GENERAL RELATIVITY
HOMEWORK SET NUMBER 4

1. If a is a constant, consider a spacetime with metric

$$d\tau^2 = e^{-2ax} dt^2 - dx^2 - dy^2 - dz^2.$$

(a) Find all of the nonzero affine connections. **(b)** Find the geodescic equation for $d^2x/d\tau^2$ and show that in the limit of instantaneous zero velocity $dx/dt = dy/dt = dz/dt = 0$, the acceleration in the x direction is constant, $d^2x/d\tau^2 = a$, as though the particle were in a uniform gravitational field. **(c)** Find the components of the Riemann tensor $R^1{}_{001}$, $R^0{}_{110}$, $R^1{}_{010}$, and $R^0{}_{101}$. The other components are zero. **(d)** From the Einstein Field Equations, find the diagonal elements of the energy momentum tensor. Is this tensor physically acceptable?

2. In a Schwarzchild metric, what is the volume contained in the spherical shell between radii r_1 and r_2? You may leave your answer as an integral.

3. Consider two concentric coplanar circles (r=constant) in Schwarzschild geometry. Suppose that the measured circumferences of these circles are C_1 and C_2. **(a)** What is the radial co-ordinate difference Δr between these circles? **(b)** What is the measured radial distance between these two circles? **(c)** For the sun, take the two circles as having circumferences $C_1 = 2\pi R_\odot$ and $C_2 = 4\pi R_\odot$. By how many centimeters does the measured radial distance between them differ from the result expected in flat space?

4. A particle is to be launched in the outward radial direction from the point $r = 4GM$ in the Schwarzschild geometry. **(a)** At what speed $dr/d\tau$ must the particle be launched if it is to reach the point $r = 8GM$ with zero speed? **(b)** How much proper time does this trip take?

5. Light orbits a Schwarzschild black hole at a radius $r = 3GM$, where as usual r is the radial co-ordinate in standard co-ordinates. **(a)** What is the orbital period of the light as measured by a distant observer? **(b)** What is the orbital period as measured by an observer at rest at $r = 3GM$?

Note that

$$\int \frac{dy}{\sqrt{1 - 1/y}} = y \cos\theta - \ln\tan(\theta/2) \quad \text{where} \quad \theta = \arcsin(y^{-1/2}).$$