CLOUDS, EXTENT III

ROSSOW (1978), ICARUS

A MORE DETAILED MODEL, COULD LOOK AT TIMESCALES:

1) CONVECTIVE UPLIFT OF CONDENSIBLE VAPOR

2) COAGULATION — 2 SMALL PARTICLES MAKE A BIGGER ONE
— BROWNIAN MOTION

3) COALESCENCE — 1 BIG PARTICLE W/ HIGHER VELOCITY ABSORBS SMALLER ONES WITH SLOWER VELOCITY

4) NUCLEATION

→ COMPETE W/ FALLOUT OF LARGE PARTICLES AND THEIR VAPORIZATION AT HIGHER T.

- REACH SOME KIND OF STEADY STATE

→ PARTICLE DISTRIBUTION → EVEN HARDER

$$n(a) = \left(\frac{a}{a_0}\right)^6 \exp\left[-6\left(\frac{a}{a_0}\right)\right], \quad a_0 = 4\,\mu m$$

EARTH'S WATER CLOUDS

→ TO MY KNOWLEDGE, NONE OF THE CURRENT EGP OR BD CLOUD MODELS THAT HAVE BEEN EXTENSIVELY COMPARED TO OBSERVATIONS EVEN TRY TO PREDICT THE PARTICLE SIZE DISTRIBUTION

CLOUD OPACITY

SCATTERING & ABSORPTION PROPERTIES CALCULATED FROM MIE THEORY

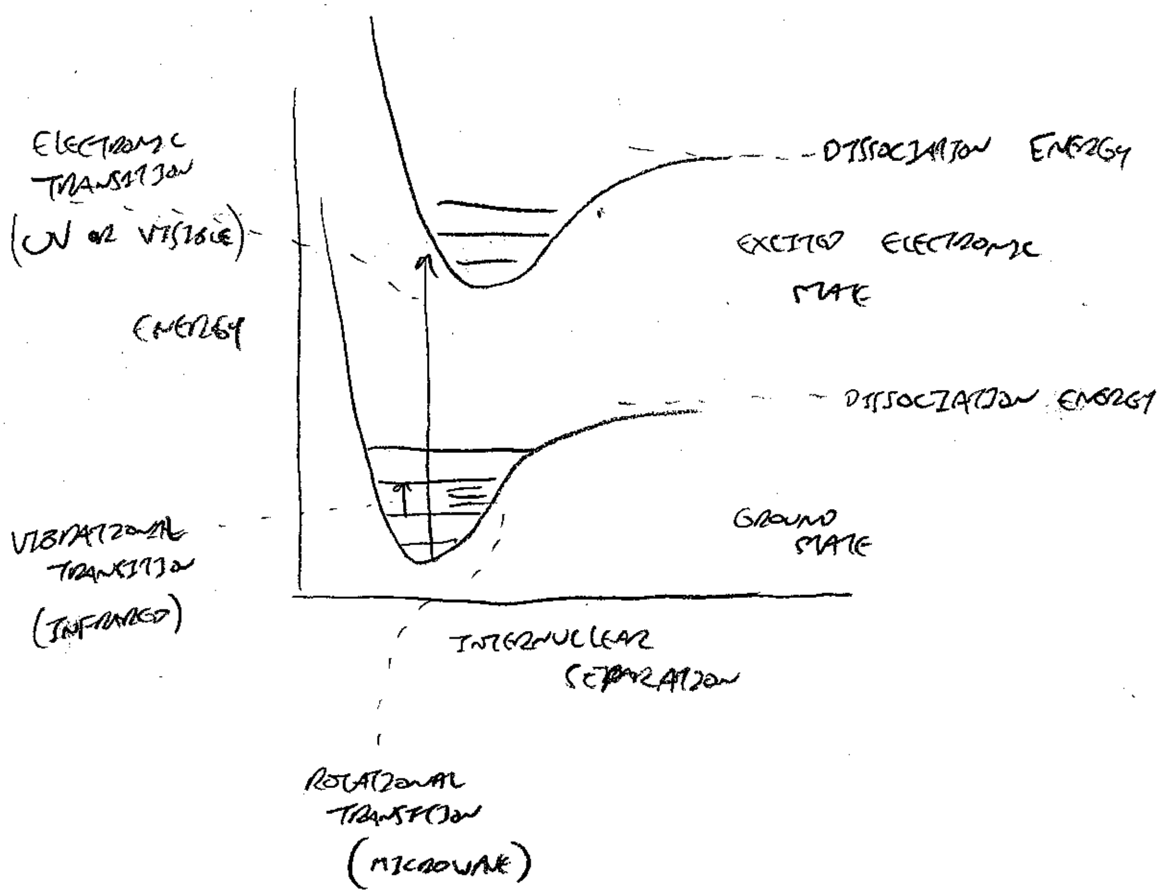
$r \sim \lambda$

$r \ll \lambda$, THEN RAYLEIGH SCATTERING

$r \gg \lambda$, THEN STANDARD GEOMETRIC OPTICS

- MIE THEORY IS JUST APPLYING MAXWELL'S EQUATIONS TO SMALL PARTICLES

MOLECULAR OPACITIES



MOLECULAR OPACITIES

WAVEFUNCTION

$$H\Psi_i = E_i \Psi_i$$

TIME-INDEPENDENT SCHR. EQ.

