Appendix Q Physics of Planetary Interiors

Q.1 Introduction

A fruitful way of studying the overall structure of a planet is by treating it as a collection of atoms in equilibrium. Several different kinds of energy are involved, and the relations between them allow the planets to be understood as a single class of object, regardless of their composition. This treatment also makes clear the relationship between a planet and other types of body in the Universe. Unlike for main-sequence stars, the composition may range from being solid to almost completely gaseous. The treatment here gives the essential physics; greater detail about the physics of the interiors of planets has been given by Cole (1984, 1986).

Q.2 Applying the Virial Theorem

The total energy of the body is the sum of the kinetic and potential energy contributions, the former being due to the translational motion of the particles of which it consists and the latter to the interactions between them. A central relationship between the kinetic and potential energies of a group of particles in an equilibrium configuration is expressed by the *virial theorem* (Appendix D). Writing E_k for the total translational kinetic energy and E_v for the total potential energy, the virial theorem states that

$$2E_k = -E_v \tag{Q.1}$$

For planets, the particles are atoms, composed of positively charged nuclei and negatively charged electrons. This means that the kinetic and potential energies can be described in terms of the magnitudes of the electric charges, the particle masses and the total mass, M_p, and the radius, R_p, of the planet as a whole. It will be shown that thermal energy plays a negligible role in considering the equilibrium, so that planets can essentially be regarded as 'cold bodies'.

Q.3 Energies Involved

We now specify explicitly the nature of the kinetic and potential energies. For the relatively low temperatures that exist in planets, as compared to those in stars, the kinetic energy can be supposed to come entirely from the motion of the electrons, with the motions of the positive nuclei negligible by comparison. Consider a hydrogen atom. Relative to the centre of mass, the momenta of the proton and electron have equal magnitude, p. Since kinetic energy is $p^2/(2m)$, it is clear that the kinetic energy of the proton can be neglected because of its relatively high mass. The potential energy has two contributions, one due to electrostatics and the other due to gravity. It is required to express these various energies in terms of the mass and radius of the body and its composition.

Q.3.1 Kinetic (Degeneracy) Energy

We consider a planetary body of mass M_P composed of N_P atoms. If m_A is the average mass of the constituent atoms, then

$$N_P = \frac{M_P}{m_A} \tag{Q.2}$$

Each atom will have Z extranuclear electrons giving ZN_p electrons in the body. Electrons are fermions (Section K.1) and so obey the Pauli Exclusion Principle of all having a different quantum mechanical state. The number of such states in the body is then equal numerically to the number of electrons there. If the volume of the planet is V_p, then the volume available to each electron in the body is

$$v_P = \frac{V_P}{ZN_p} = \frac{4\pi R_P^3}{3ZN_P}$$

Assuming that the 'cell' occupied by the electron is spherical, its radius, d, is given by the expression

$$d = \left(\frac{3\nu_P}{4\pi}\right)^{1/3} = \left(\frac{1}{ZN_p}\right)^{1/3} R_P = \left(\frac{m_A}{ZM_P}\right)^{1/3} R_P \tag{Q.3}$$

The electron kinetic energy is composed of two parts. The first part is the motion of the electron within its cell. The second part is the kinetic energy of motion of the cell, as a result of the atom moving due to its thermal energy; we show later that this kinetic energy is negligible in comparison with the first. The kinetic energy can then be expressed in terms of the dimensions of the cell alone. This energy is independent of the temperature and so is called *degenerate*. In Section K.2, an estimate of the degenerate energy was found from the Heisenberg Uncertainty Principle. Here, we use another approach.

