.

If $M(r)$ is the mass *interior* to radius r and $\rho(r)$ is the density *at* r:

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

For a sphere of constant density, $M(r) = \frac{4}{3}\pi r^3 \rho$ and one may integrate this equation to obtain

$$P_{central} = \frac{GM\rho}{2R}$$

This (top) equation is one of the fundamental equations of stellar structure. It is called the "equation of hydrostatic equilibrium". Whenever dP/dr differs from this value, matter moves.

Proof of Central Pressure Equation

We integrate this equation: $$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Assuming the star has constant density ρ_0 : $$M(r) = \frac{4}{3}\pi r^3 \rho_0$$

$$\frac{dP}{dr} = -\frac{G}{r^2}\left(\frac{4\pi r^3 \rho_0}{3}\right)\rho_0 = -\frac{4\pi G\rho_0^2 r}{3}$$

$$\int_{P_c}^{0} dP = \int_0^R -\frac{4\pi G\rho_0^2}{3}\, r\, dr = -\frac{4\pi G\rho_0^2}{3}\int_0^R r\, dr$$

$$P_c = \frac{4\pi G\rho_0^2}{3}\frac{R^2}{2} = \left(\frac{G\rho_0}{2R}\right)\left(\frac{4\pi R^3 \rho_0}{3}\right) = \boxed{\frac{GM\rho_0}{2R}}$$

Note implications for central temperature. *If an ideal gas, 75% H and 25% He, fully ionized*

$$P_c = P_{\text{ideal}}$$

$$P_c = 1.69 \rho N_A k T_c = \frac{GM\rho}{2R}$$

Solve for T_c:

$$T_c = \frac{GM}{3.38 N_A k R}$$

For example, for the sun (not really constant in density), so answer is an underestimate

$$T_c \approx \frac{(6.67 \times 10^{-8})(2.00 \times 10^{33})}{(3.38)(6.02 \times 10^{23})(1.38 \times 10^{-16})(6.96 \times 10^{10})}$$

$$= 6.8 \times 10^6 \text{ K}$$

$$T_C \propto \frac{M}{R}$$

For stars supported by ideal gas pressure and near uniform structure (not red giants)

Note that as the radius gets smaller, the central temperature *increases*.

What is the time scale for this increase (if gravity is the only source of power)?

$$\tau_{KH} = \frac{3GM^2}{10RL}$$

which is ~ 10 million years for the sun.

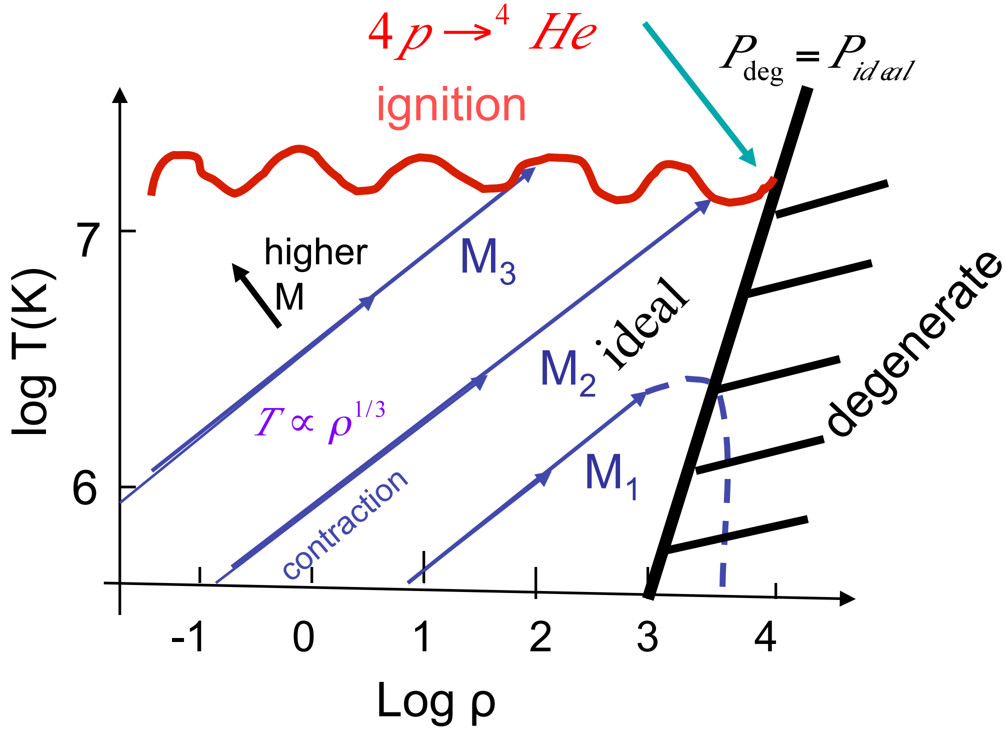
(20 - 30 My is more accurate)

<u>What is the lowest mass a star can have and still burn hydrogen?</u>

Hydrogen fusion occurs when T >10^7 K

A star must have sufficient mass such that as it contracts, its core can heat up to > 10^7 K before the core stops contracting due to degeneracy pressure

lightest star will be mass that hits this point.



To get the blue lines in the plot:

$$T \propto \frac{M}{R} \qquad M \sim \frac{4\pi}{3} R^3 \rho$$

$$\Rightarrow R \propto \left(\frac{M}{\rho}\right)^{1/3}$$

$$T \propto \frac{M\rho^{1/3}}{M^{1/3}}$$

$$T \propto M^{2/3} \rho^{1/3}$$

$T \propto \rho^{1/3}$ for a given M and

T at a given ρ is higher

for bigger M

Minimum Mass Star

Solve for condition that ideal gas pressure and degeneracy pressure are equal at 10^7 K.

$$P_{\rm deg} = P_{\rm ideal}$$

$$1.0 \times 10^{13} (\rho Y_e)^{5/3} = 1.69 \rho N_A k T$$ (assuming 75% H, 25% He by mass)

At 10^7 K, this becomes

$$1.40 \times 10^8 \rho (10^7) \approx 8.00 \times 10^{12} \rho^{5/3}$$ (taking $Y_e = 0.875$)

which may be solved for the density to get $\rho \approx 2300$ gm cm^{-3}

The total pressure at this point is

$$P_{\rm tot} \approx \frac{1}{2}(P_{\rm deg} + P_{\rm ideal}) \approx \frac{1}{2}(2P_{\rm ideal}) \approx P_{\rm ideal}$$

$$\approx 1.40 \times 10^8 (2300)(10^7) \approx 3.2 \times 10^{18} \text{ dyne cm}^{-2}$$

$$P_{\rm tot} = P_c = \left(\frac{GM\rho}{2R}\right)$$

But $R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$ i.e., $\rho = \frac{M}{4/3\ \pi R^3}$

Combining terms we have

$$3.2 \times 10^{18} \approx \frac{GM\rho}{2}\left(\frac{4\pi\rho}{3M}\right)^{1/3}$$

$$M^{2/3} \approx \frac{2(3.2 \times 10^{18})3^{1/3}}{G\rho^{4/3}(4\pi)^{1/3}}$$

and using again $\rho \approx 2300$ gm cm^{-3}

$$M \approx 8.7\, x 10^{31} \text{ gm}$$

or 0.044 solar masses.

