Lecture 10 Stellar Dynamics

Moments of the collisionless Boltzman equation

"The Jeans Equations"

The First Moment

integrate the cbe over all velocities $\int dv_1\, dv_2\, dv_3$ at a point $\vec{x}, t$:

define · space density $\nu \equiv \int f\, d^3\vec{v}$

· Bulk flow or mean streaming motion $\overline{v_i}$

$$\overline{v_i} \equiv \frac{\int f v_i\, d^3\vec{v}}{\int f\, d^3\vec{v}} = \frac{\int f v_i\, d^3\vec{v}}{\nu} \qquad \vec{\overline{v}} = (\overline{v}_1, \overline{v}_2, \overline{v}_3)$$

Integrate each term of the cbe to get:
and use the summation convention over i

$$\int \frac{\partial f}{\partial t} d^3\vec{v} + \int v_i \frac{\partial f}{\partial x_i} d^3\vec{v} - \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d^3\vec{v} = 0$$

$$\frac{\partial}{\partial t} \int f\, d^3\vec{v} + \frac{\partial}{\partial x_i} \int v_i f\, d^3\vec{v} - \frac{\partial \Phi}{\partial x_3} \iint dv_1\, dv_2 \int_{v_3=-\infty}^{v_3=+\infty} \partial f + 2\ \text{mo}$$

Iden
Term

note: $\int_{v_3=-\infty}^{v_3=+\infty} \partial f = f(+\infty) - f(-\infty) = 0 - 0 = 0$

because there are no stars moving infinitely fast.

The other 2 terms involving v_1 and v_2 go away for the same reason.

so using our definitions for the stellar density ν and the bulk motion $\vec{\bar{v}}$, we get that

$$\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i}(\nu \bar{v}_i) = 0$$

or

$$\frac{\partial \nu}{\partial t} + \mathrm{div}_3(\nu \vec{\bar{v}}) = 0$$

The first moment of the cbe is thus a scalar equation of continuity in real space.

it expresses the conservation of mass, or more precisely, the conservation of stars, since we are studying the ~~point~~ motion of test point particles in a fixed potential.

The Second Moments

multiply the CBE by v_i and integrate the equation over all velocities.

- assume $\lim_{|v_i| \to \infty} f v_i = 0$

Define $$\overline{v_i v_j} = \frac{\int v_i v_j f \, d^3\vec{v}}{\int f \, d^3\vec{v}}$$

and the velocity dispersion around the mean streaming motion $\vec{\overline{v}}$:

$$\sigma_{ij}^2 = \overline{(v_i - \overline{v_i})(v_j - \overline{v_j})} = \overline{v_i v_j} - \overline{v_i} \cdot \overline{v_j}$$

"Total Dispersion" "Streaming motion"

doing the integration, we get

$$(*) \quad \frac{\partial}{\partial t} \int f v_j \, d^3\vec{v} + \int v_i v_j \frac{\partial f}{\partial x_i} d^3\vec{v} - \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d^3\vec{v} = 0$$

We can kill the last term by using some trickery involving the divergance theorem and the vector analog of integration by parts

$$\int g \vec{\nabla} \cdot \vec{F} \, d^3\vec{x} = \int_S g \vec{F} \cdot d^2\hat{S} - \int (\vec{F} \cdot \vec{\nabla}) g \, d^3\vec{x}$$

(see homework #2)

The last term on the right hand side turns into →

$$\cancel{\frac{\partial \Phi}{\partial x_i}} \cdot -\int \frac{\partial v_j}{\partial v_i} f \, d^3\vec{v} = -\int \delta_{ij} f \, d^3\vec{v} = -\delta_{ij} \nu$$

Kronecker delta
1 if i=
0 if i

so, using the definitions for $\overline{v_j}$ and $\overline{v_i v_j}$, eqn (*)

becomes

$$\frac{\partial(\nu \overline{v_j})}{\partial t} + \frac{\partial}{\partial x_i}(\nu \overline{v_i v_j}) + \nu \frac{\partial \Phi}{\partial x_j} = 0$$

→ This equation can be put into a remarkable form.

subtract $\overline{v_j} \cdot$ (Equation of continuity), i.e.

$$\overline{v_j} \cdot \underbrace{\left(\frac{\partial \nu}{\partial t} + \frac{\partial(\nu \overline{v_i})}{\partial x_i} \right)}_{=0}$$

(so we're just subtracting zero)

$$\nu \frac{\partial \overline{v_j}}{\partial t} - \overline{v_j} \frac{\partial(\nu \overline{v_i})}{\partial x_i} + \frac{\partial(\nu \overline{v_i v_j})}{\partial x_i} = -\nu \frac{\partial \phi}{\partial x_j}$$

using $\sigma_{ij}^2 \equiv \overline{v_i v_j} - \overline{v_i}\,\overline{v_j}$ in

get:

$$\boxed{\nu \frac{\partial \overline{v_j}}{\partial t} + \nu \overline{v_i} \frac{\partial \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial(\nu \sigma_{ij}^2)}{\partial x_i}}$$

The second moment of the CBE.