The kinetic energy of the electron is written $E_k = p^2/(2m_e)$, where p is the momentum and $m_e = 9.109 \times 10^{-31}$ kg is the mass of the electron. In quantum mechanics, momentum is related to the electron de Broglie wavelength (Section I.2.2), λ, by the relationship

$$p\lambda = h$$

where $h = 6.626 \times 10^{-34}$ J s is the Planck constant. Considering the electron with a wavelike nature within an enclosure of maximum extent equal to the diameter of its cell, $2d$, then

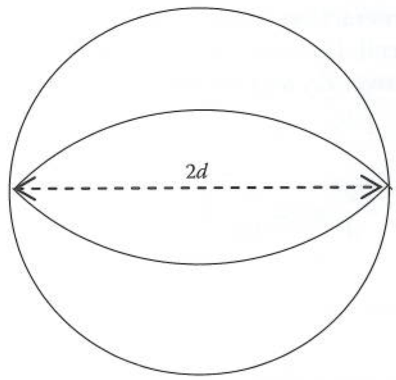

FIGURE Q.1
The longest standing wave within a sphere of diameter $2d$.

its lowest energy state will be when it exists as a standing wave, as shown in Figure Q.1, which has a wavelength of $4d$. This gives

$$E_k = \frac{h^2}{2m_e\lambda^2} = \frac{h^2}{32m_e d^2} \tag{Q.4}$$

The degenerate energy for the whole body is thus

$$E_K = E_k Z N_p \tag{Q.5}$$

Combining Equations Q.2 through Q.4 with Equation Q.5 gives

$$E_K = \gamma_K \frac{M_P^{5/3} Z^{5/3}}{m_A^{5/3} R_P^2} \tag{Q.6}$$

with

$$\gamma_K \approx \frac{h^2}{32m_e} = 1.506 \times 10^{-38}\ \text{kg m}^4\ \text{s}^{-2}$$

More precise theory gives $\gamma_K = 2.31 \times 10^{-38}$ kg$^{-2/3}$ m^4 s^{-2}, a value that we shall use hereafter.

At this point, we can confirm that we do not have to take thermal energy into account. For the Earth, taken as consisting of silicon with $Z = 14$ and $m_A = 4.6 \times 10^{-26}$ kg, the value of E_K is about 1.48×10^{34} J. If the planet had the specific heat capacity of silicon, 700 J kg^{-1} K^{-1}, then it would have to be at an average temperature of about 3.6×10^6 K for the thermal energy to equal that due to the electrons.

Q.3.2 Electrostatic Energy

The problem of summing the contributions of all electrons and nuclei to the electrostatic energy is rather complicated, but we can appeal to an approximate argument for our present purpose. Under the density conditions of normal planets, the speeds of the charges are non-relativistic, so the interaction is adequately described by electrostatics. The electrostatic energy for a single cell, E_e, is written, to a good approximation, as the interaction of the two charges, one positive, Ze, and the other negative, $-e$, separated by the size of a cell. Then

$$E_e \approx -a\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{d} = -a\frac{Ze^2}{4\pi\varepsilon_0}\left(\frac{ZM_P}{m_A}\right)^{1/3}\frac{1}{R_P} = -a\frac{Z^{4/3}M_P^{1/3}e^2}{4\pi\varepsilon_0 m_A^{1/3}R_p} \tag{Q.7}$$

where

e is the electronic charge

ε_0 is the permittivity of free space (8.854×10^{-12} F m^{-1})

a is a numerical constant close to unity

The total electrostatic energy is given by summing over all the cells: $E_e = E_e\, N_P\, Z$, that is,

$$E_e \approx -a\frac{Z^{4/3}M_P^{1/3}e^2}{4\pi\varepsilon_0 m_A^{1/3}R_p}Z\frac{M_E}{m_A} = -a\gamma_e\frac{M_P^{4/3}Z^{7/3}}{m_A^{4/3}R_p} \tag{Q.8}$$

with

$$\gamma_e = \frac{e^2}{4\pi\varepsilon_0} = 2.307\times10^{-28}\ \text{kg m}^3\ \text{s}^{-2}$$

We set $a = 1$ although a precise calculation gives $a = 0.9$.

