VIII. Photoionization and Recombination

A. Photoelectric Effect

- Bound-free absorption
- Process (fig)
  - ◇ Photon with energy $\hbar\omega$ strikes an atom with binding energy I
  - ◇ Ejected electron has energy $\hbar\omega - I > 0$
  - ◇ i.e. Absorption can occur for a continuous range of frequencies
- Goal
  - ◇ Combine our expression for the probability of absorption (w_{fi}) with the density of final states
  - ◇ Calculate the probability/time for a transition to a 'cell' of phase space
  - ◇ We will express this probability in terms of a cross-section
- Final state
  - ◇ Define for the emitted electron

$$\hbar\vec{k}_f = \vec{p}_f \tag{1}$$

  - ◇ Assume a wave function normalized to a very large volume L^3

$$|\vec{k}_f> \equiv \frac{1}{L^{3/2}}|f> \tag{2}$$

  - ◇ Energy

$$E_f = \frac{p_f^2}{2m} = \frac{\hbar^2 k_f^2}{2m} \tag{3}$$

- Density of final states
  - ◇ Usual phase space argument (periodic boundary conditions)

$$k_x = \frac{2\pi n_x}{L} \qquad \text{with } n_x \text{ any integer}$$

  - ◇ For final momentum $\hbar k_f$ propagating in direction $d\Omega$, the density of final states is

$$n^2 dn d\Omega \qquad \text{with } n^2 = n_x^2 + n_y^2 + n_z^2 = k_f^2 \left(\frac{L}{2\pi}\right)^2 \tag{4}$$

  - ◇ Fig

  - ◇ Finally

$$\begin{aligned} n^2 dn d\Omega &= g(E) dE d\Omega \\ &= \left(\frac{L}{2\pi}\right)^3 \frac{m_e}{\hbar^2} k_f \, dE \, d\Omega \end{aligned} \tag{5}$$

- Cross-section
  - ◇ Definition

$$\sigma = \frac{\text{(Energy/unit time) absorbed by atom}(i \to f)}{\text{Energy flux of the radiation field}} \tag{6}$$

  - ◇ Photoelectric effect

$$d\sigma_p = \frac{\mathrm{w_{fi}}\, g(E) d\Omega \cdot \hbar\omega}{I(\omega)} \tag{7}$$

  - ◇ Plug and chug...

$$\frac{d\sigma_p}{d\Omega} = \frac{\alpha}{2\pi} \frac{k_f}{m\hbar\omega} \left| < f | \mathrm{e}^{i\vec{k}\cdot\vec{r}}\, \hat{\varepsilon} \cdot \vec{p} \, | \phi_i > \right|^2 \tag{8}$$

- Born Approximation
  - ◇ If the final momentum is large compared to the bound state momentum, a good approximation for the wave function is

$$|f> = e^{i\vec{k}_f\cdot\vec{r}} \tag{9}$$

  - ◇ Condition for a Hydrogenic ground state

$$\frac{p_f^2}{2m} = \frac{\hbar^2 k_f^2}{2m} \gg \frac{e^2}{2a_0}$$
$$\Rightarrow \; k_f a_0 \gg 1$$

  - ◇ Calculating $d\sigma/d\Omega$ from the ground state

$$|\phi_i> = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0} \tag{10}$$

  - ◇ Introduce the momentum transfer (momentum change between the ejected electron and the photon)

$$\vec{q} \equiv \vec{k}_f - \vec{k} \tag{11}$$

  - ◇ Matrix element

$$\left|<f|e^{i\vec{k}\cdot\vec{r}}\,\hat{\varepsilon}\cdot\vec{p}\,|\phi_i>\right|^2 = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\int d^3\vec{r}\; e^{i\vec{r}\cdot(\vec{k}-\vec{k}_f)}\vec{\varepsilon}\cdot\frac{\hbar}{i}\vec{\nabla}e^{-Zr/a_0} \tag{12}$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\hbar\hat{\varepsilon}\cdot\vec{k}_f\int d^3\vec{r}\;\exp\left[-i\vec{r}\cdot\vec{q}-\frac{Zr}{a_0}\right] \tag{13}$$

$$= \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2}\hat{\varepsilon}\cdot\vec{k}_f\frac{8\pi Z\hbar}{a_0}\left[\frac{1}{Z^2/a_0^2+q^2}\right]^2 \tag{14}$$

