1) Do the Twist. *The Earth is thought to have formed by accreting planetesimals of mass $m \ll M_\oplus$. Assume planetesimals collide with the planet completely inelastically and from random directions. Take the impact velocity to be $v_i \approx f v_{orb}$, where $v_{orb} \approx 30$ km s^{-1} is the orbital velocity of the Earth around the Sun, and $f \approx 1/2$ characterizes the orbital eccentricities and inclinations of the planetesimals. Further assume planetesimals to have a single size r and mass m. Estimate r such that the nascent Earth spins at the correct rate. Provide both symbolic and numerical answers.*

This is a random walk problem. Each planetesimal brings the growing Earth angular momentum ΔL. After $N = M_\oplus/m_p$ impacts, the newly formed Earth has a net angular momentum $L = N^{1/2}\Delta L$. The $\sqrt{N}$ scaling of the total L is important. If this doesn't make sense, look up a random walk. We want to reproduce the angular momentum of Earth today, $L_\oplus = I\omega$, where $I \approx \frac{2}{5}M_\oplus R_\oplus^2$ is the moment of inertia of a uniform sphere and $\omega = 2\pi/(1 \text{ day})$ is the rotational frequency of the Earth.

What about ΔL? We want to approximate the average change in the magnitude of L, $<\Delta L>$. Each planetesimal brings $\Delta L = m_p v_i b$, where b is the impact parameter, or lever arm, of the collision. Based on the cross-section of Earth, let's approximate $b \sim \frac{2}{3}R_\oplus$. Is this consistent with the fact that Earth must grow from these planetesimals? At constant density, if the mass doubles, the radius changes by $2^{1/3} \sim 1.25$, or only an additional 25%, so we can safely neglect Earth's radius evolution as it grows. However, we can't neglect the 3D geometry of the problem, and the fact that $\vec{L}$ and $\vec{\Delta L}$ are, in fact, vector quantities. Taking this into account, the average change in the magnitude of L,

$$<\Delta L> \approx \frac{1}{\sqrt{3}}\Delta L = \frac{1}{\sqrt{3}} m_p v_i <b> \approx \frac{1}{\sqrt{3}}\left(\frac{2}{3}\right) m_p v_i R_\oplus \tag{1}$$

Put in terms of $N = M_\oplus/m_p$,

$$\Delta L \approx 0.4 M_\oplus N^{-1} v_i R_\oplus. \tag{2}$$

We can relate this back to the total L to solve for N.

$$L_\oplus = \frac{2}{5}M_\oplus R_\oplus^2 \omega = N^{1/2}(0.4 M_\oplus N^{-1} v_i R_\oplus) \tag{3}$$

Simplifying,

$$M_\oplus R_\oplus^2 \omega = N^{-1/2} M_\oplus v_i R_\oplus \tag{4}$$

or

$$N^{1/2} = \frac{v_i}{R_\oplus \omega} \implies N = \left(\frac{v_i}{R_\oplus \omega}\right)^2 \tag{5}$$

Numerically,

$$N = \left(\frac{1.5 \times 10^6 \text{ cm/s}}{6 \times 10^8 \text{ cm} \cdot 7 \times 10^{-5} \text{ s}^{-1}}\right)^2 \approx 1300 \tag{6}$$

With their number, we know their mass, $m_p = \frac{M_\oplus}{N}$. So $m_p = \frac{6\times10^{27}\text{g}}{1300} \approx 4.6 \times 10^{24}$ g. These same planetesimals each occupy a fraction $1/N$ of the Earth's volume, so

$$R_p^3 \approx N^{-1} R_\oplus^3 \implies R_p \approx N^{-1/3} R_\oplus \approx (1300)^{-1/3} \times 6 \times 10^8 \text{cm} \approx 5.5 \times 10^7 \text{cm} \tag{7}$$

2) Round rocks. *How large must an asteroid be before its self-gravity makes it round?*