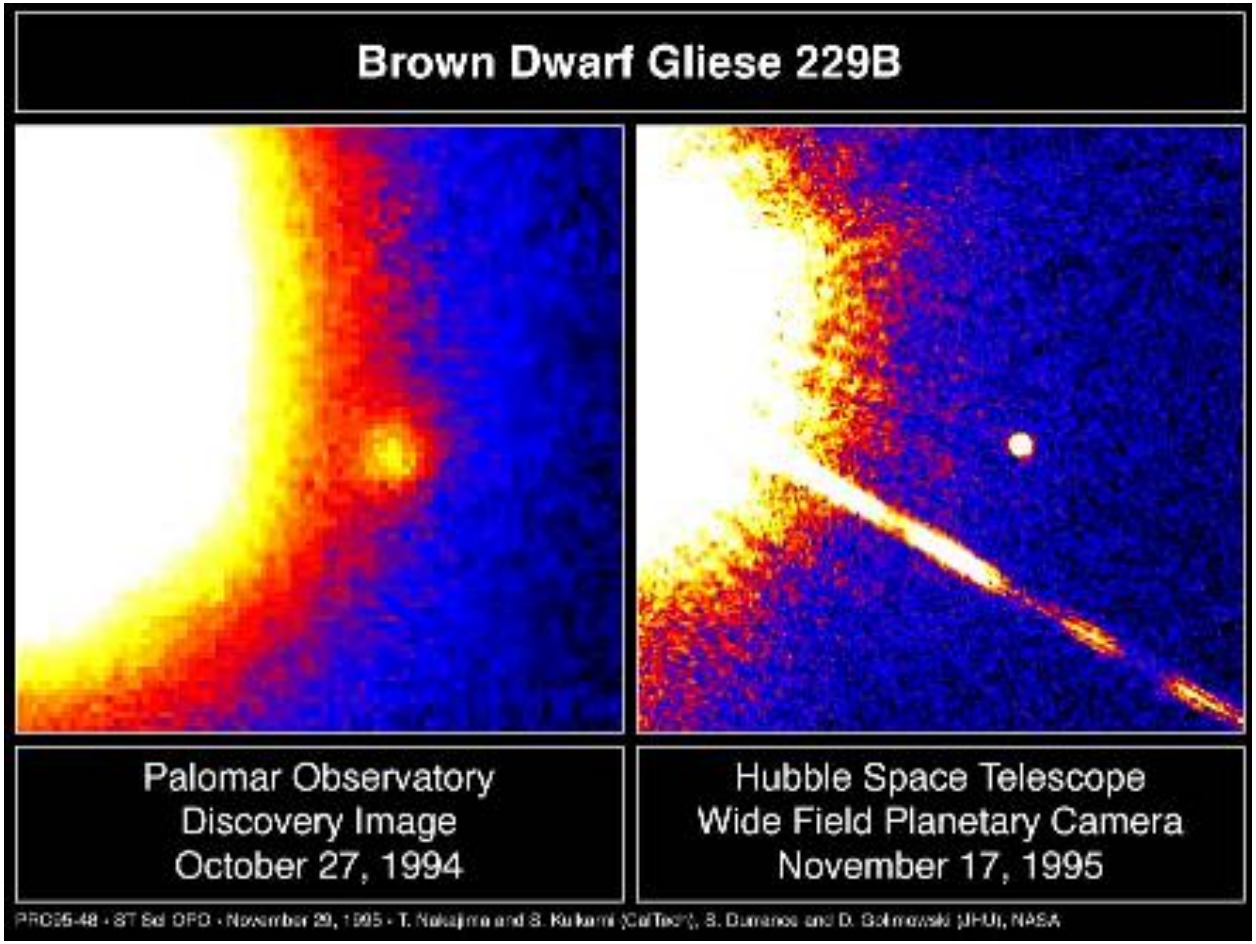
For constant density

$$P = \left(\frac{GM\rho}{2R}\right)$$

$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$$

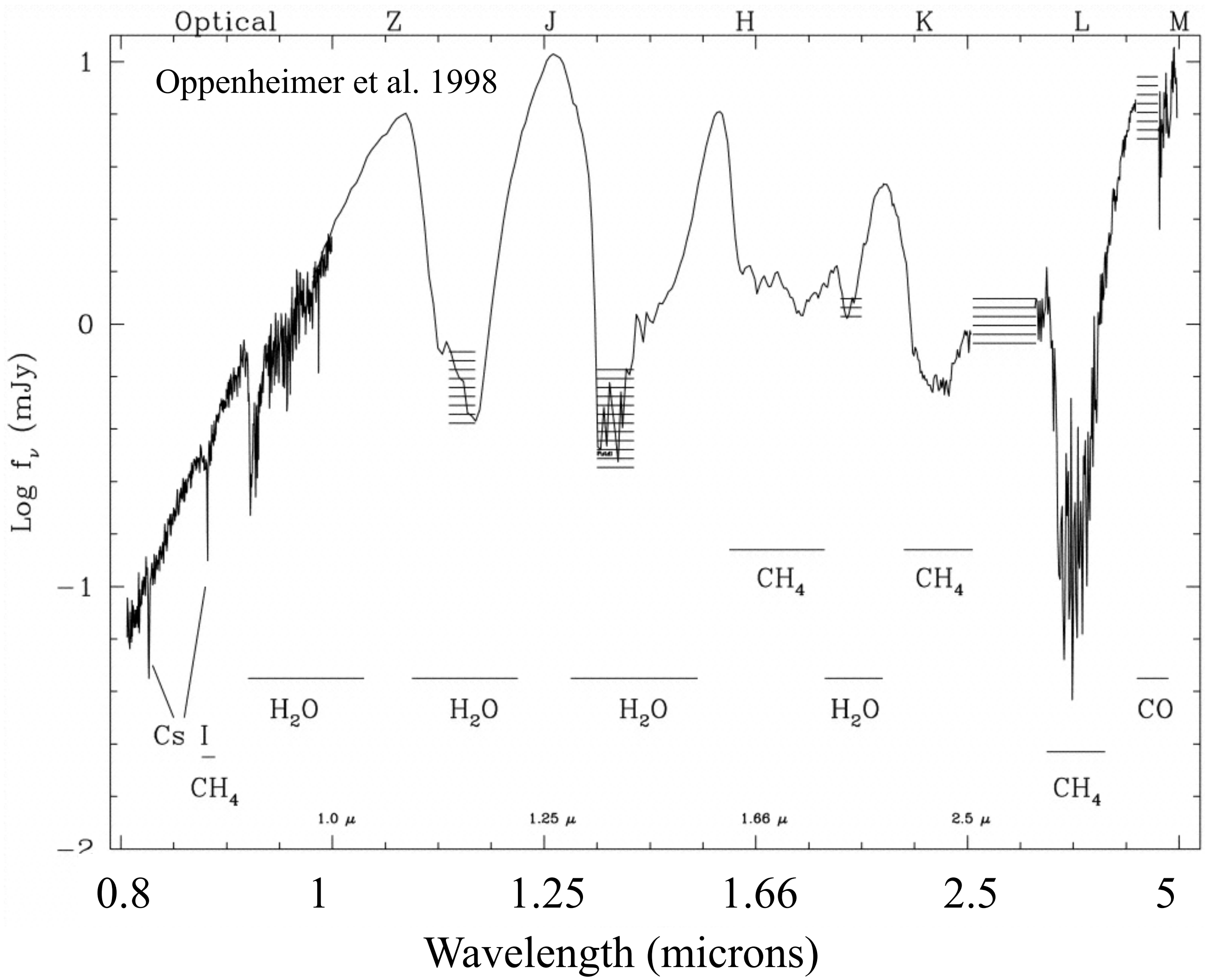
A more detailed calculation gives 0.08 solar masses.
Protostars lighter than this can never ignite nuclear reactions.
They are known as _brown dwarfs_ *(or planets if the mass is less than 13 Jupiter masses, or about 0.01 solar masses.*
[above 13 Jupiter masses, some minor nuclear reactions occur that do not provide much energy - "deuterium burning"]

Brown Dwarfs - heavier than a planet ($13\ M_{Jupiter}$) and lighter than a star



14 light years away in the constellation Lepus orbiting the low mass red star Gliese 229 is the brown dwarf Gliese 229B. It has a distance comparable to the orbit of Pluto but a mass of 20-50 times that of Jupiter. Actually resolved with the 60" Palomar telescope in 1995 using adaptive optics.

Infrared Spectrum of Gliese 229B, 1st Discovered Brown Dwarf



Note the strong absorption due to molecules (like water!).