Q.3.3 Gravitational Energy

The gravitational energy E_g is readily calculated. With b as a numerical constant of order unity,

$$E_g \approx -b\frac{GM_P^2}{R_P} = -\gamma_g\frac{M_P^2}{R_P} \tag{Q.9}$$

The value of b will depend on the distribution of mass within the body, but a typical value is 0.9 giving $\gamma_g \equiv bG = 6.0 \times 10^{-11}$ m^3 kg^{-1} s^{-2}.

Q.3.4 Energies Combined

The kinetic energy has the single contribution (Equation Q.6). The total potential energy is the sum of the expressions in Equations Q.8 and Q.9. Explicitly, the virial expression (Equation Q.1) becomes

$$2\gamma_K\frac{M_P^{5/3}Z^{5/3}}{m_A^{5/3}R_p^2} \approx \gamma_e\frac{M_P^{4/3}Z^{7/3}}{m_A^{4/3}R_p} + \gamma_g\frac{M_P^2}{R_p}$$

which is rearranged to become a relation between R_P and M_P:

$$\frac{1}{R_P} = \frac{\gamma_e}{2\gamma_K}\frac{m_A^{1/3}Z^{2/3}}{M_P^{1/3}} + \frac{\gamma_g}{2\gamma_K}\frac{M_P^{1/3}m_A^{5/3}}{Z^{5/3}} \tag{Q.10}$$

The combinations of constants on the right-hand side are

$$\frac{\gamma_e}{2\gamma_K} = 4.99\times 10^{9}\ \mathrm{m^{-1}} \quad \text{and} \quad \frac{\gamma_g}{2\gamma_K} = 1.30\times 10^{27}\ \mathrm{kg^{-2}\ m^{-1}}$$

It is seen from Equation Q.10 that for small masses the first term on the right-hand side is dominant so that $M_P^{1/3}/R_P$ = constant. This is the normal relationship between mass and radius for a body at uniform density, which would be true for a small body of uniform composition in the absence of compression effects. On the other hand, for very large masses, the second term on the right-hand side of Equation Q.10 becomes dominant, and the different relation $M_P^{1/3}R_P$ = constant then applies. The radius now decreases with increasing mass.

The plot of R_p against M_P is shown in Figure Q.2. The upper curve is for a pure hydrogen composition ($m_A = 1.673 \times 10^{-27}$ kg, $Z = 1$), while the lower curve is for a pure iron composition ($m_A = 9.28 \times 10^{-26}$ kg, $Z = 26$). It is seen that there is a maximum radius in each curve corresponding to a particular mass. The radius increases at first with mass but then decreases as the compression of the material increases with increasing gravitational force to more than compensate for the extra material.

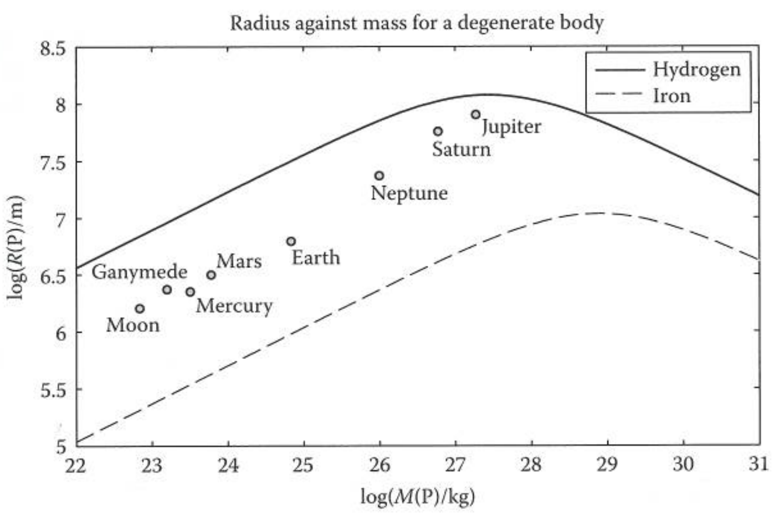

FIGURE Q.2
The radius–mass relationship for bodies made of hydrogen and of iron. According to compositions, various solar-system bodies fall between the two curves.