Lecture 9 Stellar Dynamics I

. The Nature of the Gravitational potential in a stellar system

A. The potential from distant stars dominates if the system has a lot of stars.

$$\phi_i \doteq \frac{GM_i}{r_i} \sim r_i^{-1} \qquad (\text{star } i)$$

Therefore, if we consider a uniform distribution of stars

$$\sum_{\text{shell}} \phi_i \sim \text{Vol.}_{\text{shell}} \cdot r_{\text{shell}}^{-1} \sim r_{\text{shell}}^{2}\, r_{\text{shell}}^{-1} \sim r_{\text{shell}}$$

Hence: the summed potential over the whole system is dominated by distant stars.

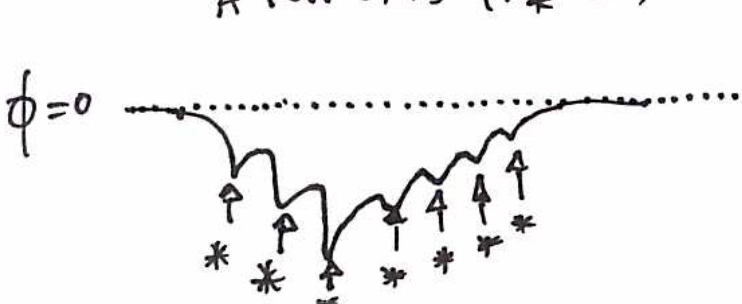
This major part is smoothly varying: "ϕ_s", since stars are distant

B. Close to individual stars, there is a non-smooth local potential, ϕ_{loc}

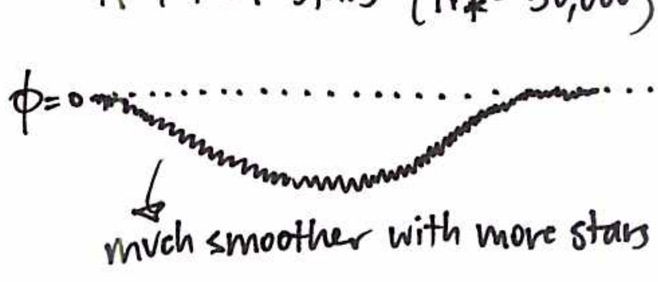
$$\phi_{tot} = \phi_s + \phi_{loc}$$

C. The relative importance of ϕ_{loc} / ϕ_s declines as the number of stars in the system increases

A few stars ($N_* = 30$):



A lot of stars ($N_* = 30{,}000$)



D. As stars move in ϕ, they follow smooth paths given by ϕ_s, plus instantaneous "scatterings" given by ϕ_{loc}.

→ scatterings are less important when N is large

II The relaxation time

A. The time for a star to change its velocity significantly as a result of encounters with individual stars. — The time for a stellar orbit to get seriously deflected from what it would have been had it followed ϕ_s

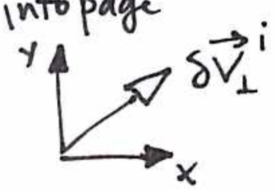
There are a number of ways of estimating the relaxation time — its definition is somewhat elastic.

Here's a definition based on <u>velocity</u> <u>perturbations</u>

· Let the velocity perturbation due to stellar encounters $\equiv \Delta\vec{v}_\perp = \sum_i \delta\vec{v}_\perp^{\,i}$

The time for $|\Delta\vec{v}_\perp| \approx |v|$ is the Relaxation Time, T_R

star's path into page



B. Estimate T_R for a system with:

M = total mass
m = mass of a single star
v = typical orbital velocity $\quad v \sim \sqrt{\frac{GM}{r}}$