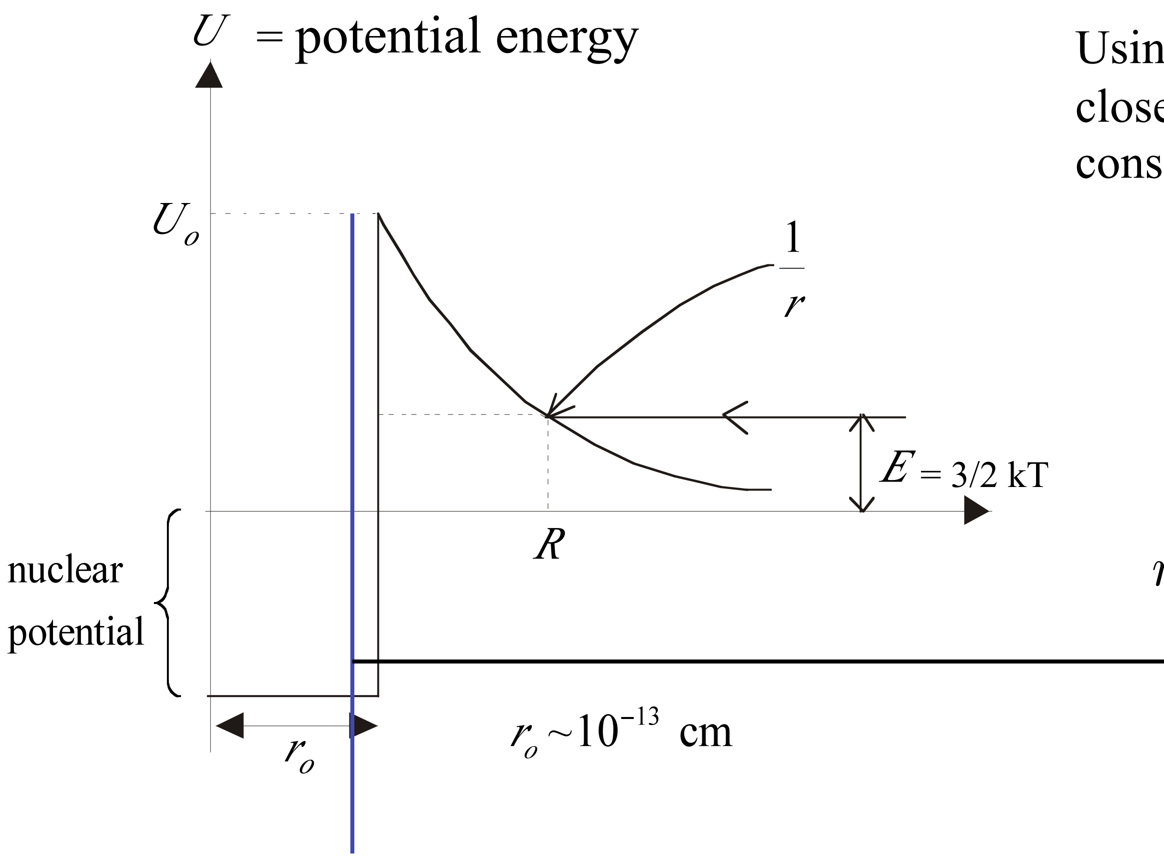
Nuclear Fusion Reactions

Main Sequence Evolution
(i.e., Hydrogen burning)

The basis of energy generation by nuclear fusion is that two reactants come together with sufficient collisional energy to get close enough to experience the strong force. This force has a range ~10^{-13} cm, i.e., about 1/100,000 the size of the hydrogen atom.

So two protons need to be no more than 10^{-13} cm apart in order to fuse together…

But before 2 protons can come close enough to form a bound state, they have to overcome their electrical repulsion.



Using classical physics, the distance of closest approach, r_{min}, is found by energy conservation:

$$KE = PE$$

$$\frac{3}{2}kT = \frac{e^2}{r_{min}}$$

$$r_{min} = \frac{2}{3}\frac{e^2}{kT} \approx 10^{-10}\ \text{cm}$$

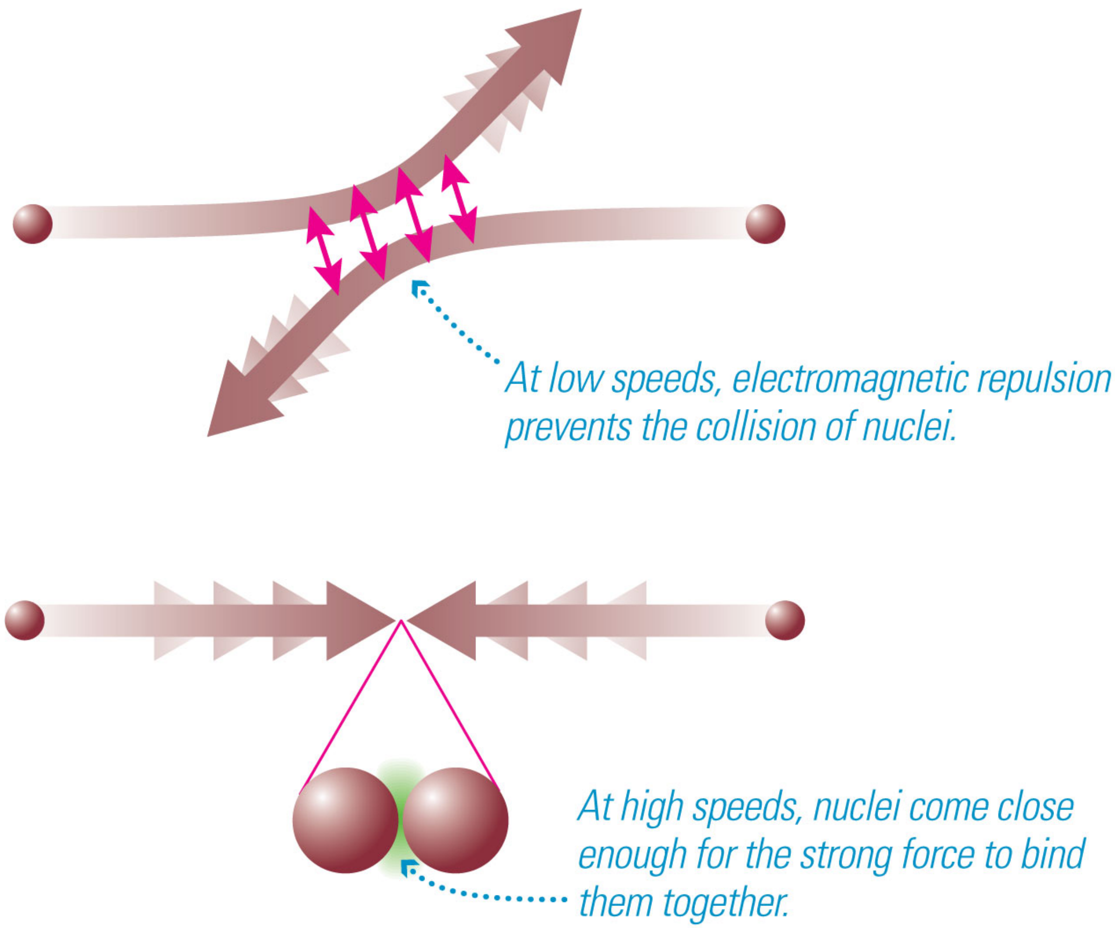
Calculation assumes $T=10^7$ K (core temperature)

So two protons could not fuse, since they are too far apart…

e = charge of proton

How is this problem reconciled?

1. Some protons are moving much faster than average, and have sufficient KE to overcome the electric repulsion. But these high-speed protons are rare…



How is this problem reconciled?

2. There is a small (but non-zero) probability that two protons can overcome their repulsion and fuse *even if their speeds are too low.* This is called quantum mechanical barrier penetration.

$$r_{\min} = \frac{2e^2}{3kT} = \frac{2e^2}{mv^2}$$

$$\begin{aligned} \text{Probability} &\propto \exp(-r_{\min}/\lambda) \\ &\propto \exp(-1/v) \\ &\propto \exp(-1/\sqrt{T}) \end{aligned}$$

$$\lambda = \frac{h}{mv}$$

Probability is highest at fast speeds *v* (or high temperatures *T*)

But when two protons do get close enough to (briefly) feel the strong force, they almost always end up flying apart again. The nuclear force is strong but the "diproton", ^2He, is not sufficiently bound to be stable.