ENERGY EIGENVALUES

i = INDIVIDUAL LEVEL

HAMILTONIAN OPERATOR → KINETIC + POTENTIAL ENERGY

$$H = \frac{-\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

WHAT IS H OF ROTATION & VIBRATION, SO THAT WE CAN FIND THE EIGENFREQUENCIES?

FOR THESE TRANSITIONS, THE MOLECULE MUST COUPLE TO THE E-M FIELD

— MUST HAVE AN ELECTRIC DIPOLES

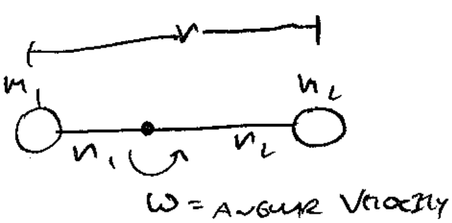
— CENTER OF MASS ≠ CENTER OF CHARGE

YES	NO DIPOLE TRANSITIONS
H_2O	H_2
O_3	N_2
CH_4	O_2
NH_3	

(BUT CAN HAVE WEAKER ELECTRIC QUADRUPOLE OR MAGNETIC DIPOLE MOMENTS, WHICH CAN LEAD TO VIBRATIONAL TRANSITIONS)

ROTATIONAL TRANSITIONS

IMAGINE SIMPLE RIGID ROTOR



$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2$$

$$L = \sum_i m_i \omega r_i^2$$

ROTATIONAL TRANSITIONS

$$E = I\omega^2 = \frac{L^2}{2I}$$ (NO POTENTIAL ENERGY)

ROTATIONAL KE

$$H\varphi = \frac{L^2\varphi}{2I} = E_J\varphi$$

$$\boxed{L^2\varphi = 2IE_J\varphi}$$

$\psi = Y_{\ell,m}(\theta,\phi)$ SOLVES THIS $\rightarrow$ SPHERICAL HARMONICS

$$L^2 Y_{\ell,m}(\theta,\phi) = J(J+1)\hbar^2 Y_{\ell,m}(\theta,\phi) = 2IE_J\varphi$$

$$E_J = J(J+1)\frac{\hbar^2}{2I}$$

$$E_J = B\,J(J+1)$$

ROTATION CONSTANT

$$\Delta E = E_J \pm E_{J-1} = 2BJ$$ EVENLY SPACED IN FREQ. OR WAVE #

For CO, $m_1 \sim m_2 \sim 12$ AMU

$r_1 \sim r_2 \sim 1$ Å

$\Delta E \sim 1.4 \times 10^{-16}$ erg $\rightarrow$ 0.014 m $\rightarrow$ MICROWAVE

IN REALITY MASSES FEEL A CENTRIFUGAL FORCE,

LEADING TO AN ADDITIONAL TERM IN THE HAMILTONIAN

VIBRATIONAL TRANSITIONS

VIBRATION ALWAYS LEADS TO ROTATION, BUT NOT VICE-VERSA

$E_{VIBR} >> E_{ROTATION}$, IGNORE ROTATION TO 1ST ORDER

DRAW ANALOGY TO SIMPLE HARMONIC OSCILLATOR

$V =$ POTENTIAL $E = \frac{1}{2}CX^2$ (POTENTIAL E ~ LIKE A PARABOLA)

