Homework Set #1

(From Sakurai, Chapter 1)

1. Matrix fun

   (a) Consider two kets $|\alpha\rangle$ and $|\beta\rangle$. Suppose $\langle a'|\alpha\rangle, \langle a''|\alpha\rangle, \ldots$ and $\langle a'|\beta\rangle, \langle a''|\beta\rangle, \ldots$ are all known where $|a'\rangle, |a''\rangle, \ldots$ form a complete set of base kets. Find the matrix representation of the operator $|\alpha\rangle\langle\beta|$ in that basis.

   (b) We now consider a spin $\frac{1}{2}$ system and let $|\alpha\rangle$ and $|\beta\rangle$ be $|s_z = \hbar/2\rangle$ and $|s_x = \hbar/2\rangle$, respectively. Write down explicitly the square matrix that corresponds to $|\alpha\rangle\langle\beta|$ in the usual (s_z diagonal) basis.

2. Using the orthonormality of $|+\rangle$ and $|-\rangle$, prove

$$[S_i, S_j] = i\varepsilon_{ijk}\hbar S_k, \qquad \{S_i, S_j\} = \left(\frac{\hbar^2}{2}\right)\delta_{ij}$$

   where

$$\begin{aligned} S_x &= \frac{\hbar}{2}\left(\,|+\rangle\langle-| \;+\; |-\rangle\langle+|\,\right) \\ S_y &= \frac{i\hbar}{2}\left(\,-|+\rangle\langle-| \;+\; |-\rangle\langle+|\,\right) \\ S_z &= \frac{\hbar}{2}\left(\,|+\rangle\langle+| \;-\; |-\rangle\langle-|\,\right) \end{aligned}$$

3. A beam of spin $\frac{1}{2}$ atoms goes through a series of Stern-Gerlach type measurements as follows:

   (a) The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms.

   (b) The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where s_n is the eigenvalue of the operator $\vec{S}\cdot\hat{n}$, with $\hat{n}$ making an angle β in the xz-plane with the respect to the z-axis.

   (c) The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

   What is the intensity of the final $s_z = -\hbar/2$ beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity? How must we orient the second measuring apparatus if we are to maximize the intensity of the final $s_z = -\hbar/2$ beam?

4. Spin $\frac{1}{2}$ and uncertainty

   (a) Compute

$$< (\Delta S_x)^2 > \equiv < S_x^2 > - < S_x >^2,$$

   where the expectation value is taken for the S_z+ state. Using your result, check the generalized uncertainty relation

$$< (\Delta A)^2 >< (\Delta B)^2 > \geq \frac{1}{4} | < [A, B] > |^2,$$

   with $A \to S_x, B \to S_y$.

   (b) Check the uncertainty relation with $A \to S_x, B \to S_y$ for the S_x+ state.

5. Evaluate the $x - p$ uncertainty product $< (\Delta x)^2 >< (\Delta p)^2 >$ for a one-dimensional particle confined between two rigid walls:

$$V = \begin{cases} 0 & \text{for } 0 < x < a, \\ \infty & \text{otherwise} \end{cases}$$

   Do this for both the ground and excited states.