GENERAL RELATIVITY
HOMEWORK SET NUMBER 3

1. Imagine a two dimensional Euclidean plane with rectilinear co-ordinates x^μ. Now imagine transforming to a new set of co-ordinates ζ^α defined using the non-orthogonal axes $x = y$ and $x = 0$. The components of ζ^α are determined by drawing lines through the point in question parallel to each of the new axes and seeing where these lines intersect the other axis. **(a)** Find the co-ordinate transform from x to ζ. **(b)** Find the metric tensor $g_{\mu\nu}$ in the new frame. **(c)** Prove that the **covariant** components of any vector A are determined by the perpendicular projections onto the axes. **(d)** Find all of the nonzero affine connections.

2. Find the Riemann tensor $R^\lambda{}_{\mu\nu\kappa}$ on the surface of a right circular cylinder of radius R, using cylindrical co-ordinates z and θ.

3. On the surface of a unit sphere, using co-ordinates θ and ϕ, find the metric and all of the nonzero Christoffel symbols.

4. On the unit sphere, consider the vector (A^θ, A^ϕ) at the point $(\theta, 0)$, where $\theta \neq 0, \pi/2$. If another vector $B^\mu = A^\mu$ at that point, and if one parallel transports B^μ around one complete loop of the small circle at constant θ, what is the value of B^μ after one circuit of the small circle?

5. On the unit sphere, consider the vector defined by $V^\theta = \cos\phi$ and $V^\phi = \sin^2\theta$. **(a)** Find all components of the covariant derivative $V^\mu{}_{;\alpha}$. **(b)** Evaluate $V^2{}_{;1;2} - V^2{}_{;2;1}$

6. On the surface of a unit sphere, find $R^\lambda{}_{\mu\nu\kappa}$, $R_{\mu\nu}$, and R.

7. In three (Euclidean) dimensions, the metric tensor is

$$ds^2 = \sin^2 r dr^2 + \zeta^2(r)\left(r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right),$$

where $\zeta(r)$ is the Riemann zeta function. For a sphere defined by $r = R$=constant, find the physically measured circumference of a great circle, the area of the sphere's surface, the radius of the sphere, and the volume of the sphere. Any integrals involving ζ need not be evaluated.

8. Assume that in 3+1 dimensions, the metric tensor is diagonal, that is that each component of the metric is $g_{\mu\nu} = F_\mu(x)\eta_{\mu\nu}$, where the $F_\mu(x)$ are four functions of position. In terms of the four functions F_μ, find $\Gamma^\alpha_{\mu\nu}$ under the assumption that all three indices α, μ, and ν are different.