mainly of elements heavier than hydrogen, where the gas pressure is dominated by electrons and hence depends on μ_e, which is very nearly 2 regardless of the detailed abundances. A homogeneous composition is also typical of young stars, since the initial stellar composition is uniform.

Another principle that enables an analytic investigation of the behaviour of stars is the representation of a star by its two extreme points – the centre and the surface (the surface is, of course, not a point in the strict sense of the word, but all points on the surface are identical by the spherical symmetry assumption). The hidden implication is that properties change monotonically between these two points. This is certainly correct for the pressure, from equation (5.1), and also for the temperature, by equation (5.3), since from equation (5.4), $F \geq 0$. The latter condition is not necessarily correct in the case of strong neutrino emission, which may turn the net q negative and may eventually lead to a temperature inversion. But we shall disregard such complications.

As a further simplification, we may represent a star by only one of the extreme points; the centre, for example. Assuming that both P and T decrease outward (and so must ρ; otherwise we would encounter the unstable situation in which heavy material lies on top of light material, resulting in a turnover), the centre of a star is the hottest and densest place. There, therefore, the nuclear reactions are fastest and, since nuclear processes dictate the evolutionary pace, the centre would be the most evolved part of the star. We should be able to learn a great deal about the evolution of a star by considering its central point alone. This will be the subject of Chapter 7. The surface of the star (the global stellar characteristics) is important from an entirely different point of view – it is the only "point" whose model-derived properties can be directly compared with observations. In some cases, global quantities and relations between them may be obtained, as we shall see in Chapter 7, without solving the set of structure equations.

For now, we shall consider several simple models based on the principle of a uniform property.

5.3 Polytropic models

The first pair of stellar structure equations, (5.1)–(5.2), is linked to the second pair, (5.3)–(5.4), by the dependence of pressure on temperature. If the pressure were only a function of density (and composition, of course), the first pair would be independent and could be solved separately, meaning that the hydrostatic configuration would be independent of the flow of heat through it. Analytic solutions of this form are more than a century old.

Multiplying equation (5.1) by r^2/ρ and differentiating with respect to r, we have

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr}. \tag{5.8}$$

Substituting equation (5.2) on the right-hand side, we obtain

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{5.9}$$

We now consider equations of state of the form

$$P = K\rho^\gamma, \tag{5.10}$$

where K and γ are constants, known as *polytropic equations of state*. It is customary to define the corresponding *polytropic index*, denoted by n, as

$$\gamma = 1 + \frac{1}{n}. \tag{5.11}$$

Thus the equation of state of a completely degenerate electron gas is polytropic, with an index of 1.5 ($\gamma = 5/3$) in the nonrelativistic case and 3 ($\gamma = 4/3$) in the extreme relativistic limit. An ideal gas, too, may be described by a polytropic equation of state under certain conditions; we shall encounter such cases later on. Substituting equations (5.10)–(5.11) into equation (5.9), we obtain a second-order differential equation:

$$\frac{(n+1)K}{4\pi G n}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho. \tag{5.12}$$

The solution $\rho(r)$ for $0 \leq r \leq R$, called a *polytrope*, requires two boundary conditions. These are $\rho = 0$ at the surface ($r = R$), which follows from $P(R) = 0$, and $d\rho/dr = 0$ at the centre ($r = 0$), since hydrostatic equilibrium implies $dP/dr = 0$ there (see Section 2.3). Hence a polytrope is uniquely defined by three parameters: K, n, and R, and it enables the calculation of additional quantities as functions of radius, such as the pressure, the mass, or the gravitational acceleration.