In class, we calculated how high the highest mountain on Earth could be. Let's use that same approach. Rocks can sustain a pressure $P \sim 2 \times 10^8$ N/m^2. So we set $P = \rho g h$. Let's say $h < R_a$ defines roundness.

$$P = \rho \left(\frac{GM_a}{R_a^2} \right) R_a = \frac{GM_a \rho}{R_a} \tag{8}$$

but $M_a = \frac{4}{3}\pi R_a^3 \rho$, so

$$P = G \left(\frac{4}{3}\pi R_a^3 \rho \right) \rho R_a^{-1} = \frac{4}{3}\pi G R_a^2 \rho^2 \implies R_a = \left(\frac{3P}{4\pi G \rho^2} \right)^{1/2} \tag{9}$$

Numerically,

$$R_a = \left(\frac{3 \times 2 \times 10^8 \text{ N/m}^2}{4\pi \times 6 \times 10^{-11} \, \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2} \left(3000 \, \frac{\text{kg}}{m^3} \right)^2} \right)^{1/2} \approx 300 \text{ km} \tag{10}$$

3) Food Power.

A. Would the energy of all the calories you have consumed as food be enough to eject you from the solar system? To escape from the solar system you would need to escape from the Earth, then escape from the Sun.

A 20yr old has lived for about 7000 days, eating about 2000 kcal a day. In Joules, this is about $6 \times 10^{10} J$. In contrast, the gravitational binding energy per kilogram of an object on the surface of the earth is

$$\frac{U_e}{m} = G\frac{M_e}{r} \approx 5 \times 10^7 \text{ J/kg} \tag{11}$$

and relative to the sun (at earth's distance)

$$\frac{U_{\text{total}}}{m} \approx \frac{U_\odot}{m} = G\frac{M_\odot}{r} \approx 10^9 \text{ J/kg} \tag{12}$$

Thus for an approximately 75kg (165 lbs) person, the food they have consumed is more than enough energy to escape the earth (4×10^9 J)—and borderline/just-about-enough to escape from the solar system (7×10^{10} J).

B. NASA pays about \$20,000 per kilogram to get rockets away from Earth. For comparison, 1kW hr=3.6×10^6 J costs about 15 cents and 1 gal gas $\sim 1 \times 10^8$ J costs about \$ 1.50. Is NASA wasting our money by sending rockets away from the Earth?

From part 'A,' we see that about $10^8 J$ are required per kilogram to escape the earth. Using the conversions given in the problem, this translates to about \$2 of electricity; or about \$1 of gasoline. Comparing this to $\$20,000$ from NASA—it would seem that something is amiss! What are we not considering? We'll get to the answer in the next question, but think about this: if you were standing on a metal platform, and you kept lighting matches and holding them below the platform (to produce thrust), how long would it take to make it to space?

C. There are about 1000 kcal in a bacon double cheeseburger, if that was being perfectly converted to thrust, at what rate would you have to eat them to maintain escape velocity near earth's surface?

This is basic projectiles stuff. Remember that the definition of power is work over time, i.e. $P = \frac{\Delta E}{t}$. Near the surface of the earth, the work done is the change in potential energy, i.e. $\Delta E = mg\Delta h$; thus:

$$P = \frac{\Delta E}{t} = \frac{mg\Delta h}{t} = mg\frac{\Delta h}{t} = mg \times v \tag{13}$$

Or in other terms, power is force times velocity, where the force required is to resist gravity (and thus maintain a constant velocity). The escape velocity is found by equating kinetic energy to potential energy

$$\frac{1}{2}mv^2 = G\frac{Mm}{r} \quad \rightarrow \quad v = \sqrt{\frac{2GM}{r}} = \sqrt{2gr} \tag{14}$$

Where we have substituted the acceleration of gravity at the surface of the earth. Plugging this back into equation 16, we see that

$$P = m\sqrt{2r}g^{3/2} \tag{15}$$

Thus for a 75 kg person, the power required is something like $10^7 J/s \approx 2400$ kcal/s, or two and a half bacon double cheeseburgers *per second*.