*One must also have, while the protons are briefly together, a **weak** interaction.*

$$p + p \rightarrow ({}^2He) \rightarrow {}^2H + e^+ + \nu_e$$

That is, a proton turns into a neutron, a positron, and a neutrino. The nucleus ^2H, deuterium, is permanently bound.

The rate of hydrogen burning in the sun is thus quite slow because:

- The protons that fuse are rare, only the ones with about ten times the average thermal energy

- Even these rare protons must penetrate a barrier to go from 10^{-10} cm to 10^{-13} cm and the probability of doing that is exponentially small

- Even the protons that do get together generally fly apart unless a weak interaction occurs turning one to a neutron while they are briefly togther

and that is all quite good because if two protons fused every time they ran into each other, the sun would explode.

Now we discuss the actual fusion reactions that power low-mass stars like the Sun while on the main sequence.

Some nuclear physics notation:

$$p = \text{proton}$$
$$e^+ = \text{positron}$$
$$\nu = \text{neutrino}$$
$$\gamma = \text{energy (photon)}$$

$$^2\text{H} = \text{hydrogen (H) nucleus with 1 proton and 1 neutron}$$

$$^A\text{X} = \text{nucleus of element X}$$

$$A = \text{number of protons} + \text{number of neutrons}$$

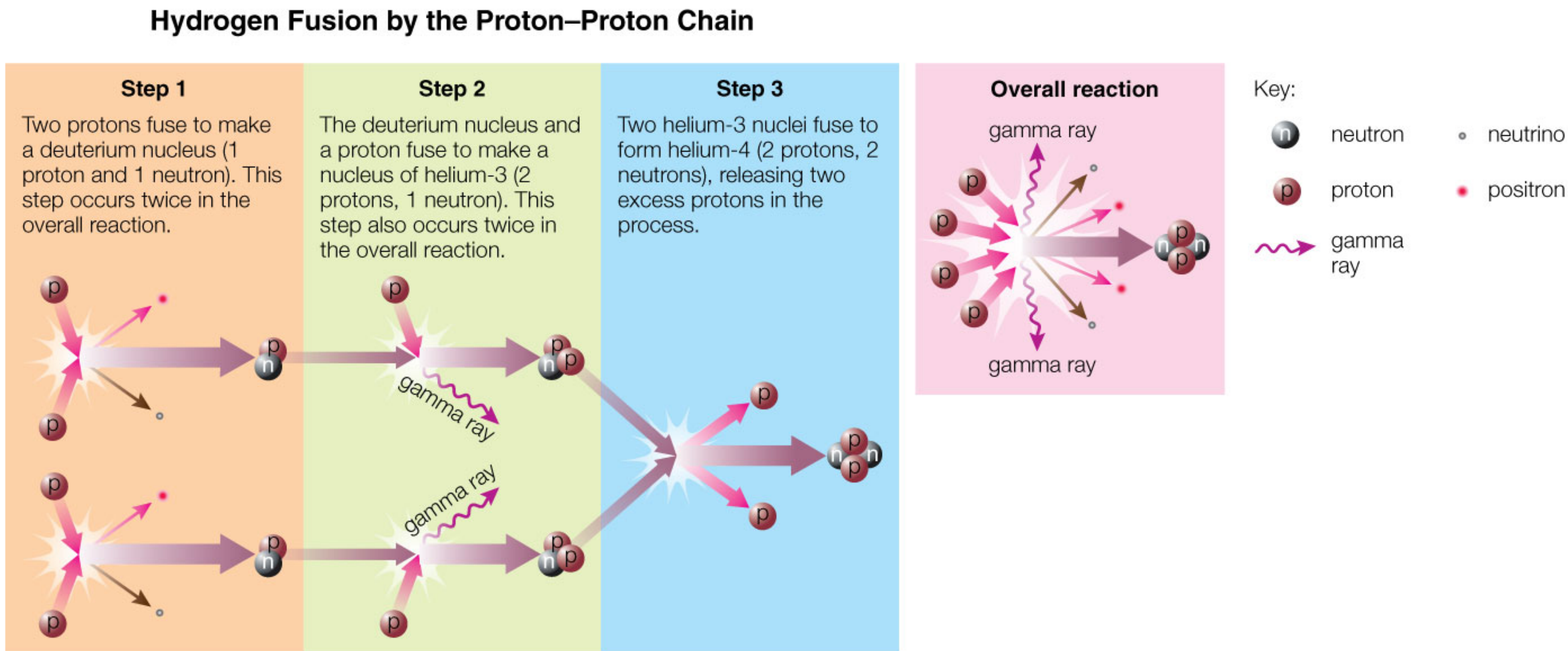
The big picture: The proton-proton I (ppI) chain is the primary set of hydrogen fusion reactions in low-mass stars like the Sun

$$\begin{aligned} p + p &\rightarrow {}^2\mathrm{H} + e^+ + \nu + \text{energy} \\ {}^2\mathrm{H} + p &\rightarrow {}^3\mathrm{He} + \text{energy} \\ {}^3\mathrm{He} + {}^3\mathrm{He} &\rightarrow {}^4\mathrm{He} + p + p + \text{energy} \end{aligned}$$

The bigger picture: Hydrogen fuses into helium, producing energy

$$p + p + p + p \rightarrow {}^4\mathrm{He} + \text{energy}$$

The big picture: The ppI chain is the primary set of hydrogen fusion reactions in low-mass stars like the Sun



Nuclear reaction shorthand:

I(j,k)L

I = Target nucleus — j = incident particle

L = Product nucleus — k = outgoing particle or energy

E.g., pp1

$$p(p,e^+\nu)^2H(p,\gamma)^3He(^3He,2p)^4He$$

E.g., the main CNO cycle (later)

$$^{12}C(p,\gamma)^{13}N(e^+\nu)^{13}C(p,\gamma)^{14}N(p,\gamma)^{15}O(e^+\nu)^{15}N(p,^4He)^{12}C$$

THE PP 1 CYCLE

Bethe 1939
(Nobel 1967)

MeV = mega electron volt

$$p + p \rightleftarrows (2p) \rightarrow {}^2H + e^+ + \nu_e + 0.42\,\text{MeV}$$

nb. strong and weak interaction

where 1 MeV = 1.602×10^{-6} erg and the energy comes off in the form of radiation and the kinetic energy of the products.

In shorthand this can also be written $p(p,e^+\nu)^2H$.

So now we have protons, ^{4}He, and ^{2}H. Next

$$^2H + p \rightarrow {}^3He + \gamma + 5.49\,\text{MeV}$$

no weak interaction needed, very fast

or ^{2}H(p,γ)^{3}He. This may be followed by either

$$a)\,{}^3He + {}^3He \rightarrow {}^4He + 2p + 12.86\,\text{MeV}$$ pp1

or

^{4}Li is unbound

$$b)\,{}^3He + {}^4He \rightarrow {}^7Be + \gamma + 1.59 MeV$$ pp2,3

or in shorthand $^3He(^3He,2p)^4He$ or $^3He(\alpha,\gamma)^7Be$. In the sun the former is much more likely than the latter.

Neglecting for now reaction b), the total effect is

$$4p \rightarrow {}^4He + 2e^+ + 2\nu_e + 24.68 \text{ MeV}$$

The neutrinos carry away 0.26 MeV (on the average each) and are lost (more about these later). The positrons annihilate with two electrons to give an additional 2 x 0.511 MeV x 2 = 2.04 MeV. Thus

$$4p \rightarrow {}^4He + 26.20 \text{ MeV}$$

Q or q = energy produced per gram of material

How many ergs per gram is this?

$$Q_{pp} = (26.2\,\mathrm{MeV})(1.602 \times 10^{-6}\,\mathrm{erg/MeV})(N_A/4)$$

$$Q_{pp} = (26.20)(1.602 \times 10^{-6})(6.02 \times 10^{23})/4$$
$$= 6.4 \times 10^{18}\,\mathrm{erg\,g^{-1}}$$

i.e. per gram of H burned

Implication for the sun

$$\tau_{\mathrm{nuc}} = \frac{(Q_{pp})(\text{Mass of burnable H in Sun})}{L_\odot}$$
$$= 3.4 \times 10^{17}\,\mathrm{sec}$$

$(0.7)(0.15)(1M_\odot)$

or 10.4 *billion* years!