It is convenient to define a dimensionless variable θ in the range $0 \leq \theta \leq 1$ by

$$\rho = \rho_c\theta^n, \tag{5.13}$$

to obtain equation (5.12) in a simpler form,

$$\left[\frac{(n+1)K}{4\pi G\rho_c^{\frac{n-1}{n}}}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n. \tag{5.14}$$

Obviously, the coefficient in square brackets on the left-hand side of equation

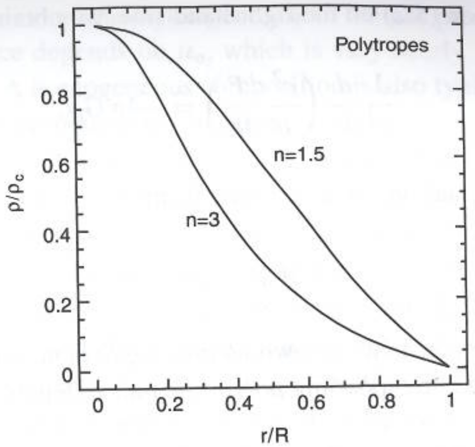

Figure 5.1 Normalized polytropes for $n = 1.5$ and $n = 3$.

(5.14) is a constant having the dimension of length squared,

$$\left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}\right] = \alpha^2, \tag{5.15}$$

which can be used in order to replace r by a dimensionless variable ξ,

$$r = \alpha\xi. \tag{5.16}$$

Substituting equation (5.16) into equation (5.14), we now obtain the well-known Lane–Emden equation of index n,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n, \tag{5.17}$$

subject to the boundary conditions: $\theta = 1$ and $d\theta/d\xi = 0$ at $\xi = 0$. Equation (5.17) can be integrated starting at $\xi = 0$; for $n < 5$, the solutions $\theta(\xi)$ are found to decrease monotonically and have a zero at a finite value $\xi = \xi_1$, which corresponds to the stellar radius,

$$R = \alpha\xi_1. \tag{5.18}$$

Examples of solutions (ρ/ρ_c as a function of r/R), for $n = 1.5$ and $n = 3$, are given in Figure 5.1. As shown, the structure of a polytrope depends only on n. A polytrope of index 3 describes a star in which the mass is strongly concentrated at the centre, whereas a polytrope of index 1.5 describes a more even mass distribution.

Table 5.1 Polytropic constants

n	D_n	M_n	R_n	B_n
1.0	3.290	3.14	3.14	0.233
1.5	5.991	2.71	3.65	0.206
2.0	11.40	2.41	4.35	0.185
2.5	23.41	2.19	5.36	0.170
3.0	54.18	2.02	6.90	0.157
3.5	152.9	1.89	9.54	0.145

The total mass M of a polytropic star is given by

$$M = \int_0^R 4\pi r^2 \rho\, dr = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \xi^2\theta^n d\xi. \tag{5.19}$$

From equation (5.17) we have

$$M = -4\pi\alpha^3\rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) d\xi = -4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}. \tag{5.20}$$

Exercise 5.2: Solve the Lane–Emden equation analytically for (a) $n = 0$ and (b) $n = 1$ and find ξ_1 and $M(R)$ in each case.

In later discussions we shall often resort to general relations between stellar properties resulting from a polytropic equation of state. These follow easily from equation (5.20). Eliminating α between equations (5.18) and (5.20), we obtain a linear relation between the central density and the average density $\bar{\rho}$,

$$\rho_c = D_n\bar{\rho} = D_n\frac{M}{\frac{4\pi}{3}R^3}, \tag{5.21}$$

which is generally valid. Only the constant D_n derives from the solution of equation (5.17) and depends on the value of n:

$$D_n = -\left[\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-1}. \tag{5.22}$$

Values of D_n for various n can be found in Table 5.1.

