A.1 Angular Momentum

A. Eigenstates and eigenvalues

- Generic operators J^2, J_z
  - ◇ Desire simultaneous eigenstates $|\lambda m\rangle$

$$\begin{aligned} J^2|\lambda m\rangle &= \lambda\hbar^2|\lambda m\rangle \\ J_z|\lambda m\rangle &= m\hbar|\lambda m\rangle \end{aligned}$$

  - ◇ Restrictions on λ, m

$$\begin{aligned} < J^2 > \geq 0 \;&\Rightarrow\; \lambda \geq 0 \\ < J^2 - J_z^2 > &= < J_x^2 + J_y^2 > \geq 0 \\ &= (\lambda - m^2)\hbar^2 \\ &\Rightarrow \quad -\sqrt{\lambda} \leq m \leq \sqrt{\lambda} \end{aligned}$$

- Introduce $J_\pm \equiv J_x \pm iJ_y$
  - ◇ Similar flavor as $a, a^\dagger$ from the SHO
  - ◇ Note

$$\begin{aligned} &[J_z, J_\pm] \;=\; \pm\hbar J_\pm \\ &J_- J_+ = J^2 - J_z^2 - \hbar J_z \\ &J_+ J_- = J^2 - J_z^2 + \hbar J_z \\ &[J_\pm, J^2] = 0 \end{aligned}$$

- Magic

$$J^2 J_\pm|\lambda m\rangle = \lambda\hbar^2 J_\pm|\lambda m\rangle$$

  - ◇ $J_\pm|\lambda m\rangle$ are eigenkets of J^2 with identical eigenvalues!

$$\begin{aligned} J_z J_\pm|\lambda m\rangle &= J_\pm J_z|\lambda m\rangle \;\pm\; \hbar J_\pm|\lambda m\rangle \\ &= (m \pm 1)\hbar J_\pm|\lambda m\rangle \end{aligned}$$

  - ◇ $J_\pm|\lambda m\rangle$ are eigenstates of J_z with eigenvalues $m' = m \pm 1$
- Identify $J_\pm$ as Raising/Lowering Operators

$$J_\pm|\lambda m\rangle = c_\pm(\lambda, m)\;|\lambda, m \pm 1\rangle$$

- $\diamond$ Restrict $|m| < \sqrt{\lambda}$
- $\diamond$ $\exists\ m_\pm$ s.t. $J_\pm|\lambda m_\pm\!> = 0$

$$\begin{aligned} J_-J_+|\lambda m_+\!> &= 0 = (\lambda\hbar^2 - m_+^2\hbar^2 - m_+\hbar^2)|\lambda m_+\!> \\ &\Rightarrow\ \lambda = m_+(m_+ + 1) \\ J_+J_-|\lambda m_-\!> &= 0 = (\lambda\hbar^2 - m_-^2\hbar^2 + m_-\hbar^2)|\lambda m_-\!> \\ &\Rightarrow\ \lambda = m_-(m_- - 1) \end{aligned}$$

- $\diamond$ Therefore, $m_- = -m_+$
- $\diamond$ Travel from m_- to m_+ with repeated applications of J_+

$$\begin{aligned} &\Rightarrow\ m_+ - m_- = n, \quad \text{an integer} \\ &\Rightarrow\ m_+ = n/2 \end{aligned}$$

- m, λ
  - $\diamond$ $m_+ = n/2 \equiv j = 0, \frac{1}{2}, 1, \ldots$
  - $\diamond$ $\lambda = j(j+1)$ with $m_j = -j, -j+1, \ldots, j-1, j$
- Altogether now

$$J^2|jm_j\!> = j(j+1)\hbar^2|jm_j\!> \tag{1}$$

$$J_z|jm_j\!> = m_j\hbar|jm_j\!> \tag{2}$$

  - $\diamond$ $j =$ integer or half-integer
  - $\diamond$ $m_j = -j, j+1, \ldots, j-1, j$
  - $\diamond$ $2j+1$ states for each j
  - $\diamond$ For L, identify $j \to \ell, m_j \to m$
- Spherical Harmonics
  - $\diamond$ $\mathcal{Y}_{\ell m}(\theta, \phi)$
  - $\diamond$ Represented by $|\ell m\!>$
  - $\diamond$ $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}, \quad L_z|\ell m\!> = m\hbar|\ell m\!>$
    - ▲ Expand $\mathcal{Y}_{\ell m} = f_{\ell m}(\theta)g_m(\phi)$

$$g(\phi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{im\phi}$$

    - ▲ Trust me

$$f_{\ell\ell}(\theta) = (\sin\theta)^\ell$$

- ▲ Apply J_- to get the other $f_{\ell m}(\theta)$ functions
- Finally

$$Y_{\ell m}(\theta,\phi) = \left[\frac{(\ell-|m|)!}{(\ell+|m|)!}\frac{2\ell-1}{4\pi}\right]^{\frac{1}{2}} (-1)^{(m+|m|)/2}\,\mathcal{P}_\ell^m(\cos\theta)\,\mathrm{e}^{im\phi} \tag{3}$$