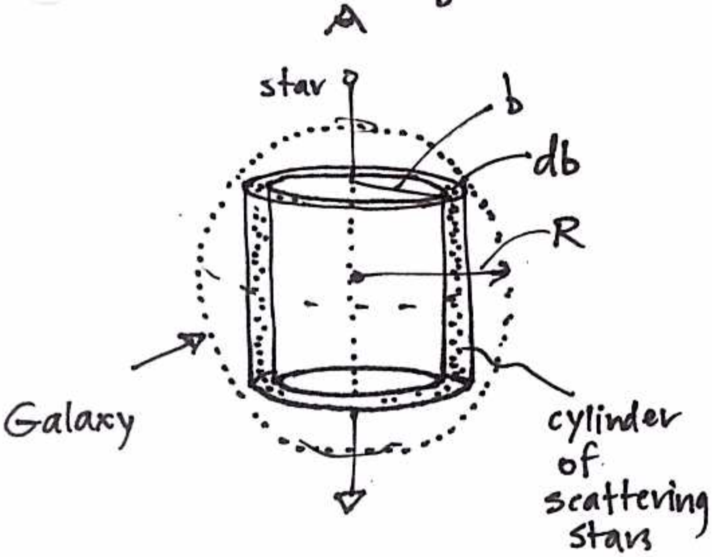
b = impact parameter

Outline computation of T_R (see BT ch. 4):

$|\delta v_\perp| \cong \frac{2Gm}{bv}$ ←

1) Make a rough estimate for $\delta\vec{v}_\perp^{\,i}$ for a single encounter with a star at distance r. Assume $\delta\vec{v}_\perp$ per encounter is small.

Thus, $|\delta v_\perp|$ is roughly equal to the force at closest approach, $\frac{2GM}{b^2}$ multiplied by the duration of the force b/v.



2) The surface density of stars in our hypothetical stellar system is $\sigma \sim \frac{N}{\pi R^2}$. One trip across the galaxy causes a star to suffer

$$\delta n = \frac{N}{\pi R^2} 2\pi b\, db = \frac{2N}{R^2} b\, db$$

scattering encounters with stars in the cylinder b to $b+db$.

3) Add up the $\delta \vec{v}_\perp^{\,i}$ s in quadrature from all scatterers in the cylinder at distance b. Quadrature because they pull in random directions, so you're adding steps as in a random walk.

vadrature add
or random
process:

$$\sum_N |\delta \vec{v}_\perp^{\,i}|^2 = N \langle |\delta \vec{v}_\perp| \rangle^2$$

square grows as N

$v_\perp^2$ changes by an average amount

$$\delta v_\perp^2 \simeq \left(\frac{2GM}{bv}\right)^2 \frac{2N}{R^2} b\, db$$

This breaks down for large deflections, i.e. at $b \leq b_{min} = \frac{GM}{v^2}$

$\int \frac{1}{r} dr$ means each decade produces an equivalent amount of deflection.

$$\Delta v_\perp^2 = \int_{b_{min}}^{R} \delta v_\perp^2 \simeq 8N \left(\frac{GM}{Rv}\right)^2 \ln\left(\frac{R}{b_{min}}\right)$$

call: $\dfrac{R}{b_{min}} = \Lambda$, note: $v^2 \simeq \dfrac{GM}{R} \approx \dfrac{GNm}{R}$ { i.e. stars are travelling at "orbital velocity" }

$|\Delta v_\perp|^2$ on a single crossing is: $\dfrac{|\Delta v_\perp|^2}{v^2} = \dfrac{8 \ln \Lambda}{N}$

4) If the star makes many crossings, v^2 changing by $|\Delta v_\perp|^2$ each time, the number of crossing times required for the velocity to change by order of itself is

$$n_{relax} = \frac{N}{8 \ln \Lambda}$$

$$\rightarrow \quad T_R = n_{relax} \cdot t_{cross} = n_{relax} \cdot \frac{R}{v}$$

$$\Lambda = \frac{R}{b_{min}} \cong \frac{Rv^2}{Gm} \simeq N \quad \rightarrow \quad \boxed{T_R = \frac{N}{8 \ln N}\, t_{cross}}$$

see def. prev. page

C. Typical relaxation times compared to ages:

		$N/8\ln N$	Age$/t_{cross}$
Open cluster	$N = 100$	3	100 ← Relaxed
Globular cluster	$N = 10^4$	130	10^5 ← Relaxed
Galaxy	$N = 10^{11}$	5×10^8	100 ← Unrelaxed

D. When the assumption of equal masses is relaxed, the effective relaxation time is even shorter by a factor of ~ 10. (for a salpeter type IMF) ← This is more realistic

E. Consequences of stellar relaxation: 1) leads to a slow redistribution (equipartition) of energy and momentum among the light and heavy stars: "secular evolution".