    - ▲ In step 2, we integrated by parts noting $\hat{\varepsilon}\cdot\vec{k} = 0$
    - ▲ The result is simply the Fourier transfor of a radial exponential
  - ◇ Define angles
    - ▲ Fig

▲ Define

$$\cos\alpha \equiv \frac{\vec{k}_f \cdot \vec{\varepsilon}}{k_f} = \sin\theta\cos\phi$$

◇ Gather and conquer

$$\frac{d\sigma_p^B}{d\Omega} = 32\hbar k_f \alpha \, \frac{(\hat{\varepsilon}\cdot\vec{k}_f)^2}{m\omega} \frac{Z^5}{a_0^5} \frac{1}{[\,Z^2/a_0^2 + q^2\,]^4} \tag{15}$$

$$= \frac{32 e^2 Z^5 \ \cos^2\!\alpha \ (k_f a_0)^3}{mc\omega(Z^2 + q^2 a_0^2)^4} \tag{16}$$

◇ Non-relativistic Born approximation

▲ Compare (swindle?)

$$\frac{k}{k_f} = \frac{\omega}{c}\cdot\frac{1}{k_f} = \frac{\hbar\omega}{\hbar c k_f} \sim \frac{\hbar^2 k_f^2}{2mc\hbar k_f} = \frac{\hbar k_f}{2mc} = \frac{v_f}{c} \ll 1$$

▲ Therefore

$$q^2 = k_f^2 + k^2 - 2\vec{k}_f\cdot\vec{k} \ \approx \ k_f^2\,[\,1 - (v_f/c)\cos\theta\,] + \mathcal{O}(v_f^2/c^2) \tag{17}$$

◇ Finally

▲ To first order in v_f/c and taking $q^2 a_0^2 \gg 1$

$$\frac{d\sigma_p^B}{d\Omega} = \frac{32 e^2 Z^5}{mc\omega(k_f a_0)^5} \frac{\sin^2\theta\cos^2\phi}{[\,1 - (v_f/c)\cos\theta\,]^4} \tag{18}$$

▲ Integrating over θ and ϕ

$$\sigma^B_{photo} = 2^8 \frac{\pi}{3} \alpha\, a_0^2\, Z^5 \left(\frac{I_H}{\hbar\omega}\right)^{7/2} \tag{19}$$

○ I_H is the ionization potential

○ Note $\sigma = 0$ for $\hbar\omega < I_H$

○ Also, our expression only holds for $\hbar\omega \gg Z^2 I_H$

◇ Cross-section from excited states

▲ If $|i>$ is an 'ns' state, multiply by $1/n^3$

▲ Therefore, the cross-section is much smaller for excited states

▲ General expression

$$\sigma^B_{photo}(n,\ell) = \frac{2^9 \pi^7 m e^{10} Z^4}{3\sqrt{3} c h^6 n^5} \frac{g(\omega, n, \ell, Z)}{\omega^3} \tag{20}$$

○ g is the bound-free Gaunt factor

○ Convenient way of sweeping QM under the rug

- ◦ Near threshold ($\hbar\omega \approx I_{n\ell}$), g is unity to 20%

- Dipole Approximation
  - ◇ Valid for $\hbar\omega \approx I_H$
  - ◇ Same approximation as we used to calculate dipole radiation for absorption and induced emission
  - ◇ Instead of $|f\!> = e^{i\vec{k}_f\cdot\vec{r}}$, let $e^{i\vec{k}\cdot r} = 1$ assuming

$$ka_0 \ll 1 \qquad \text{(automatically satisfied near threshold)}$$

  - ◇ Differential cross-section

$$\frac{d\sigma_p^D}{d\Omega} = \frac{\alpha}{2\pi}\frac{m\omega k_f}{\hbar}\left(\hat{\varepsilon}\cdot\vec{r}\right)_{fi} \tag{21}$$

  - ◇ For photoionization out of the ground state (1s)

$$|i\!> = |n = 1; \ell = 0; m = 0; m_s = \pm\!> \tag{22}$$
$$|f\!> = \text{"appropriate final state"} \tag{23}$$