What does this equation mean?

$$\frac{\partial \bar{v}_j}{\partial t} + \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = -\frac{\partial \Phi}{\partial x_j} - \frac{1}{\nu}\frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i}$$

it looks exactly the same as Eulers Equation for the velocity of a fluid.

Gradient of the pressure

Gradient of the potential

$$\underbrace{\frac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v}}_{} = -\frac{1}{\rho}\nabla p - \nabla \Phi$$

TOTAL change in velocity of a fluid element as it moves along

change in velocity at a particular point (fixed) over a tiny interval of time

Change in the velocity that is being "blown in"

imagine that there is no pressure

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho}\nabla p - \nabla\Phi$$

$$\frac{D\vec{v}}{Dt} = -\nabla\Phi$$

$$a = \frac{F}{m}$$ ......

so the second moment of the CBE is a nasty way of saying $F = ma$.

stress tensor

$$\nu \frac{\partial \bar{v}_j}{\partial t} + \nu \bar{v}_i \frac{\partial \bar{v}_j}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial(\nu \sigma_{ij}^2)}{\partial x_i}$$

change in the momentum of the mean streaming motion

acceleration of gravity

plays the role of pressure.

Spilling coherent motion into velocity dispersion leads to a decrease ~~associated~~ of momentum associated with streaming motion.

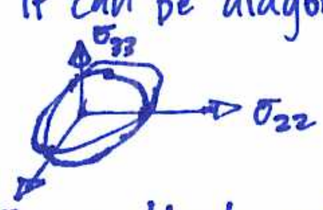
The pressure provided by the velocity dispersion is anisotropic.

The velocity ellipsoid

The velocity dispersion $\sigma_{ij}^2 = \overline{v_i v_j} - \bar{v}_i \bar{v}_j = \overline{v_j v_i} - \bar{v}_j \bar{v}_i = \sigma_{ji}^2$

is symmetric. Hence it can be diagonalized

That is at any given point,



We can choose

Principle axes σ_{11}, σ_{22}, σ_{33} so that $\sigma_{ij}^2 = \sigma_{ii}^2 \delta_{ij}$ no sum conv.

There exists an ellipsoid centered on $(\bar{v}_1, \bar{v}_2, \bar{v}_3)$ with the same principle axes.

put this as an example comparison to my distributions.

consider the anisotropic Gaussian distribution function centered on same $\bar{v}$ as follows

$$f(v_1, v_2, v_3) = \text{const}\; e^{-\frac{(v_1-\bar{v})^2}{\sigma_{11}^2} - \frac{(v_2-\bar{v}_2)^2}{\sigma_{22}^2} - \frac{(v_3-\bar{v}_3)^2}{\sigma_{33}^2}}$$

This gaussian distribution function has the same stress tensor $\nu \sigma_{ij}^2$

Such a gaussian form is not required by the CBE, but in practice is found in many stellar systems because Gaussian distribution functions maximize entropy and so are a natural equilibrium state for systems in which <u>any</u> processes can redistribute energy and momentum.

The value and limitations of the Jean's Equations

- can relate observables like $\vec{v}$, ν, and σ_{ij}^2 to the gravitational potential $\left(\frac{\partial \Phi}{\partial x_i}\right)$. "Weighs galaxies".

But:

The Jeans equations describe a massless "tracer population" in an external potential. Need to add Poisson's equation to get Φ from ρ

$$\nabla^2 \phi = 4\pi G \rho$$

no feedback for self consistency between the potential and the density that creates it.

- No equation of state to relate σ to ν, as ideal gas equation would relate T, p to ρ for ideal gas. So you have to assume σ_{ij}, that is $f(\vec{v})$. Every different assumption leads to a differant solution

  → solutions to Jean's equations depend on $f(\vec{v})$ and are thus non-unique.

Higher moments can be taken $\Rightarrow$ BBGKY hierarchy. However, each moment introduces a higher-order tensor (like $T_3 = (v_i - \bar{v}_i)(v_j - \bar{v}_j)(v_k - \bar{v}_k)$) for

which you need to make assumptions. The equations never close. To close, you have to assume some form for the N^{th} velocity tensor. That is, you have to assume some form for $f(\vec{v})$.

