Astro 2

Quiz 3 Math Extra Problems

$v_{esc} = \sqrt{\frac{2GM}{R}}$

i=initial, f=final

1. What happens to the escape velocity from the surface of a planet if you:
   a. Increase the mass of the planet by a factor of 9 but keep the radius constant?

      $V_{esc} \propto \sqrt{M}$ $\quad V_{esc,i} \propto \sqrt{M_i}$ $\quad V_{esc,f} \propto \sqrt{M_f}$ $\quad M_f = 9M_i$

      $V_{esc,f} \propto \sqrt{M_f} = \sqrt{9M_i} = \sqrt{9}\sqrt{M_i} = 3V_{esc,i}$

      Vesc increases by a factor of 3

   b. Decrease the mass of the planet to 1/16 of the original mass but keep the radius the same? $V_{esc} \propto \sqrt{M}$, $V_{esc,i} \propto \sqrt{M_i}$, $V_{esc,f} \propto \sqrt{M_f}$, $M_f = \frac{1}{16}M_i$

      $V_{esc,f} \propto \sqrt{M_f} = \sqrt{\frac{1}{16}M_i} = \sqrt{\frac{1}{16}}\sqrt{M_i} = \frac{1}{4}V_{esc,i}$

      Vesc becomes 1/4 the original value

   c. Increase the radius by a factor of 25 but keep the mass of the planet the same?

      $V_{esc} \propto \sqrt{\frac{1}{R}}$, $V_{esc,i} \propto \sqrt{\frac{1}{R_i}}$, $V_{esc,f} \propto \sqrt{\frac{1}{R_f}}$, $R_f = 25R_i$

      $V_{esc,f} \propto \sqrt{\frac{1}{R_f}} = \sqrt{\frac{1}{25R_i}} = \sqrt{\frac{1}{25}}\sqrt{\frac{1}{R_i}} = \frac{1}{5}V_{esc,i}$

      Vesc becomes 1/5 the original value

   d. Keep the mass of the planet the same but shrink the radius to 1/81 of its original size? $V_{esc} \propto \sqrt{\frac{1}{R}}$, $V_{esc,i} \propto \sqrt{\frac{1}{R_i}}$, $V_{esc,f} \propto \sqrt{\frac{1}{R_f}}$, $R_f = \frac{1}{81}R_i$

      $V_{esc,f} \propto \sqrt{\frac{1}{R_f}} = \sqrt{\frac{1}{\frac{1}{81}R_i}} = \sqrt{81}\sqrt{\frac{1}{R_i}} = 9V_{esc,i}$

      Vesc increases by a factor of 9

   e. Increase the mass of the planet by a factor of 2 and increase the radius by a factor of 8? $V_{esc} \propto \sqrt{\frac{M}{R}}$, $V_{esc,i} \propto \sqrt{\frac{M_i}{R_i}}$, $V_{esc,f} \propto \sqrt{\frac{M_f}{R_f}}$, $M_f = 2M_i$, $R_f = 8R_i$

      $V_{esc,f} \propto \sqrt{\frac{M_f}{R_f}} = \sqrt{\frac{2M_i}{8R_i}} = \sqrt{\frac{2}{8}}\sqrt{\frac{M_i}{R_i}} = \sqrt{\frac{1}{4}}\sqrt{\frac{M_i}{R_i}} = \frac{1}{2}V_{esc,i}$

      Vesc decreases by 1/2

   f. Increase the mass of the object that is trying to escape by a factor of 6, but keep everything else the same?