Getting back to part 'B,' this is the amount of energy that would need to be perfectly converted to thrust *per second*. Using energy at a rate *less* than this, will be insufficient to escape the earth, no matter the net energy used. So if you're trying to propel yourself using a single match.... you're out of luck, no matter the amount of energy used. This is why it costs NASA so much money to get things into space. An amount of energy only a factor of 10 less than the entire amount of energy you have consumed in your life, needs to be converted to thrust on the scale of an hour. That's no easy task.

Keep in mind, if you are constantly exerting a sufficient force to resist gravity, you can escape at *any* velocity. The smaller the velocity you choose, however, the longer you have to exert the force—and the more energy you will use. In actuality, the space shuttle exerts a much larger force to maintain a large acceleration, gradually increasing its velocity to a little bit below escape velocity–required for circular orbit. Thus the true force would be larger, and the average velocity a good deal smaller.

4) Asteroid Rotation. *For many asteroids, we can't tell whether they are a single, solid "rock" or a self-gravitating "rubble pile" (See Figure 1). We're going to examine an asteroid's lightcurve to see if we can derive any constraints. Asteroids exhibit time-varying brightness due to their asymmetric shapes. As they spin, they present a different size reflecting surface between the sun and our telescopes. (I like to think of spinning a potato on its axis, and imagining how much light reflects back to me from its surface...)*

A. The asteroid 2867 Steins' lightcurve is attached (Figure 2). What is its rotation period?

The lightcurve modulates based on the asteroid's non-spherical shape. It repeats with $\sim \frac{1}{4}$ day peroid.

B. Steins orbits the sun in a relatively circular orbit at about 2 AU from the sun. A typical asteroid has an albedo, or reflectivity of about 0.3. At our closest approach to Steins we make a measurement of the flux coming from the asteroid to be about 2×10^{-12} erg s^{-1} cm^{-2}. *What is its radius?*

The solar flux at 2 AU is

$$F = \frac{L_\odot}{4\pi d^2} = \frac{4 \times 10^{33} \text{ erg/ s}}{4\pi (2 \text{ AU})^2} = 3.5 \times 10^5 \text{ s}^{-1} \text{ cm}^{-2} \tag{16}$$

The cross-section of the asteroid is πR_a^2 and it reflects with an efficiency of 0.3, so the total reflected light is

$$L_a = 0.3\pi R_a^2 \, \frac{L_\odot}{4\pi d^2_{\text{sun-asteroid}}}. \tag{17}$$

By the time it reaches Earth, the flux is $L_a/4\pi d^2_{\text{a-E}}$:

$$F = \frac{0.3\pi R_a^2}{4\pi d^2_{\text{asteroid-Earth}}} \frac{L_\odot}{4\pi d^2_{\text{sun-asteroid}}} = 2 \times 10^{-12} \text{ erg s}^{-1} \text{ cm}^{-2} \tag{18}$$

Solving for R_a, I get 2.6×10^5 cm $= 2.6$ km

C. Can Steins be a rubble pile, or must it be a solid body? Given the size you derived what is the critical rotation rate?

The key here is to imagine a pebble on the asteroid's surface. Does the gravitational force hold it down, or does the centrifugal force throw it off?

$$a_{\text{grav}} = \frac{f}{m} = \frac{GM_a}{R_a^2}. \tag{19}$$

With $\rho = 2$ g/ cm^3, I get $a_{\text{grav}} = 0.14$ cm/ s^2.

The centrifugal force is

$$a_{\text{cent}} = \Omega^2 R_a \approx 0.022 \text{ cm/ s}^2. \tag{20}$$

So gravity could certainly hold together a loose pile of rocks at this rotation rate. The critical rate is when $a_{\text{cent}} = a_{\text{grav}}$, corresponding to

$$\Omega^2 = \frac{GM_a}{R_a^3} \implies \Omega \approx 7 \times 10^{-4}\ \text{s}^{-1} \tag{21}$$

or a rotational period of about 22 minutes.