- Total cross-section

$$\sigma^D_{photo} = \frac{2^9\pi^2}{3}\alpha a_0^2\left(\frac{I_H}{\hbar\omega}\right)^4 f(\eta) \tag{24}$$

  - ◇ where

$$f(\eta) \equiv \frac{\exp\left[\,-4\eta\cot^{-1}\eta\,\right]}{1-\exp\left[\,-2\pi\eta\,\right]} \tag{25}$$

$$\eta \equiv \left(\frac{I_H}{E_f}\right)^{\frac{1}{2}} = \left(\frac{I_H}{\hbar\omega - I_H}\right)^{\frac{1}{2}} \tag{26}$$

  - ◇ For general atomic K-shell (n=1) absorption, replace

$$I_H \to Z^2 I_H$$
$$a_0 \to a_0 Z$$
$$\eta \to \left[\,Z^2 I_H/(\hbar\omega - Z^2 I_H)\,\right]^{\frac{1}{2}}$$
$$\alpha \to 2\alpha \qquad \{\text{2 K-shell electrons}\}$$

  - ◇ In the Born limit, the Dipole approximation becomes $\eta \ll 1$

$$f_{Born}(\eta) \to \frac{1}{2\pi\eta} \approx \frac{1}{2\pi Z}\left(\frac{I_H}{\hbar\omega}\right)^{-\frac{1}{2}} \tag{27}$$

    - ▲ This gives the same result as Equation (19)

◇ Near threshold, $\hbar\omega \approx I_H \;\Rightarrow\; \eta \gg 1$

$$f(\eta) \approx \mathrm{e}^{-4+4/3\eta^2} \approx \mathrm{e}^{-4}\left(1+\frac{4}{3\eta^2}\right) \tag{28}$$

$$\approx \mathrm{e}^{-4}\left(1+\frac{1}{\eta^2}\right)^{4/3} = \mathrm{e}^{-4}\left(\frac{\hbar\omega}{I_H}\right)^{4/3} \tag{29}$$

- Altogether

$$\sigma \propto \begin{cases} 0 & \hbar\omega < I \\ \nu^{-8/3} & \hbar\omega \approx I \\ \nu^{-3} & \hbar\omega \gtrsim I \\ \nu^{-7/2} & \hbar\omega \gg I \end{cases} \tag{30}$$

B. Radiative Recombination

- Fasten your seatbelts!
- Consider an e^- incident on a proton. In the non-relativistic case, the following can occur
  1. Nothing $\leftarrow$ Most probable since protons are small
  2. $\mathrm{e}^- + \mathrm{p} \rightarrow \mathrm{e}^- + \mathrm{p}$ $\leftarrow$ Coloumb scattering – uninteresting in our course
  3. $\mathrm{e}^- + \mathrm{p} \rightarrow \mathrm{H} + \gamma$ $\leftarrow$ Radiative recombination
  4. $\mathrm{e}^- + \mathrm{p} \rightarrow \mathrm{e}^- + \mathrm{p} + \gamma$ $\leftarrow$ Bremsstrahlung
  5. $\mathrm{e}^- + \mathrm{p} \rightarrow \mathrm{n} + \nu_e$ $\leftarrow$ Inverse β decay – need weak interaction theory
- Our process

$$\mathrm{e}^- + A^{(n)+} \rightarrow A^{(n-1)+} + \gamma \tag{31}$$

- Contrast with dielectric recombination (not considered here) where no photon is emitted but another e^- is excited

- Key: Milne's Relation
  - ◇ Can relate the radiative recombination cross-section to photoionization via detailed balance
  - ◇ Need to assume thermal equilibrium and the phase-space distribution function $f(v)$ for the electrons
  - ◇ Let $\sigma_{rec}(v)$ be the cross-section for recombination for electrons of velocity v
    - ▲ Number of recombinations per unit time per unit volume due to thermal electrons in speed range dv

$$n_+ n_e \sigma_{rec} f(v) v dv \tag{32}$$

    - ▲ n_+ is the ion number density, n_e is the electron density
  - ◇ Number of ions photoionized per time per volume per frequency $d\nu$

$$\frac{4\pi}{h\nu} n_n \sigma_{photo} \left(1 - \mathrm{e}^{-h\nu/kT}\right) u(\nu, T) d\nu \tag{33}$$