  - ◇ $\mathcal{P}_\ell^m$: Associated Legendre polynomials
  - ◇ $Y_{\ell m}(\theta,\phi)$: Rank ℓ tensor
  - ◇ Convenient relation

$$\mathcal{Y}_{\ell,-m} = (-1)^m Y_{\ell m}^* \tag{4}$$

  - ◇ Listing a few

$$\begin{aligned}
\mathcal{Y}_{00} &= \frac{1}{\sqrt{4\pi}}\\
\mathcal{Y}_{11} &= -\sqrt{\frac{3}{8\pi}}\mathrm{e}^{i\phi}\sin\theta\\
\mathcal{Y}_{10} &= \sqrt{\frac{3}{4\pi}}\cos\theta\\
\mathcal{Y}_{22} &= \sqrt{\frac{15}{32\pi}}\mathrm{e}^{2i\phi}\sin^2\theta\\
\mathcal{Y}_{21} &= -\sqrt{\frac{15}{8\pi}}\mathrm{e}^{i\phi}\sin\theta\cos\theta\\
\mathcal{Y}_{20} &= \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right)
\end{aligned}$$

B. Addition of Angular Momentum

- General case

$$\vec{J} \;=\; \vec{J}_1 \;+\; \vec{J}_2 \tag{5}$$

  - ◇ $\vec{J}_1, \vec{J}_2$ could be $\vec{L}, \vec{S}$
  - ◇ Or $\vec{J}_1 = \vec{L}+\vec{S}$ and $\vec{J}_2 = \vec{I}$ with $\vec{I}$ the nuclear spin
  - ◇ Or addition of spin for two electrons $\vec{S}_1, \vec{S}_2$
- Guiding principles
  - ◇ Can only specify one component of each of the vectors we are adding (e.g. $\hat{z}$)

- $\diamond$ $\vec{J}$ must obey our rules of quantization for angular momentum

$$[J_x, J_y] = i\hbar J_z$$
$$[J_x, J^2] = 0$$

- $\diamond$ J_1^2, J_2^2 also commute with J^2 and J_z
  - ▲ Find simultaneous eignestates of $\{J^2, J_z, J_1^2, J_2^2\}$
  - ▲ But not J_{1z}, J_{2z}
  - ▲ Label the states $|jmj_1j_2>$ or $|jm>$ for short
  - ▲ Original basis: $|j_1j_2m_1m_2>$

- Formal solution

$$|jm> = \sum_{m_1,m_2} |m_1m_2><m_1m_2|jm> \tag{6}$$

  - $\diamond$ $<m_1m_2|jm>$ – Clebcsh-Gordon coeffecients
  - $\diamond$ Similarly

$$|m_1m_2> = \sum_{jm} |jm><jm|m_1m_2> \tag{7}$$

- Rules:

(a) m values

$$m = m_1 + m_2 \tag{8}$$

(b) j values (for $j_1 \geq j_2$)

$$j = j_1 + j_2, j_1 + j_2 - 1, \ldots j_1 - j_2 \tag{9}$$

(c) Find coefficients using C-G tables or calculate explicitly
  - $\diamond$ Apply the lowering operator J_-
  - $\diamond$ $J_-|jm> = \sqrt{j(j+1) - m(m-1)}\hbar|j, m-1>$

- Example: Two spin 1/2 particles
  - $\diamond$ j value: $j = 1, 0$
  - $\diamond$ m value: Sum of m_1 and m_2
  - $\diamond$ Explicitly
    - ▲ $|jm> = |11> \Rightarrow m_1 = m_2 = +\frac{1}{2}$

$$|11> = |++>$$

▲ $|jm\rangle = |10\rangle \Rightarrow m_1 = +\frac{1}{2}, -\frac{1}{2}; m_2 = -\frac{1}{2}, +\frac{1}{2}$

$$|10\rangle = \frac{1}{\sqrt{2}}|+-\rangle + \frac{1}{\sqrt{2}}|-+\rangle$$

▲ $|jm\rangle = |1-1\rangle \Rightarrow m_1 = m_2 = -\frac{1}{2}$

$$|1-1\rangle = |--\rangle$$

▲ $|jm\rangle = |00\rangle \Rightarrow m_1 = +\frac{1}{2}, -\frac{1}{2}; m_2 = -\frac{1}{2}, +\frac{1}{2}$ and orhogonal to $|10\rangle$