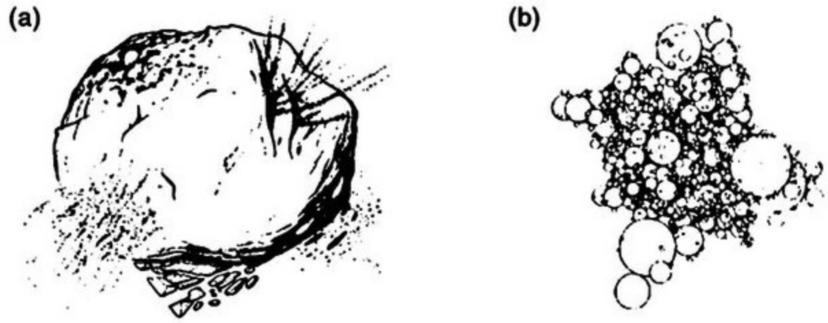

Figure 1. Different conceptions of asteroid makeup. Some asteroids are monolithic rocks, case (a), while others are agglomerates of loose rocks and dust, case (b).

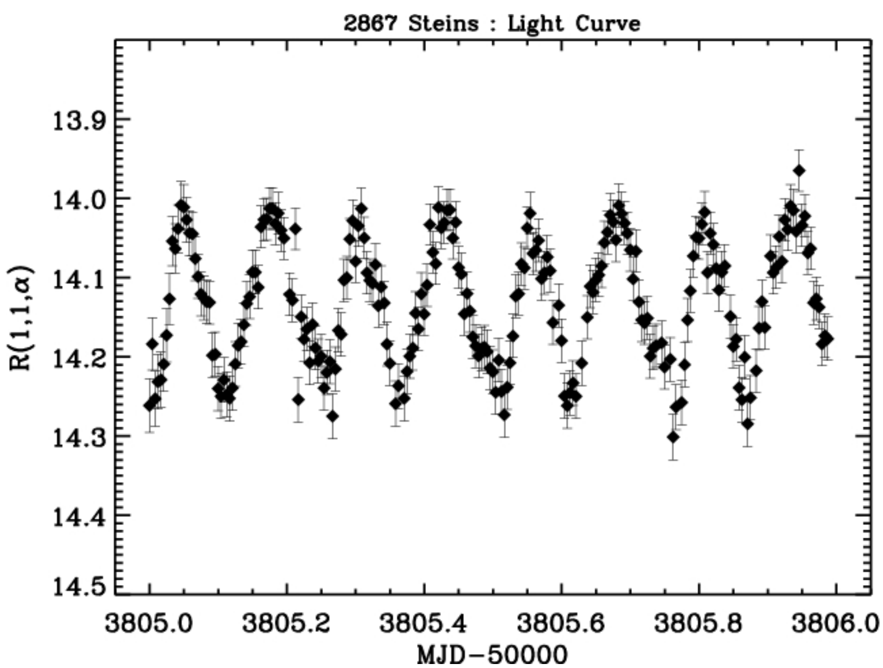

Figure 2. Asteroid 2867 Steins lightcurve.

5) Exoplanets.

A. Estimate the magnitude of the RV for a planet of radius R_p, mass M_p in a circular orbit of radius a and inclination i ... around a star of mass M_s. What would be the RV of the sun due to the earth (looking in the orbital plane)? Due to jupiter?