    - ▲ n_n is the neutral atom density with electron in level n
    - ▲ Factor $1 - \mathrm{e}^{-h\nu/kT}$ gives the net photoionization rate with stimulated recombinations subtracted out
  - ◇ Identifying $f(v)$ with the Maxwellian velocity distribution

$$\begin{aligned} \frac{\sigma_{photo}}{\sigma_{rec}} &= \frac{n_+ n_e}{n_n} \mathrm{e}^{h\nu/kT} \frac{f(v) c^2 h}{8\pi m \nu^2} \\ \text{with } f(v) &= 4\pi \left(\frac{m}{2\pi kT}\right)^{3/2} v^2 \exp\left(\frac{-mv^2}{2kT}\right) \end{aligned}$$

  - ◇ Using Saha's equation (without derivation)

$$\frac{n_+ n_e}{n_n} = \left(\frac{2\pi mkT}{h^2}\right)^{3/2} \frac{g_e g_+}{g_n} \mathrm{e}^{-I_H/kT} \tag{34}$$

    - ▲ Derived from simple statistical mechanics (partition functions)
    - ▲ g_e, g_+, g_n are the appropriate degeneracies
  - ◇ Finally, using $h\nu = \frac{1}{2}mv^2 + I_n$

$$\sigma_{rec}(v, n) = 2\left(\frac{h\nu}{mc}\right)^2 \frac{g_n}{g_e g_+} \frac{\sigma_{photo}}{v^2} \tag{35}$$

    - ▲ Generally: $g_e = 2, g_+ = 1$ and $g_n = 2n^2$
    - ▲ Also, quick approximation for $n = 1$ and Hydrogen

$$\begin{aligned} v \sim \alpha c, \quad I_H &\sim mc^2\alpha^2 \\ \Rightarrow \sigma_{rec}(v, n) &\sim \alpha^2 \sigma_{photo}(\nu) \end{aligned}$$

- To evaluate σ_{rec} or $d\sigma_{rec}$, we simply consider σ_{photo} or $d\sigma_{photo}$ as before

$$d\sigma_{photo} = \frac{\alpha}{2\pi}\frac{v_f}{\hbar^2\omega}d\Omega\left|<f|\mathrm{e}^{i\vec{k}\cdot\vec{r}}\,\hat{\varepsilon}\cdot\vec{p}|i>\right|^2 \tag{36}$$

  - ◇ Now $|i>$ is the free particle and $|f>$ is the bound state

- Assume dipole approximation: $\mathrm{e}^{i\vec{k}\cdot\vec{r}} = 1$

  - ◇ Initial wave function → Should asymptotically approach an incoming plane wave at large distances

$$\begin{aligned} |i> &\rightarrow \frac{1}{\sqrt{V}}\mathrm{e}^{i\vec{k}_i\cdot\vec{r}} \ \text{ at } r\rightarrow\infty \\ &= \sqrt{\frac{\pi}{2Vk_i r}}\sum_{\ell=0}^{\infty}(2\ell+1)\,i^\ell\,P_\ell(\cos\theta)\,J_{\ell+\frac{1}{2}}(kr) \end{aligned}$$

  - ◇ Appropriate representation of $|i>$ are the Hydrogenic energy eigenfunctions which asymptotically approach the above expression

$$|i> = \frac{\sqrt{2\pi\hbar}}{2k_i}\sum_{\ell=0}^{\infty}(2\ell+1)i^\ell P_\ell(\cos\theta)\frac{\Gamma(\ell+1-i\eta)}{|\Gamma(\ell+1-i\eta)|}R_{w\ell}(r) \tag{37}$$

    - ▲ where

$$\begin{aligned} \eta &= \sqrt{\frac{I_n}{E_i}} \\ R_{w\ell} &= \text{eigenfunctions with energy E=W} \end{aligned}$$

- For transitions to the ground state ($1s$), only $\ell = 1$ term contributes (same as A_{fi} selection rule; $\Delta\ell = \pm 1$)