2) leads to gaussian velocity distribution functions — maximum entropy

lose energy

→ Heavy stars sink to the center of the system, light stars thrown into the halo

gain energy

→ Core collapse / halo expansion

→ Tidal stripping of outer halo accelerates mass loss

→ Fate of collapsing core depends on core velocity dispersion and total mass

⇒ "Gravo-thermal catastrophe" entropy increase by core shrinkage and halo expansion

Globular Clusters ⟺ Formation of binaries → core "bounce" + re-expansion repeats until the whole system evaporates, leading a single degenerate binary(s)

→ white dwarf analogy

Galactic Nuclei ⟺ Runaway stellar collisions due to finite stellar sizes → supernova unbinding or formation of a black hole

II → Dynamics of UNRELAXED systems – the collisionless Boltzmann Equ

$\phi = \phi_s$!

A. System: a large number of stars moving in a smooth potential
N is so large that phase space is well-populated & "distribution function" applies.

Distribution function: $f(\vec{x},\vec{v},t)\, d^3\vec{x}\, d^3\vec{v}$ = Nº of particles/vol. in phase space : $(\vec{x},\vec{v})$

B. Since $\phi = \phi_s$, neighboring particles in phase space (same $\vec{x}$, same $\vec{v}$) move together, so the motion of particles in phase space can be described by a "fluid approximation"

Define phase space coordinates

$$\vec{w} = (\vec{x},\vec{v}) = (w_1 \cdots w_6)$$

phase space velocity

$$\dot{\vec{w}} = (\dot{\vec{x}},\dot{\vec{v}}) = (\vec{v}, -\nabla\phi)$$

C. Flow is smooth as stars do not jump discontinuously from one region of space to another – no collisions

⇒ phase space density of stars obeys a continuity equation in $\vec{w}$ space:

phase-space "current"

$$\frac{\partial f}{\partial t} + \mathrm{div}_6 (f\dot{\vec{w}}) = 0$$

expand: $$\frac{\partial f}{\partial t} + f\, \mathrm{div}_6\, \dot{\vec{w}} + \dot{\vec{w}} \cdot \vec{\nabla}_6 f$$

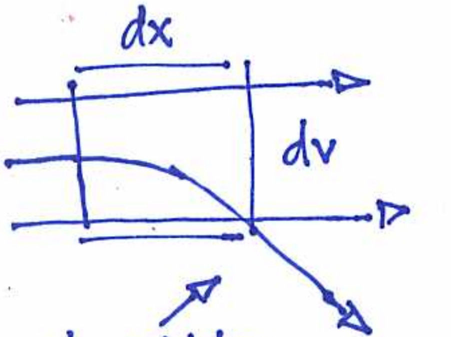
consider the $\text{div}_6 \dot{\vec{w}}$ term. This is the divergance of the flow in phase space. We can show that it is incompressible $\text{div}_6 \dot{\vec{w}} = 0$

$$\text{div}_6 \dot{\vec{w}} = \sum_{\alpha=1}^{6} \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} = \sum_{1}^{3} \left(\frac{\partial v_\alpha}{\partial x_\alpha} + \frac{\partial \dot{v}_\alpha}{\partial v_\alpha} \right) = 0$$

any velocity can be at any position.

This is zero because v_α and x_α are independent coordinates v_α does not depend uniquely on x_α

How does acceleration (i.e. force) depend on velocity? We have assumed a potential, so by assumption $\frac{\partial \dot{v}_\alpha}{\partial v_\alpha} = 0$, as long as the force on a star is independent of velocity



Not allowed. Violates conservation of particle number in the phase space volume dx dv

Examples where not so:

- Frictional drag
- Collisions: colliding particles share momenta, so force you experience depends on your velocity

CONCLUDE: Flow in phase space is incompressible as long as the forces do not depend on velocity in the direction of the force.

Final Result: Collisionless Boltzmann Equation

$$\boxed{\begin{aligned} &\frac{\partial f}{\partial t} + \dot{\vec{w}} \cdot \vec{\nabla}_6 f = 0 \\ &\frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla}_3 f - \vec{\nabla}_3 \phi \cdot \frac{\partial f}{\partial \vec{v}} = 0 \end{aligned}}$$

..or..