$$|00\rangle = \frac{1}{\sqrt{2}}|+-\rangle - \frac{1}{\sqrt{2}}|-+\rangle$$

C. Wigner-Eckart Theorem

- Rotation operator
  - ◇ Analogous to translation and time-evolution operators
  - ◇ Generate rotation

$$\mathcal{D}(R) = 1 - i\epsilon\frac{\vec{J}\cdot\hat{n}}{\hbar} \tag{10}$$

- Vector operator: $\vec{V}$
  - ◇ Analagous to classical vector
  - ◇ Transforms like the following under a rotation

$$V_i \rightarrow \sum_j R_{ij}V_j$$

  - ◇ Expectation value of $\vec{V}$ changes under a rotation like

$$<\alpha|V_i|\alpha> \rightarrow <\alpha|\mathcal{D}^\dagger(R)V_i\mathcal{D}(R)|\alpha> = \sum_j R_{ij}<\alpha|V_j|\alpha>$$

  - ◇ Commutation relation

$$[V_i, J_j] = i\varepsilon_{ijk}\hbar V_k \tag{11}$$

- Spherical Tensors
  - ◇ General definition

$$T_q^{(k)} = \mathcal{Y}_{\ell=k}^{m=q}(\vec{V}) \tag{12}$$

▲ k = rank

▲ q = magnetic quantum number

- $\diamond$ Irreducible $\rightarrow$ Not composed of a set of tensors with different behaviour under rotations
- $\diamond$ Cartesian tensor

$$U_iV_j = \frac{\vec{U}\cdot\vec{V}}{2}\delta_{ij} + \frac{(U_iV_j - U_jV_i)}{2} + \left(\frac{U_iV_j + U_jV_i}{2} - \frac{\vec{U}\cdot\vec{V}}{3}\delta_{ij}\right)$$

  - ▲ Each of the above terms behaves differently under rotations
  - ▲ Specifically, as angular momenta with $\ell = 0, 1, 2$ respectively
- $\diamond$ Specific spherical tensors

$$\mathcal{Y}_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \rightarrow T_0^{(1)} = \sqrt{\frac{3}{4\pi}}V_z$$
$$\mathcal{Y}_1^{\pm1} = \mp\sqrt{\frac{3}{4\pi}}\frac{x \pm iy}{\sqrt{2}r} \rightarrow T_{\pm1}^{(1)} = \sqrt{\frac{3}{4\pi}}\left(\mp\frac{V_x \pm iV_y}{\sqrt{2}}\right)$$
$$\mathcal{Y}_2^{\pm2} = \sqrt{\frac{15}{32\pi}}\frac{(x \pm iy)^2}{r^2} \rightarrow T_{\pm2}^{(2)} = \sqrt{\frac{15}{32\pi}}\,(V_x \pm iV_y)^2$$

- $\diamond$ Commutation relations

$$[J_z, T_q^{(k)}] = \hbar q T_q^{(k)} \tag{13}$$
$$[J_\pm, T_q^{(k)}] = \hbar\sqrt{(k \mp q)(k \pm q + 1)}T_{q\pm1}^{(k)} \tag{14}$$

- Wigner-Eckart Theorem

$$<\alpha', j'm'|T_q^{(k)}|\alpha, jm> = <jk; mq|jk; j'm'>\frac{<\alpha'j'|T^{(k)}|\alpha j>}{\sqrt{2j+1}} \tag{15}$$

  - $\diamond$ First factor on RHS is the Clebsch-Gordon coeffcient for adding j to k to get j' $\rightarrow$ Purely geometrical
  - $\diamond$ Second factor does depend on dynamics, but makes no reference to the magnetic quantum numbers m, m' and q
  - $\diamond$ Application: Evaluate $<\alpha'j'|T^{(k)}|\alpha j>$ for one (educated) choice of m, m', q and use C-G to relate all other terms
- Selection Rules
  (a) m-selection rule

$$<\alpha', j'm'|T_q^{(k)}|\alpha, jm> = 0 \quad \text{unless } m' = q + m \tag{16}$$

  (b) Triangle relation

$$|j - k| \leq j' \leq j + k \tag{17}$$

- Examples

(a) Tensor of rank 0, i.e. scalar $T_0^{(0)} = C$

$$<\alpha', j'm'|C|\alpha, jm> = \delta_{jj'}\delta_{mm'}\frac{<\alpha'j'|C|\alpha j>}{\sqrt{2j+1}} \tag{18}$$

(b) Vector operator $\to$ Rank 1 tensor. The spherical components of $\vec{V}$ are $V_{q=\pm1,0}$. Selection rules

$$\Delta m \equiv m - m' = \pm 1, 0$$

$$\Delta j \equiv j' - j = \begin{cases} \pm 1 \\ 0 \end{cases}$$

$0 \to 0$ transition is forbidden