  - ◇ Same integrals as for photoelectric effect in Dipole approx!
  - ◇ Milne relation

$$\begin{aligned} \sigma_{rec}(v;n=1) &= 2\left(\frac{h\nu}{mc}\right)^2\frac{1}{v^2}\sigma_{photo}(v) \\ &= 2\left(\frac{h\nu}{mc}\right)^2\frac{m}{2(\hbar\omega-I_\nu)}\frac{2^9\pi^2}{3}\alpha a_0^3\left(\frac{I_H}{\hbar\omega}\right)^4 f(\eta) \\ &= \frac{2^8\pi^2}{3}\alpha^3 a_0^2\left(\frac{I_H}{\hbar\omega}\right)^2\frac{I_H}{\hbar\omega-I_H}f(\eta) \end{aligned} \tag{38}$$

  - ◇ With our previous expressions of $f(\eta)$

$$\sigma_{rec}(v;n=1) = \begin{cases} 1/v^2 & \text{for } \frac{1}{2}mv^2 \ll I_H \\ 1/v^5 & \text{for } \frac{1}{2}mv^2 \gg I_H \end{cases} \tag{39}$$

- Arbitrary n, Z with Milne

$$\sigma_{rec}(v,n) = \frac{2^5 Z^4}{3\sqrt{3}} \alpha^3 \pi a_0^2 \frac{I_H^2}{E\hbar w_n} \frac{1}{n^3} \tag{40}$$

  - ◇ where

$$\hbar\omega_n = \frac{Z^2 I_H}{n^2} + E$$
$$E = \frac{1}{2} m v^2$$

  - ◇ The approximation breaks down for $\frac{1}{2}mv^2 \gg Z^2 I_H/n^2$ (i.e. the Born approximation limit)
  - ◇ But $\sigma(v)$ is very small for large v $[\sigma(v) \propto v^{-5}]$
- Recombination rate:
  - ◇ Definition: Total recombination rate

$$\alpha_r = \sum_n < \sigma_n v >_{Maxwellian} \tag{41}$$

  - ◇ One can sum (approximately) over n using the Euler-Maclaurin sum formula
  - ◇ For Hydrogenic ions

$$\alpha_r(T) = 2^6 \alpha^3 \pi a_0^2 \left(\frac{2kT}{27\pi m} \right)^{\frac{1}{2}} \frac{Z^2 I_H}{kT} \phi \left(\frac{Z^2 I_H}{kT} \right) \tag{42}$$

    - ▲ where $\phi(y)$ has the asymptotic form

$$\phi(y) \to \begin{cases} \frac{1}{2}\,[\,1.735 + \ln y - (6y)^{-1}\,] & \text{for } y \gg 1 \\ y\,[\,-1.202 \ln y - 0.298 + \mathcal{O}(y)\,] & \text{for } y \ll 1 \end{cases} \tag{43}$$

    - ▲ Therefore

$$\alpha_r(T) \propto \begin{cases} T^{-\frac{1}{2}} & \text{for } kT \ll Z^2 I_H \\ T^{-\frac{3}{2}} & \text{for } kT \gg Z^2 I_H \end{cases} \tag{44}$$

- To find the line emission following recombination, first calculate $\alpha_r(T) =< \sigma_{rec}(v,n) v >$ and then use the $A_{n'\ell' n\ell}$ to do the branching ratio calculation
  - ◇ 'Classic' Balmer decrement calculation
  - ◇ c.f. J. Miller, 1974 Annual Rev. A[2]
- Continuum recombination emission
  - ◇ For Hydrogen $(n_+ = n_p)$

◇ Total energy emitted per unit volume per unit frequency $d\omega$

$$\frac{dP_{rec}}{dV d\omega} = \sum_n n_e n_p \frac{\hbar^2 \omega_n}{m} \sigma_{rec}(v,n) f(v) \tag{45}$$

$$= \sum_n n_e n_p \frac{\hbar^2 \omega_n}{m} \frac{2^5}{3\sqrt{3}} \alpha^3 \pi a_0^2 \frac{I_H^2}{E\hbar\omega_n} \frac{1}{n^3} \cdot 4\pi \left(\frac{m}{2\pi kT} \right)^{\frac{3}{2}} v^2 \exp\left[-\frac{mv^2}{2kT} \right] \tag{46}$$

$$= \sum_{n=1}^{\infty} n_e n_p \frac{2^8 \pi^2 \alpha^3 a_0^2 I_H^2 \hbar}{3\sqrt{3} m^2 n^3} \left(\frac{m}{2\pi kT} \right)^{\frac{3}{2}} \exp\left[-\frac{\hbar\omega - I_H/n^2}{kT} \right] \tag{47}$$