Again:

The big picture: The ppI chain is the primary set of fusion reactions in low-mass stars like the Sun

$$\begin{aligned} p+p &\rightarrow {}^{2}\mathrm{H}+e^{+}+\nu+\text{energy} \\ {}^{2}\mathrm{H}+p &\rightarrow {}^{3}\mathrm{He}+\text{energy} \\ {}^{3}\mathrm{He}+{}^{3}\mathrm{He} &\rightarrow {}^{4}\mathrm{He}+p+p+\text{energy} \end{aligned}$$

The bigger picture: Hydrogen fuses into helium, producing energy

$$p+p+p+p \rightarrow {}^{4}\mathrm{He} + \text{energy}$$

H. A. Bethe (1906-2005)
Nobel Prize 1967

NUCLEAR REACTION RATES

Because of the exponential dependence of barrier penetration on energy and charge, the rates of nuclear fusion reactions are extremely temperature sensitive. It also obviously takes a higher temperature to fuse heavier isotopes that have larger charge.

In general, $\epsilon \propto T^n$

For the pp 1 cycle

ϵ = energy produced per second per gram of material

$$\epsilon_{pp} = 0.076\rho X_H^2 (T_c/10^7)^4 \text{ erg g}^{-1}\text{ s}^{-1}$$

where X_H is the mass fraction of hydrogen.

Assume that the sun is hot enough to run the pp 1 cycle only in the inner 10% of its mass

$$
\begin{aligned}
L &\approx 0.1\, M_\odot\, \epsilon_{pp} \\
&\approx 7.4 \times 10^{30} \rho (T_c/10^7)^4 \\
&\approx 7.4 \times 10^{30} (100)(1.5)^4 \\
&\approx 3.7 \times 10^{33}\ \mathrm{erg\ s^{-1}}
\end{aligned}
$$

$X_H \sim 0.7$

actually less

which is pretty close to correct

How much mass burned per second?
(i.e., how much hydrogen fuses every second?)

$$\frac{dM}{dt} = \frac{L_\odot}{Q_{pp}} = \frac{3.8 \times 10^{33}\,\mathrm{erg/s}}{6.4 \times 10^{18}\,\mathrm{erg/g}}$$
$$= 5.9 \times 10^{14}\,\mathrm{g/s} = 650\ \mathrm{million\ tons/sec}$$

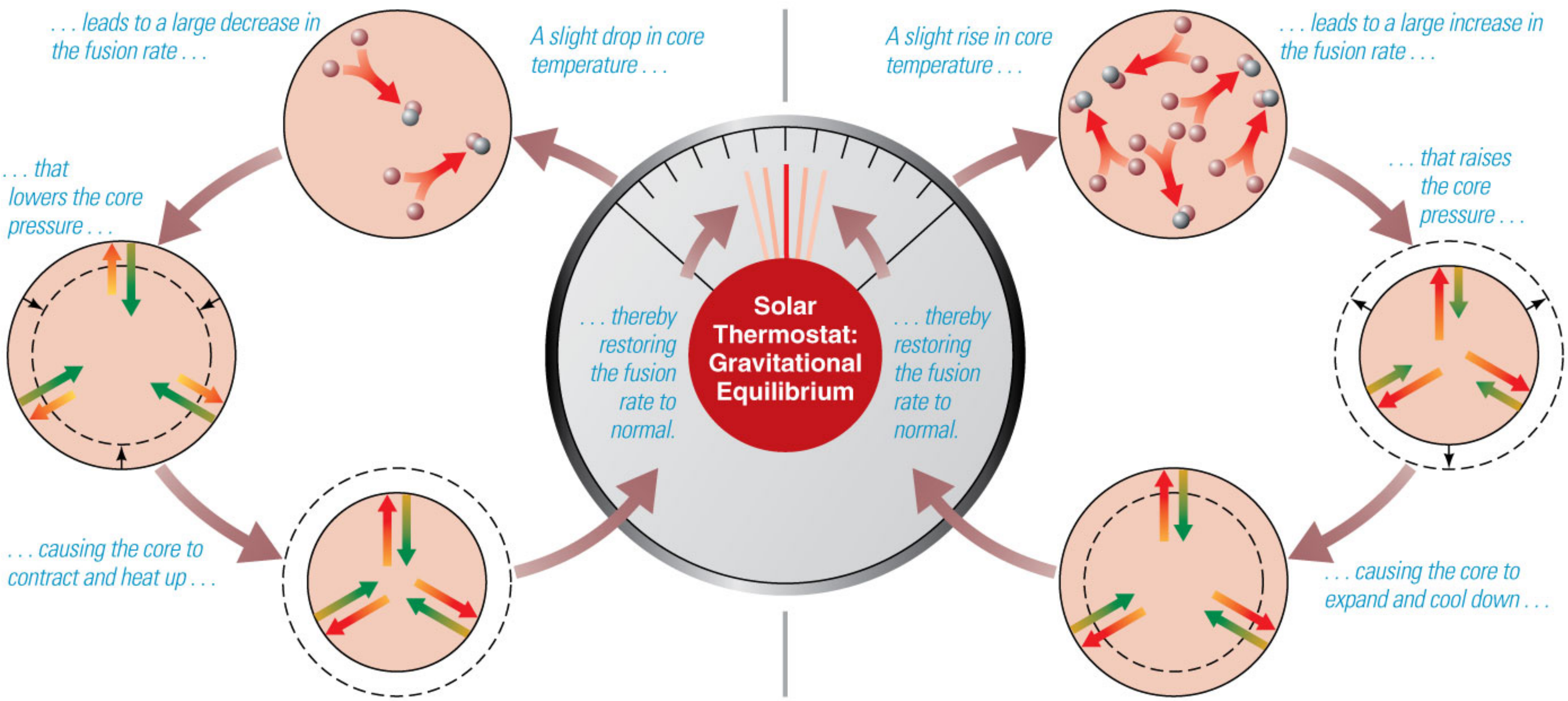
How much mass-energy does the Sun lose each year?
(i.e., how much mass is converted to energy during fusion?)

$$\frac{dM}{dt} = \frac{L_\odot}{c^2} = \frac{3.8 \times 10^{33}\,\mathrm{erg/s}}{(3.0 \times 10^{10}\,\mathrm{cm/s})^2} = 4.3 \times 10^{12}\,\mathrm{g/s}$$

or about $7 \times 10^{-14}\ M_\odot$ per year

Why are stars stable in the first place?

It's a balance between fusion and gravitational contraction



STELLAR STABILITY

$$P_{cent} \propto T$$

$$P_{cent} \sim \frac{GM\rho}{R} \Rightarrow T \propto \frac{1}{R}$$

$$\epsilon_{nuc} \propto T^n \; (\mathrm{n} \approx 4)$$

So if start to run reactions faster

$$T\uparrow \Rightarrow P\uparrow$$
$$\Rightarrow R\uparrow$$ because an ideal gas expands when heated
$$\Rightarrow T\downarrow \Rightarrow \epsilon\downarrow$$

Similarly if the rate of reactions declines for some reason,

$$T\downarrow \Rightarrow P\downarrow$$
$$\Rightarrow R\downarrow$$
$$\Rightarrow T\uparrow \Rightarrow \epsilon\uparrow$$