$\Rightarrow$ We've seen that the 2^{nd} moment of the CBE is equivalent to Euler's momentum equation for a fluid. This is conventionally derived assuming a collisional system with pressure. Yet the equations are identical save the anisotropic "pressure", $\nu\sigma_{ij}$

How can the equations for collisional and collisionless systems be so similar when the microphysics is so different.

Understanding the Jean's Equations

x term of the second Jean's equation

if the dispersion is increasing down stream then the bulk flow must be converting into dispersion.

$$\frac{\partial \bar{V}_x}{\partial t} = -\bar{V}_x \frac{\partial \bar{V}_x}{\partial x} - \bar{V}_y \frac{\partial \bar{V}_x}{\partial y} - \bar{V}_z \frac{\partial \bar{V}_z}{\partial z} - \frac{\partial \phi}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial x}\nu\sigma_{xx}^2 - \frac{1}{\nu}\frac{\partial}{\partial y}\nu\sigma_{yx}^2 - \frac{1}{\nu}\frac{\partial}{\partial z}\nu\sigma_{zx}^2$$

Bulk velocity being blown in from 3 orthogonal directions

now:

$$\sigma_{xx}^2 = \overline{V_x V_x} - \bar{V}_x \bar{V}_x = \frac{1}{\nu}\int V_x V_x f d^3\vec{V} - \frac{1}{\nu}\int f V_x d^3 V \cdot \frac{1}{\nu}\int f V_x$$

$$\sigma_{yx}^2 = \overline{V_y V_x} - \bar{V}_y \bar{V}_x = \frac{1}{\nu}\int V_y V_x f d^3\vec{V} - \frac{1}{\nu}\int f V_y d^3\vec{V} \cdot \frac{1}{\nu}\int f V_x d^3\vec{V}$$

$$\sigma_{zx}^2 = \overline{V_z V_x} - \bar{V}_z \bar{V}_x = \frac{1}{\nu}\int V_z V_x f d^3\vec{V} - \frac{1}{\nu}\int f V_z d^3\vec{V} \cdot \frac{1}{\nu}\int f V_x d^3\vec{V}$$

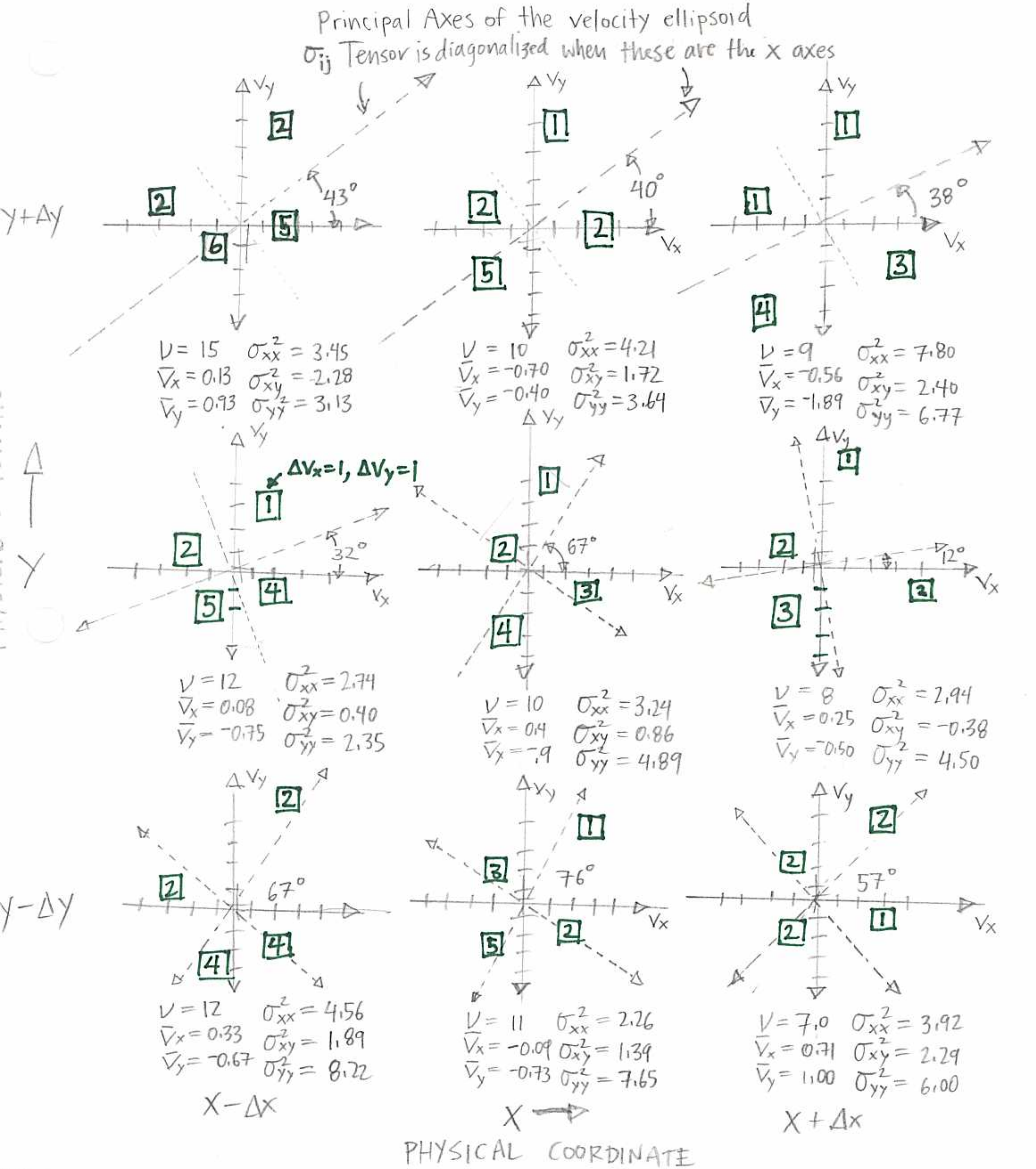
so the equation in its fully expanded form is:

$$\frac{\partial \bar{V}_x}{\partial t} = -\bar{V}_x \frac{\partial \bar{V}_x}{\partial x} - \bar{V}_y \frac{\partial \bar{V}_x}{\partial y} - \bar{V}_z \frac{\partial \bar{V}_x}{\partial z} - \frac{\partial \phi}{\partial x} - \frac{1}{\nu}\frac{\partial}{\partial x}\left[\int V_x V_x f d^3\vec{V} - \frac{1}{\nu}\left[\int f V_x d^3 V\right]^2\right]$$

$$-\frac{1}{\nu}\frac{\partial}{\partial y}\left[\int V_y V_x f d^3\vec{V} - \frac{1}{\nu}\left[\int f V_y d^3\vec{V}\right]\left[\int f V_x d^3\vec{V}\right]\right]$$

$$-\frac{1}{\nu}\frac{\partial}{\partial z}\left[\int V_z V_x f d^3\vec{V} - \frac{1}{\nu}\left[\int f V_z d^3\vec{V}\right]\left[\int f V_x d^3\vec{V}\right]\right]$$

Consider the following chunk of phase space: x, y, V_x, V_z (a 4-Dimensional plot!)



Di[illegible]ence [illegible]proximation the change [illegible]n the streaming velocity

$$\Delta \bar{V}_x = \Delta t \cdot \left[-\bar{V}_x \cdot \frac{\Delta \bar{V}_x}{\Delta x} - \bar{V}_y \cdot \frac{\Delta \bar{V}_y}{\Delta y} - \frac{\Delta \phi}{\Delta x} - \frac{1}{\nu}\frac{\Delta(\nu \sigma_{xx}^2)}{\Delta x} - \frac{1}{\nu}\frac{\Delta(\nu \sigma_{xy}^2)}{\Delta y} \right]$$

Diagonalizing the dispersion tensor:

symmetric $\sigma_{xy} = \sigma_{yx}$ Eigen value

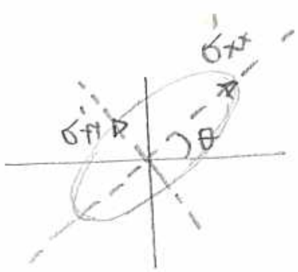
$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix} \begin{bmatrix} V_{1x} \\ V_{1y} \end{bmatrix} = \lambda \begin{bmatrix} V_{1x} \\ V_{y} \end{bmatrix}$$

Eigenvector

In order for solutions to exist

$$\det \begin{bmatrix} \sigma_{xx} - \lambda & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} - \lambda \end{bmatrix} = 0$$

$$\lambda = \frac{\sigma_{xx} + \sigma_{yy} \pm \sqrt{(\sigma_{xx} + \sigma_{yy})^2 - 4(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)}}{2}$$



$$\left.\begin{aligned} \lambda_+ &= \sigma'_{xx} \\ \lambda_- &= \sigma'_{yy} \end{aligned}\right\}$$ These are the lengths of the axes of velocity ellipsoid.

The eigenvectors are the columns of the rotation matrix corresponding to the angle which causes $\sigma'_{xy} \to 0$