▲ where $\frac{1}{2} m v^2 = \hbar\omega_n - I_H/n^2$

◇ Fig

- Total emissivity

$$\frac{dP_{rec}}{dV} = \sum_{n=1}^{\infty} \int_{I_H/n^2\hbar}^{\infty} d\omega \frac{dP_{rec}}{dV d\omega} \tag{48}$$

$$= \sum_{n=1}^{\infty} \frac{2^7 \pi \alpha^3 a_0^2 I_H^2}{3\sqrt{3} n^3} \left(2\pi m kT \right)^{-\frac{1}{2}} n_e n_p \tag{49}$$

$$= n_e n_p 2^7 \pi \alpha^3 a_0^2 I_H^2 \left(2\pi m kT \right)^{-\frac{1}{2}} \frac{\zeta(3)}{3\sqrt{3}} \tag{50}$$

◇ $\zeta(x)$ is the Riemann zeta function

C. Bremsstrahlung

- Radiation due to the acceleration of a charge in the Coulomb field of another charge
  ◇ Non-relativistic treatment $\quad v/c \ll 1$

- ◇ Classical ($v/c \ll Z\alpha$) and quantum ($v/c \gg Z\alpha$)

- Astrophysical implication → Dominant cooling process at very high T
- Like charge Bremsstrahlung (i.e. $\mathrm{e}^- + \mathrm{e}^-$)
  - ◇ No emission in dipole approximation
  - ◇ Center of mass is a constant of the motion

$$\frac{d^2}{dt}\vec{d} = \frac{d^2}{dt}\sum_i e\vec{r}_i \propto \frac{d}{dt}\left(\frac{d}{dt}\sum_i m_i\vec{r}_i\right) = 0 \tag{51}$$

- The case we will consider:

$$\mathrm{e}^- + \mathrm{ion} \rightarrow \mathrm{e}^- + \mathrm{ion} + \gamma \tag{52}$$

  - ◇ γ has energy: $\hbar\omega = E_i - E_f$
  - ◇ Let our ion have charge Ze
- Classical treatment (E+M)
  - ◇ $v/c \ll Z\alpha$
  - ◇ Treat ion as a fixed particle owing to its large mass
  - ◇ Small-angle scatter regime ⇒ Deviation of the electron path from a straight line is negligible
    - ▲ Dipole moment

$$\vec{d} = -e\vec{r} \tag{53}$$

$$\ddot{\vec{d}} = -e\dot{\vec{v}} \tag{54}$$

    - ▲ Figure

  - ◇ Classical E+M: Probability of emitting a photon per unit frequency (Electric dipole)

$$\frac{dW}{d\omega} = \frac{8\pi\omega^3}{3c^3\hbar}|\hat{d}(\omega)|^2 \tag{55}$$

    - ▲ $\hat{d}(\omega)$ is the Fourier transform of the dipole moment

- ▲ Note: The Fourier transform of $\ddot{d}$ is $-\omega^2\hat{d}(\omega)$
- ▲ Therefore

$$-\omega^2\hat{d}(\omega) = -\frac{e}{2\pi}\int_{-\infty}^{\infty} \dot{\vec{v}}\, \mathrm{e}^{i\omega t}\, dt \tag{56}$$

◇ Electron is in close contact with the ion for time

$$\tau \approx \frac{b}{v} \tag{57}$$

◇ Evaluate $\hat{d}(\omega)$

- ▲ If $\omega\tau \gg 1$, the exponential in Equation (56) oscillates rapidly and the integral vanishes
- ▲ If $\omega\tau \ll 1$, the exponential is approximately unity

$$\hat{d}(\omega) \sim \begin{cases} \frac{e}{2\pi\omega^2}\Delta v & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{58}$$

◇ Putting it together

$$\frac{dW}{d\omega} = \begin{cases} \frac{2e^2}{3\pi c^3\hbar\omega}|\Delta v|^2 & \omega\tau \ll 1 \\ 0 & \omega\tau \gg 1 \end{cases} \tag{59}$$