Thus a tight equilibrium is maintained

Important exception:
Degenerate matter

$$+ \; \tau_{nuc} \lesssim \tau_{diff}$$

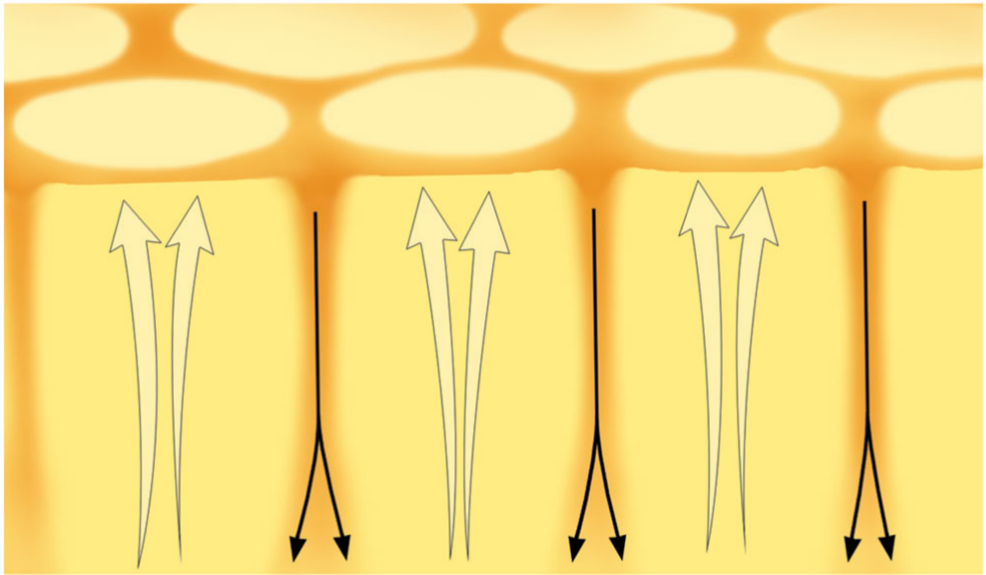
Energy Transport in Stars

3 ways energy can be transported in stars:

1. Convection
   Energy is transported by moving material
   Occurs where the temperature gradient is large or material is very opaque
   Analogy: boiling water, convective cells in the atmosphere

2. Radiation (or radiative diffusion)
   Energy (photons) diffuse through the star (slowly) by themselves

3. Conduction
   Energy is transported via collisions among electrons in a (solid) material
   Analogy: Heating a metal rod
   Important in white dwarfs, not normal stars



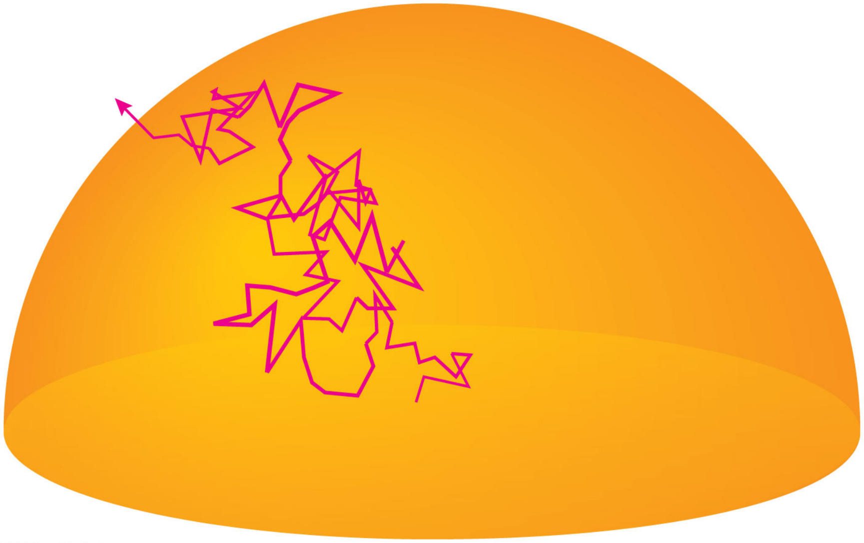
a This diagram shows convection beneath the Sun's surface: hot gas (yellow arrows) rises while cooler gas (black arrows) descends around it.

Convection:
Hot material rises, carrying energy up to surface

Cold material sinks

Radiative Diffusion:
A photon "random walks" from the core to the surface of the star, losing energy as it travels

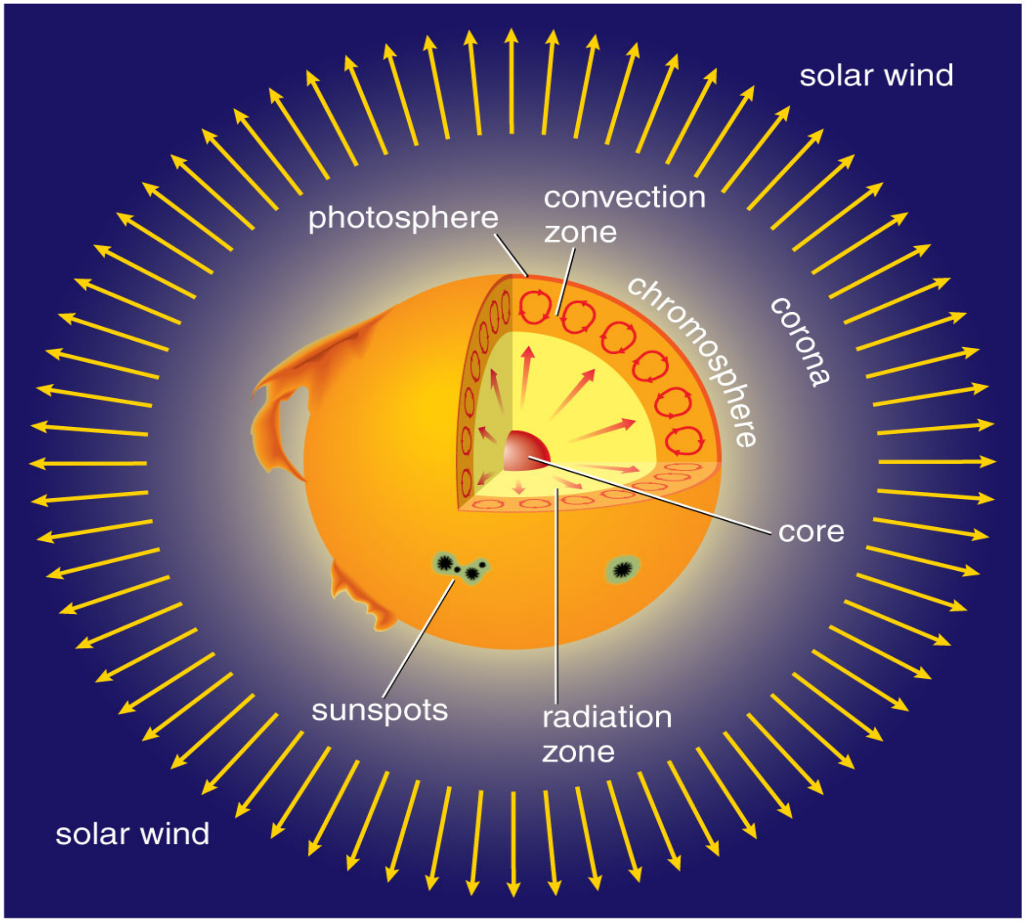
This takes 1000s of years!