Assume circular motion about a center of mass which is a distance a from the planet (m); and the total separation between the planet and the star (M) is r. For a circular orbit, the centripetal force is due to gravity:

$$G\frac{mM}{r^2} = \frac{Mv^2}{r-a} \tag{22}$$

The stars mass is in the R.H.S. because we are looking for its velocity. Using the center of mass equation, we can find that $d = \frac{m}{m+M}r$, and thus

$$v_{RV} = \sqrt{\frac{G}{r}\left(\frac{m^2}{m+M}\right)} \tag{23}$$

The velocity apparent to the observer at an angle i (where $i = 0$ is looking along the plane of the orbit) is then

$$v_{RV} = \sqrt{\frac{G}{r}\left(\frac{m^2}{m+M}\right)}\cos i \tag{24}$$

For the sun: $M_\odot \approx 2 \times 10^{33} g$, and the earth has $M_e \approx 5 \times 10^{27} g$ at a radius $R_e \approx 10^{13}$cm, jupiter: $M_J \approx 2 \times 10^{30} g$ at $R_J \approx 8 \times 10^{13}$cm. Therefore RV's (with $G \approx 6.7 \times 10^{-8}$cgs) are about $V_e \approx 10$cm/s ≈ 0.1mph; $V_J \approx 1300$cm/s ≈ 30mph

B. Estimate the percent drop in luminosity at peak transit.

This is just geometry. If the area of the star ($\pi R_\star^2$) is covered by the entire area of the planet (πR_p^2), the luminosity will drop by the ratio of the square of their radii:

$$\Delta L = \frac{R_p^2}{R_\star^2} \tag{25}$$

E.g., for the sun (w/ radius $R_\odot \approx 7 \times 10^{10} cm$), the earth ($R_e \approx 6 \times 10^8 cm$) would cause an approximately 0.01% drop in luminosity. You could get a similar result accounting for eccentricity by instead using quantitative versions of Kepler's laws for elliptical motion.

C. What is the probability that you could view the planet by the transit method? The probability is simply that of the planet crossing in front of its parent star—from your perspective at some inclination i. Clearly if the angle between your line of sight and the plane of the orbit (again, i) is 0, you will see the transit, if the angle is $\pi/2$ you will not. The critical angle i_c (in which the C.O.M. of the planet barely grazes the top of the star) is determined by geometry:

$$\sin i_c = \frac{R_\odot}{a} \tag{26}$$

Where a is the distance between the planet and the star. The probability that a randomly oriented system is within this angle is $2i_c/\pi$:

$$P_{\text{transit}} = \frac{2\sin^{-1}\left(\frac{R_\odot}{a}\right)}{\pi} \approx \frac{2R_\odot}{a\pi} \tag{27}$$

For the earth-sun system, this probability would be about 0.4%.

D. Consider a transit survey which looks at a fixed 10 square degrees of the sky, with a sensitivity of 1ppm for stars of apparent magnitude 12 and brighter. How many planets could such a survey detect?

The first step is determining how many mag 12 and brighter stars can be seen in 10 square degrees. Our sun is mag 12 at about 230 pc; therefore the sun fulfills this requirement if it is within a volume of 5×10^7 pc^3 of us. The milky-way has a volume of about 3×10^{11} pc^3 with a total of about 10^{11} stars[1]. Thus there are about 10^7 magnitude 12 or brighter stars in the sky—which in total is about 41,000 square degrees. We can expect something vaguely like 10^3 such stars in the given field of view.... phew.

From the calculation in part 'c,' we would be able to detect earth—and therefore presumably all of the 3-4 inner planets in our solar system—as long as they pass in front of the host star. Lets say half of those 10^3 stars have similar systems to ours, and the other half have no planets; there would then be about 2000 viable planets, with only about 0.4% transiting.... thus about 8 planets [2].

[1] This assumes that the stars in the MW are distributed isotropically relative to us—which clearly isn't the case (the center of the galaxy has more stars). The center of the galaxy, is however, almost 10,000 pc away which is significantly larger than the volume we're considering, thus our assumption is (at least somewhat) valid

[2] The actually field-of-view of Kepler is far denser than average, increasing the number of observable main-sequence stars by two orders of magnitude—-otherwise, our results are completely consistent with NASA predictions (about 400 earth-like planets).