      $V_{esc} = \sqrt{\frac{2GM}{R}}$, $m_f = 6m_i$

      no m in equation, mass of escaping object is not a factor in escape velocity

      Vesc stays the same

2. The escape velocity from the surface of Jupiter is 60,000 m/s. If the mass of Jupiter was quadrupled and its radius shrinks to 1/16 of its original size, what would the new escape velocity be? $V_{esc} \propto \sqrt{\frac{M}{R}}$, $V_{esc,J} \propto \sqrt{\frac{M_J}{R_J}}$, $V_{esc,f} \propto \sqrt{\frac{M_f}{R_f}}$, $M_f = 4M_J$, $R_f = \frac{1}{16}R_J$

$V_{esc,J} = 60{,}000 \text{ m/s}$

$$V_{esc,f} \propto \sqrt{\frac{M_f}{R_f}} = \sqrt{\frac{4M_J}{\frac{1}{16}R_J}} = \sqrt{4 \cdot 16}\sqrt{\frac{M_J}{R_J}} = 8\, V_{esc,J} = 8\,(60{,}000 \text{ m/s})$$

The new Vesc would be $\boxed{420{,}000 \text{ m/s}}$

3. If you start from the orbit of the Earth, you have to go 42,000 m/s to escape the Sun's gravity. How fast do you have to do if you start at the orbit of Saturn, which is 9 times farther away from the Sun than Earth is? $V_{esc} \propto \sqrt{\frac{1}{R}}$, $V_{esc,f} \propto \sqrt{\frac{1}{R_f}}$, $V_{esc,i} \propto \sqrt{\frac{1}{R_i}}$

$R_f = 9R_i$, $V_{esc,i} = 42{,}000 \text{ m/s}$

$$V_{esc,f} \propto \sqrt{\frac{1}{R_f}} = \sqrt{\frac{1}{9R_i}} = \sqrt{\frac{1}{9}}\sqrt{\frac{1}{R_i}} = \frac{1}{3}V_{esc,i} = \frac{1}{3}(42{,}000 \text{ m/s})$$

You must travel at $\boxed{14{,}000 \text{ m/s}}$

4. Derive the formula for the Schwarzschild Radius, starting from the formula for escape velocity.

$$V_{esc} = \sqrt{\frac{2GM}{R}}$$

R_{sch} is the radius where $V_{esc} = c$ so set $V_{esc} = c$ and $R = R_{sch}$ and solve for R_{sch}

$$c = \sqrt{\frac{2GM}{R_{sch}}} \Rightarrow c^2 = \frac{2GM}{R_{sch}} \Rightarrow R_{sch} = \frac{2GM}{c^2}$$

5. How does the Schwarzschild Radius change if you:
   a. Increase the mass of your black hole by a factor of 3?

   $R_{sch} \propto M$, $R_{sch,i} \propto M_i$, $R_{sch,f} \propto M_f$, $M_f = 3M_i$

   $R_{sch,f} \propto M_f = 3M_i \propto 3R_{sch,i}$

   R_{sch} increases by a factor of 3

   b. Decrease the mass of the black hole to ⅕ of the original mass?

   $R_{sch} \propto M$, $R_{sch,i} \propto M_i$, $R_{sch,f} \propto M_f$, $M_f = \frac{1}{5}M_i$

   $R_{sch,f} \propto M_f = \frac{1}{5}M_i \propto \frac{1}{5}R_{sch,i}$

   R_{sch} becomes $\frac{1}{5}$ its original size

6. The Schwarzschild Radius of a $5M_\odot$ black hole is 15,000 m. As the black hole sucks in mass, the Schwarzschild Radius increases. By the time the Schwarzchild radius is 45,000 m, how much mass has the black hole eaten? Give your answer in units of $M_\odot$.

$R_{sch} \propto M$, $R_{sch,i} \propto M_i$, $R_{sch,f} \propto M_f$, $M_i = 5M_\odot$

$R_{sch,f} = 45{,}000 \text{ m}$ $R_{sch,i} = 15{,}000 \text{ m}$ $\Rightarrow R_{sch,f} = 3R_{sch,i}$

$M_f \propto R_{sch,f} = 3R_{sch,i} \propto 3M_i$

$M_f = 3M_i = 3(5M_\odot) = 15M_\odot$

Black hole grew from $5M_\odot$ to $15M_\odot$, which means it consumed $\boxed{10M_\odot}$