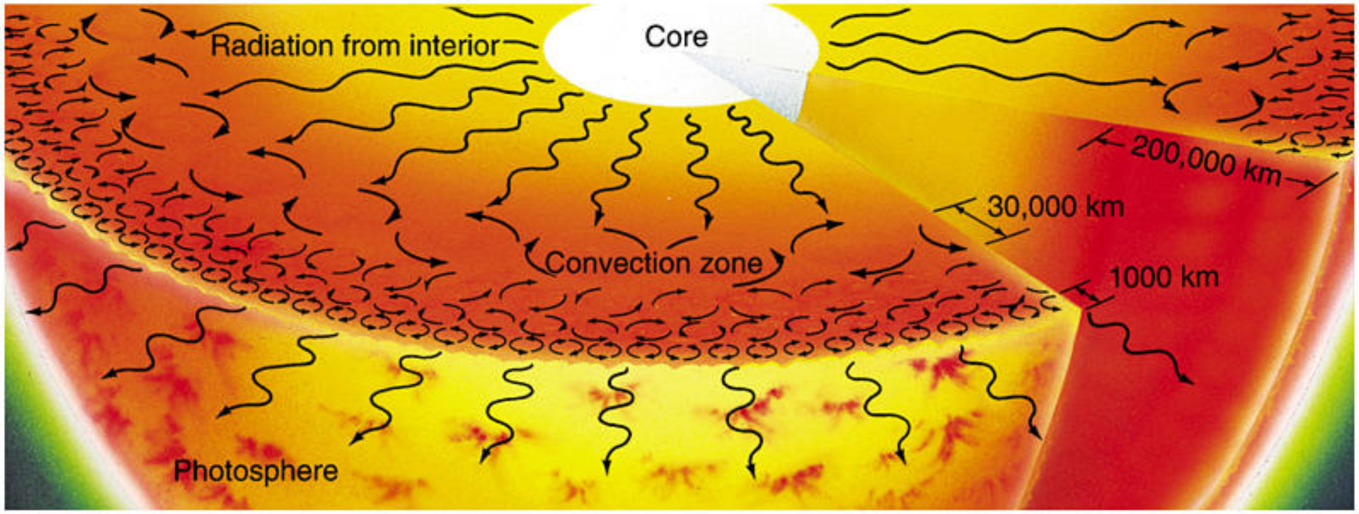


Energy may be transported in more than one way through a star's interior.

Which ways depend on the mass of the star.

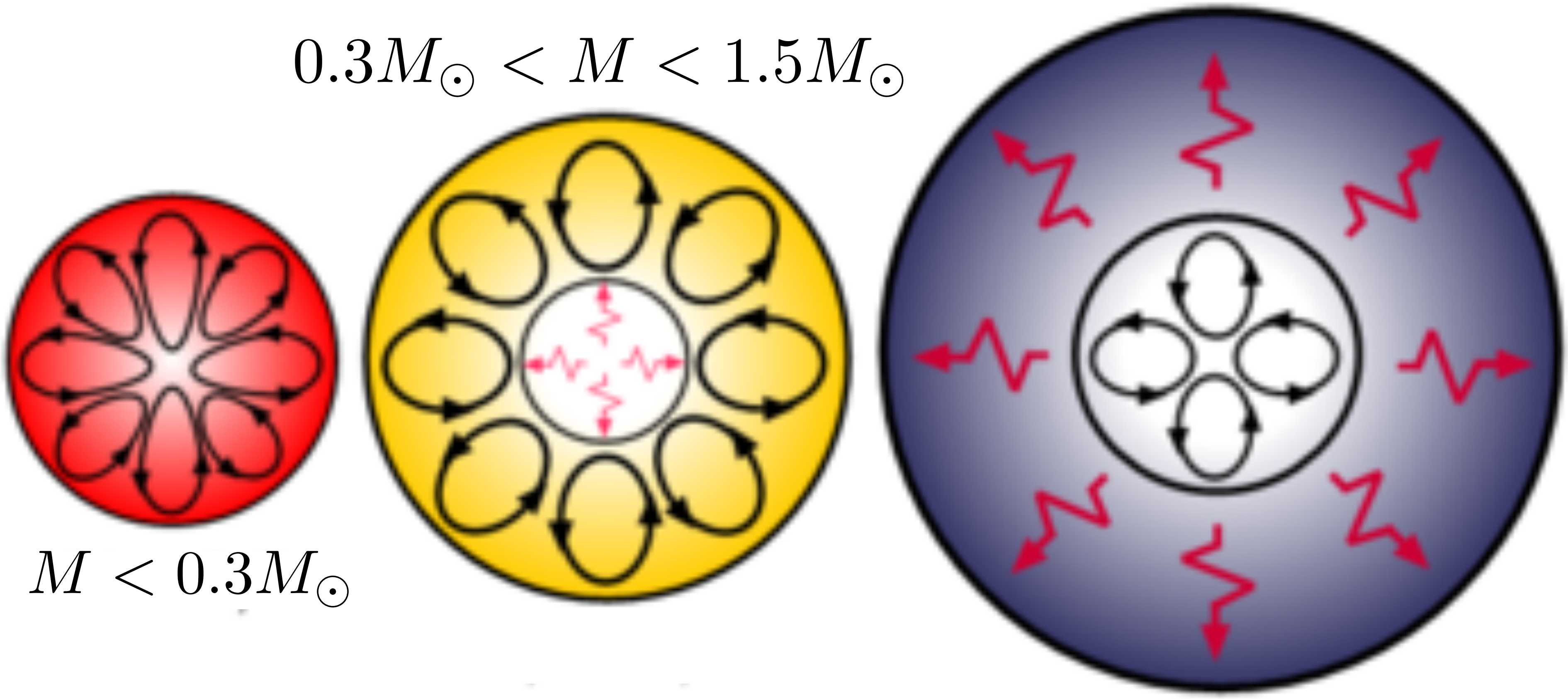
For the Sun, radiation dominates throughout the interior. Convection occurs in the outer envelope.





Interior Structure of Main Sequence Stars

$M < 0.3 M_\odot$	Star completely convective
$0.3 M_\odot < M < 1.5 M_\odot$	Core radiative; envelope convective
$M > 1.5 M_\odot$	Core convective; envelope radiative



Black loops = convection
Red arrows = radiation

Why the different internal transport mechanisms?

In the outer envelope, the opacity (ability to absorb energy) of the gas is important:

- Low-mass stars: Gas is cool enough for neutral hydrogen to exist, which has high opacity, heating the gas and causing it to move (convection)

- High-mass stars: Gas is hot enough for ionized hydrogen, which has lower opacity, allowing photons to pass through unimpeded (radiation)

In the interior, the temperature dependence of the energy production rate is the dominant factor:

- Low-mass stars: pp chain has T^4 dependence, which is weak enough such that radiation can carry away the energy

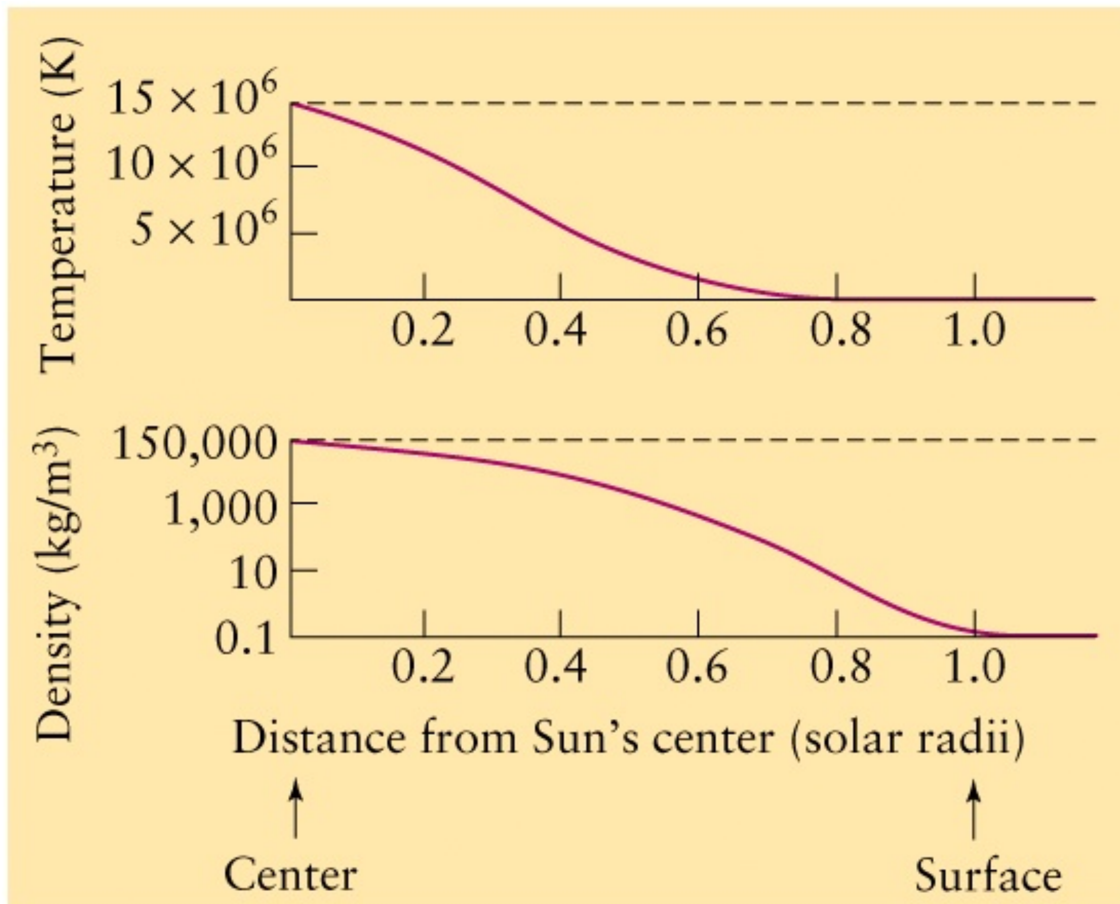
- High-mass stars: CNO cycle has T^{20} dependence, meaning much more energy produced closer to core than right above it, resulting in convection