Using equation (5.20) again, but now eliminating ρ_c with the aid of equation (5.15) and substituting α from equation (5.18), we obtain a relation between the stellar mass and radius, which may be expressed in terms of two constants, M_n

and R_n, in the form

$$\left(\frac{GM}{M_n}\right)^{n-1}\left(\frac{R}{R_n}\right)^{3-n} = \frac{[(n+1)K]^n}{4\pi G}. \tag{5.23}$$

The values of the constants $M_n = -\xi_1^2(d\theta/d\xi)_{\xi_1}$ and $R_n = \xi_1$ vary with the polytropic index n in the range from 1 to 10, as listed in Table 5.1. We note that $n = 3$ is a special case: the mass becomes independent of radius and is uniquely determined by K,

$$M = 4\pi M_3\left(\frac{K}{\pi G}\right)^{3/2}. \tag{5.24}$$

Thus for a given K, there is only one possible value for the mass of a star that will satisfy hydrostatic equilibrium. Another special case is $n = 1$, for which the radius is independent of mass and is uniquely determined by K:

$$R = R_1\left(\frac{K}{2\pi G}\right)^{1/2}. \tag{5.25}$$

Between these limiting values of n, $1 < n < 3$, we have from equation (5.23)

$$R^{3-n} \propto \frac{1}{M^{n-1}}, \tag{5.26}$$

meaning that the radius decreases with increasing mass: the more massive the star, the smaller (and hence denser) it becomes.

A final important relation is obtained between the central pressure and the central density by substituting K from the mass-radius relation (5.23) in equation (5.10), $P_c = K\rho_c^{1+\frac{1}{n}}$, whence

$$P_c = \frac{(4\pi G)^{\frac{1}{n}}}{n+1}\left(\frac{GM}{M_n}\right)^{\frac{n-1}{n}}\left(\frac{R}{R_n}\right)^{\frac{3-n}{n}}\rho_c^{\frac{n+1}{n}}. \tag{5.27}$$

Eliminating R between equations (5.27) and (5.21), and assembling all n-dependent coefficients into one constant B_n, reduces equation (5.27) to

$$P_c = (4\pi)^{1/3}B_n G M^{2/3}\rho_c^{4/3}. \tag{5.28}$$

The remarkable property of this relation is that it depends on the polytropic equation of state *only* through the value of B_n, which, as we see from Table 5.1, varies very slowly with n. It therefore constitutes an almost universal relation, and as with the upper limit derived in Exercise 2.2 (Section 2.3).

Exercise 5.3: For a given mass M and central pressure P_c, which polytrope yields a bigger star: that of index 1.5 or that of index 3?

Exercise 5.4: *Capella* is a binary star discovered in 1899, with a known orbital period, which enables the determination of the mass and radius of the brighter component: $M = 8.3 \times 10^{30}$ kg and $R = 9.55 \times 10^9$ m. Assuming that the star can be described by a polytrope of index 3, find the central pressure and the central density. Check whether the central pressure satisfies inequality (2.18).

5.4 The Chandrasekhar mass

Stars that are so dense as to be dominated by the degeneracy pressure of the electrons (discussed in Chapter 3) would be accurately described by a polytrope of index $n = 1.5$, with $K = K_1$ of equation (3.35). We know from observations that such compact stars exist – they are the *white dwarfs* mentioned in Chapter 1, which have masses comparable to the Sun's and radii not much larger than the Earth's. Their average density is thus higher than 10^8 kg m^{-3} (10^5 g cm^{-3}), about five orders of magnitude higher than the average density of the Sun. We might learn some more about these stars by investigating the properties of this particular polytrope. From equation (5.23), the relation between mass and radius becomes

$$R \propto M^{-1/3}. \tag{5.29}$$

The density, therefore, increases as the square of the mass,

$$\bar{\rho} \propto MR^{-3} \propto M^2. \tag{5.30}$$

Imagine now a series of such degenerate gaseous spheres with higher and higher masses. The radii will decrease along the series and the density will increase in proportion to M^2. Eventually, the density will become so high that the degenerate electron gas will turn to be relativistic, departing from the simple $n = 1.5$ polytrope. As the density increases (the radius tending to zero), the correct equation of state will approach the form (3.38), still a polytrope, but of index $n = 3$, with $K = K_2$. We have seen, however, that in such a case there is only one possible solution for M, uniquely determined by K. Hence our series of degenerate gaseous spheres in hydrostatic equilibrium ends at this limiting mass. The existence of