- ▲ Expanding with $q^2 = m^2\Delta v^2/\hbar^2$

$$dW = \frac{2}{3\pi}\alpha\lambda_c^2 q^2\frac{d\omega}{\omega} \tag{60}$$

◇ Cross-section: Convolve the photon emission probability with the elastic cross-section

$$d\sigma_{Brem} = \int d\sigma_{elastic}(k_i, q)\; dW(\omega, q) \tag{61}$$

- ▲ Elastic Coloumb cross-section
  - ∘ Derived from Classical E+M

$$d\sigma_{elastic}(k_i, q) = \left(\frac{m}{2\pi\hbar^2 k_i}\right)^2 |V(\tilde{q})|^2\; 2\pi q dq \tag{62}$$

  - ∘ $V(\tilde{q})$ is the Fourier transform of the Coulomb potential
  - ∘ k_i is the initial momentum of the electron with velocity v_i
  - ∘ If $v_i \ll Z\alpha c$ and $\hbar\omega \sim \frac{1}{2}mv_i^2$, one integrates Equation (61) to get the Kramers-Gaunt approxmation

▲ Bremsstrahlung cross-section

$$d\sigma_{Brem} = \frac{16\pi}{3\sqrt{3}} Z^2 \alpha^5 a_0^2 \left(\frac{c}{v_i}\right)^2 \frac{d\omega}{\omega} g \tag{63}$$

- g is a 'fudge factor' called the Gaunt factor which hides our ignorance of QM
- In the limit $v/c \ll Z\alpha$

$$g = \begin{cases} 1 & \omega \gg \frac{mv_i^3}{Ze^2} \\ \frac{\sqrt{3}}{\pi} \ln \frac{2mv_i^3}{\Gamma Z e^2 \omega} & \omega \ll \frac{mv_i^3}{Ze^2} \end{cases} \tag{64}$$

(a) Γ is Euler's constant, ≈ 0.5772

(b) The limits on ω are from semi-classical arguments on the minimum impact parameter

(c) Even the low frequency Gaunt factor is roughly unity

- Quantum Mechanics (Born Approximation)
  - ◇ $v/c \gg Z\alpha$
  - ◇ Non-relativistic $v/c \ll 1$
  - ◇ Usual cross-section

$$\frac{d\sigma_{Brem}}{d\Omega} = \frac{\alpha\omega}{2\pi m^2 c^2} |\langle f| e^{-i\vec{k}\cdot\vec{r}} \hat{\varepsilon} \cdot \vec{p} |i\rangle|^2 \tag{65}$$

▲ Wave functions are normalized to 1 particle/cm^2/s

▲ $e^{-i\vec{k}\cdot\vec{r}}$ term implies emission

◇ Wave functions: $|i\rangle$ and $|f\rangle$ are both unbound states

▲ This is a more complicated situation

▲ Note our matrix element implies

$$\langle f| e^{-i\vec{k}\cdot\vec{r}} \hat{\varepsilon} \cdot \vec{p} |i\rangle \propto \delta\left(\vec{k}_i - \vec{k}_f - \vec{k}\right) \tag{66}$$

▲ This is kinematically <u>impossible</u> because the ion must undergo some momentum change

▲ Therefore, we cannot ignore the influence of the ion altogether

◇ Non-relativistic Born approx

$$E_i = \frac{p_i^2}{2m}, \quad E_f = \frac{p_f^2}{2m}, \quad \hbar\omega = \frac{p_i^2 - p_f^2}{2m}$$
$$\Rightarrow \hbar k \ll |\vec{p}_f - \vec{p}_i|$$

▲ Treat the Coloumb potential of the ion as a fixed perturbation (time independent)

▲ Apply first order perturbation theory to the wave functions (non-degenerate)

⋄ Perturbation Theory (Review Appendix A.3)

$$V^{(1)} = V_{Coulomb} \tag{67}$$

▲ Expand the free particle wave functions to first order in the perturbation

$$|\tilde{f}> = |f^{(0)}> + |f^{(1)}> \tag{68}$$

▲ Standard solution

$$|f^{(1)}> = \sum_{k \neq f} \frac{|k^{(0)}><k^{(0)}|V^{(1)}|f^{(0)}>}{E_k - E_f} \tag{69}$$

$$\equiv \sum_{k \neq f} \frac{V^{(1)}_{kf}|k^{(0)}>}{E_k - E_f} \tag{70}$$

▲ Similar expression for $|i>$

⋄ Evaluate our matrix to first order (Set $\Delta \equiv e^{-i\vec{k}\cdot\vec{r}}\hat{\varepsilon}\cdot\vec{p}$)

$$<\tilde{f}|\Delta|\tilde{i}> = <f^{(0)}|\Delta|i^{(0)}> + <f^{(0)}|\Delta|i^{(1)}> + <f^{(1)}|\Delta|i^{(0)}> \tag{71}$$