http://burro.astr.cwru.edu/Academics/Astr221/StarPhys/transport.html

How do we know the interior structure of a star?

Astronomers use mathematical models to calculate temperature, density, luminosity, etc. as a function of radius, assuming a mass and chemical composition

Example model for the Sun



Model of present-day Sun

Table showing how *L, T*, and ρ change with radius (or mass enclosed)

Radiative

Convective

Mass ($M_\odot$)	Radius ($R_\odot$)	Luminosity ($L_\odot$)	Temperature (10^6 °K)	Density (g cm^{-3})
0.0000	0.000	0.0000	15.513	147.74
0.0001	0.010	0.0009	15.48	146.66
0.001	0.022	0.009	15.36	142.73
0.020	0.061	0.154	14.404	116.10
0.057	0.090	0.365	13.37	93.35
0.115	0.120	0.594	12.25	72.73
0.235	0.166	0.845	10.53	48.19
0.341	0.202	0.940	9.30	34.28
0.470	0.246	0.985	8.035	21.958
0.562	0.281	0.997	7.214	15.157
0.647	0.317	0.992	6.461	10.157
0.748	0.370	0.9996	5.531	5.566
0.854	0.453	1.000	4.426	2.259
0.951	0.611	1.000	2.981	0.4483
0.9809	0.7304	1.0000	2.035	0.1528
0.9964	0.862	1.0000	0.884	0.042
0.9999	0.965	1.0000	0.1818	0.00361
1.0000	1.0000	1.0000	0.005770	1.99×10^{-7}

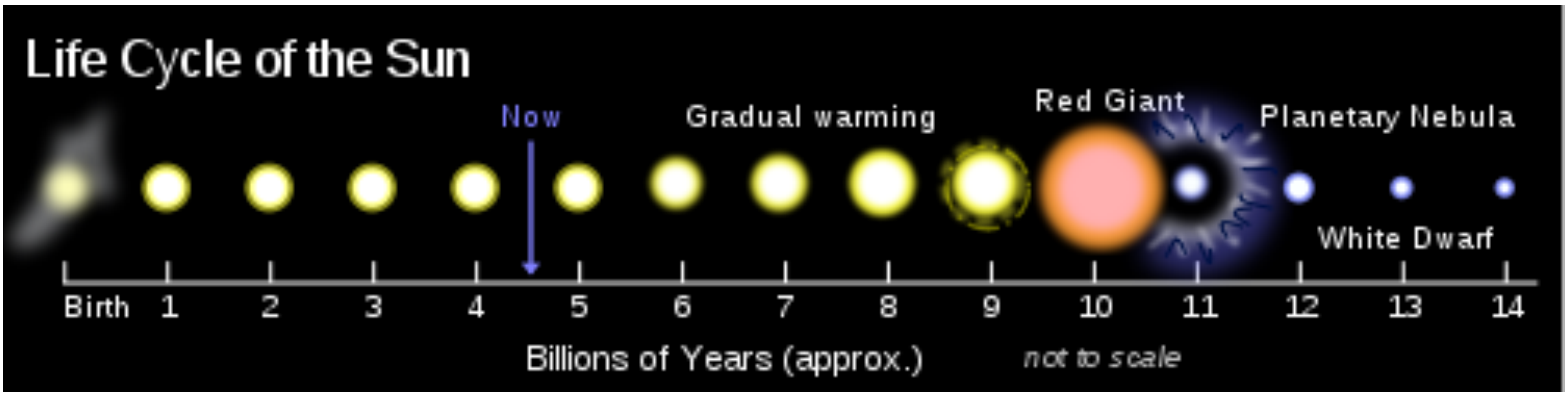
*Adapted from Turck-Chièze et al. (1988).
Composition X = 0.7046, Y = 0.2757, Z = 0.0197

Evolution of the Sun with time

Time (10^9 years)	Luminosity ($L_\odot$)	Radius ($R_\odot$)	$T_{central}$ (10^6 °K)	
Past				
0	0.7688	0.872	13.35	zero age main sequence
0.143	0.7248	0.885	13.46	
0.856	0.7621	0.902	13.68	
1.863	0.8156	0.924	14.08	
2.193	0.8352	0.932	14.22	
3.020	0.8855	0.953	14.60	
3.977	0.9522	0.981	15.12	
Now				
4.587	1.000	1.000	15.51	
Future				
5.506	1.079	1.035	16.18	
6.074	1.133	1.059	16.65	
6.577	1.186	1.082	17.13	
7.027	1.238	1.105	17.62	
7.728	1.318	1.143	18.42	
8.258	1.399	1.180	18.74	
8.7566	1.494	1.224	18.81	
9.805	1.760	1.361	19.25	↘ Red Giant

*Adapted from Turck-Chiéze et al. (1988).
Composition $X = 0.7046$, $Y = 0.2757$, $Z = 0.0197$.
Present values are $R_\odot$ and $L_\odot$.
**For time t before the present age $t_\odot = 4.6 \times 10^9$ years,
$L/L_\odot \approx 1/[1+0.4(1-t/t_\odot)]$

Evolution of the Sun



Optional Slide

Note that the units of pressure and energy density are the same

$$\frac{\text{erg}}{\text{cm}^3} = \frac{\text{dyne cm}}{\text{cm}^3} = \frac{\text{dyne}}{\text{cm}^2}$$

Ideal gas $n\,k\,T$ is the pressure but

$3/2\ n\,k\,T$ is the energy density

Radiation $\frac{1}{3} a T^4$ is the pressure

$a\,T^4$ is the energy density

Optional Slide

Why is $L \propto M^3$?

True even if star is not supported by P_{rad}
Note this is not the total heat content, just the radiation.

$$\text{Luminosity} \approx \frac{\text{Heat content in radiation}}{\text{Time for heat to leak out}} = \frac{E_{radiation}}{\tau_{diffusion}}$$

$$E_{\text{radiation}} = \frac{4}{3}\pi R^3 (aT^4) \propto R^3 T^4 \propto R^3 \left(\frac{M}{R}\right)^4 \propto \frac{M^4}{R} \qquad (\text{Since } T \propto \frac{M}{R})$$

$$\tau_{\text{diffusion}} = \frac{R^2}{l_{\text{mfp}} c} \qquad l_{\text{mfp}} = \frac{1}{\kappa\rho}$$

κ is the "opacity" in $cm^2\ gm^{-1}$

Assume κ is a constant

$$M \approx \frac{4}{3}\pi R^3 \rho \Rightarrow \rho \propto \frac{M}{R^3}$$

$$l_{\text{mfp}} \propto \frac{1}{\rho} \Rightarrow l_{\text{mfp}} \propto \frac{R^3}{M} \qquad \tau_{\text{diffusion}} \propto \frac{R^2 M}{R^3} \propto \frac{M}{R}$$

$$\boxed{L \propto \frac{M^4/R}{M/R} \propto M^3}$$

Other powers of M possible
When κ is not a constant but varies with temperature and density