▲ $|f^{(0)}>$ and $|i^{(0)}>$ describe free particles

▲ $<f^{(0)}|\Delta|i^{(0)}>$ is zero by our kinematic argument

▲ Expanding...

$$<\tilde{f}|\Delta|\tilde{i}> = \sum_{k \neq i} \frac{\Delta_{fk} V_{ki}}{E_i - E_k} + \sum_{k \neq f} \frac{V_{fk}\Delta_{ki}}{E_f - E_k} \tag{72}$$

▲ Fig

▲ The demonstrators correspond to Feymann propogators!

⋄ Finally

▲ Integrating (magic)

$$\frac{d\sigma}{d\omega} = \frac{16}{3} Z^2 \alpha^5 a_0^2 \left(\frac{c}{v_i}\right)^2 \frac{1}{\omega} \ln \frac{v_i + v_f}{v_i - v_f} \tag{73}$$

▲ Recall, this holds for $Z\alpha \ll v/c \ll 1$

▲ $1/\omega$ dependence $\Rightarrow$ Infrared divergence

▲ Updating our Gaunt factor

$$g = \begin{cases} 1 & \omega \gg \frac{mv_i^3}{Ze^2} \\ \frac{\sqrt{3}}{\pi} \ln \frac{2mv_i^3}{\Gamma Ze^2\omega} & \omega \ll \frac{mv_i^3}{Ze^2} \\ \frac{\sqrt{3}}{\pi} \ln \frac{v_i+v_f}{v_i-v_f} & v_i/c \gg Z\alpha \text{ (for all } \omega) \end{cases} \tag{74}$$

- Quantum Mechanics (Sommerfield factor)
  - ◇ If v_i or $v_f \sim Z\alpha c$, must use exact Coulomb functions for $|f>$ and $|i>$
  - ◇ Result

$$g \rightarrow f_{Somm}\, g = \frac{\sqrt{3}}{\pi} \ln\left(\frac{v_i+v_f}{v_i-v_f}\right) \frac{v_i}{v_f} \frac{1-\exp(-2\pi Z\alpha c/v_i)}{1-\exp(-2\pi Z\alpha c/v_f)} \tag{75}$$

- Energy emitted per unit time per frequency
  - ◇ Bremsstrahlung is a cooling mechanism
  - ◇ Assume thermal equilibrium for our charges with densities n_e, n_Z
  - ◇ Integrate $d\sigma/d\omega$ over a Maxwellian

$$\frac{dP_{Brem}}{d\omega dV} = \frac{16}{3\sqrt{3}} n_e n_Z \alpha^5 Z^2 a_0^2 c\hbar \left(\frac{2\pi mc^2}{kT}\right)^{\frac{1}{2}} \bar{g}(\omega, T) \mathrm{e}^{-\hbar\omega/kT} \tag{76}$$

$$\bar{g}(\omega, T) = \begin{cases} \sqrt{\frac{3}{\pi}} \left(\ln \frac{4kT}{\hbar\omega} - 0.577\right) & \hbar\omega \ll kT \\ 1 & \hbar\omega \gg kT \end{cases} \tag{77}$$

  - ◇ Nearly exponential spectrum

- Total Bremsstrahlung cooling
  - ◇ Integrate over $d\omega$

$$\frac{dP_{Brem}}{dV} = \frac{16}{3\sqrt{3}} \sum_Z n_e n_Z\, \alpha^5 Z^2 a_0^2 c \left(2\pi mc^2 kT\right)^{1/2} \bar{g}(T) \tag{78}$$

where $\bar{g}(T) \approx 1.2$ (79)

  - ◇ Comparing radiative recombination with Bremsstrahlung

$$\frac{dP_{Brem}/dV}{dP_{rec}/dV} \sim \frac{kT}{2I_H} \tag{80}$$

  - ◇ For a hydrogen (+ Helium) plasma, recombination rarely dominates the cooling compared to lines and Bremsstrahlung