$X =$ DIST. FROM CENTER OF MASS

$$E_V = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + V(x)$$

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$$

$$\boxed{E_V = \frac{p^2}{2\mu} + \frac{1}{2}Cx^2}$$

$$H\psi = E\psi$$

$$\frac{-\hbar^2}{2\mu}\frac{d^2\psi}{dx^2} + \frac{1}{2}Cx^2\psi = E_V\psi$$

$$E_V = \left(V + \frac{1}{2}\right)h\left[\frac{1}{2\pi}\sqrt{\frac{C}{\mu}}\right]$$

$$E_V = h\nu_0\left(V + \frac{1}{2}\right) \quad (V = 0, 1, 2, 3 \ldots)$$

VIBRATIONAL QUANTUM #

$$\nu_0 = \frac{1}{2\pi}\sqrt{\frac{C}{\mu}} = \text{FUNDAMENTAL FREQUENCY}$$

$$\Delta V = \pm 1$$

LEADS TO ALL TRANSITIONS HAVE THE SAME FREQUENCY

SINCE SIMPLE HARMONIC OSCILLATOR IS A SIMPLIFICATION, THIS IS NOT ACTUALLY TRUE

VIBRATIONAL II

WE CAN ESTIMATE ν_0 FOR THE CO MOLECULE:

$$C \simeq 1860 \ \frac{N}{m^2}$$

$$\nu_0 = 6.4 \times 10^{13} \ Hz \longrightarrow \boxed{4.3 \ \mu m}$$

USUALLY YOU OBSERVE ν_0 TO GET AT C.

COMBINED RO-VIBRATIONAL SPECTRUM

FOR A GIVEN VIBRATIONAL TRANSITION ΔV, THERE ARE THREE BRANCHES, P, Q, R DUE TO ROTATION

P: $\Delta J = -1$

Q: NO ROTATION — EMPTY FOR DIATOMIC MOLECULES

R: $\Delta J = +1$

SEAGER FIG. 8.6 + 8.7

MOLECULAR ELECTRONIC TRANSITIONS

$$a \sim 1 \ \text{Å} = \text{MOLECULAR SIZE}$$

$$p_e = \frac{\hbar}{a}$$

$$E_e = \frac{p_e^2}{m_e} = \frac{\hbar^2}{m a^2} \sim \text{FEW eV} \rightarrow \text{UV TO VISIBLE}$$

ATMOSPHERIC ORIGIN

PRIMARY & SECONDARY ATMOSPHERES

PRIMARY ATMOSPHERES — OBTAINED FROM THE SOLAR NEBULA

PRIMARILY H_2 + He, WITH A He/H RATIO THAT IS ASSUMED TO BE EXACTLY THE SAME AS THE SUN'S BULK RATIO

$Y \sim 0.275 \pm 0.005$, FROM HELIOSEISMOLOGY

$Y_{JUP} = 0.238 \pm 0.004$

$Y_{SAT} = 0.18 - 0.25$

$Y_{UR} \approx 0.25 - 0.30$

$Y_{NEP} \approx 0.25 - 0.30$

CH_4 IS THE ONLY "SPECTROSCOPICALLY ACTIVE" MOLECULE FOUND IN ALL FOUR PLANETS — THEY ARE SO COLD THAT ALL OTHERS CONDENSE. EVEN CH_4 PARTIALLY CONDENSES IN URANUS & NEPTUNE.

CH_4 ABUNDANCES

J: 4-5 X SOLAR

S: 5-12 X SOLAR

U: \
N: / 30-60 X SOLAR

PRIMARY ATMOSPHERES

JOVIAN ABUNDANCES WELL-DETERMINED FROM THE GALILEO ENTRY PROBE

SATURN C-ABUNDANCE DETERMINED FROM WIDE-λ COVERAGE OVER NEAR & MID IR FROM CASSINI

* THE ALLURE OF WARM, YOUNG JUPITERS, OR HIGHLY IRRADIATED JUPITERS, IS THAT ALL CNOS SPECIES ARE IN VAPOR FORM, AND THEN READILY DETECTABLE.

SECONDARY ATMOSPHERES

— IF H/He WERE OBTAINED FROM THE NEBULA, THEY ARE READILY LOST, DUE TO LOW g, LOW μ, HIGH T.

— HOWEVER, ACCRETION AND RADIOACTIVE DECAY LEADS TO HIGH TEMPERATURES WHICH RELEASES GAS FROM ROCKS

GENERIC SOLAR-SYSTEM-ROCKY COMPOSITION

OUTGASSING

H_2O	CO_2	SO_2	N_2	NOBLE GASES
58%	23%	13%	5%	1%

SECONDARY ATMOSPHERES, II

	VENUS	EARTH	MARS	
H_2O	$<0.1\%$	$<1\%$	$<0.1\%$	LOST, OR OCEANS, OR INTO ROCKS
CO_2	97%	$<1\%$	95%	MAIN COMPONENT IF NOT LOST INTO H_2O AND ROCKS
SO_2	$<1\%$	$<0.1\%$	$<0.1\%$	LOST BACK INTO ROCKS, ALSO IN VENUS ATM.
N_2	2%	78%	3%	STICKS AROUND
NOBLES	0.5%	1%	2%	STICKS AROUND
O_2	—	21%	—	

LEFT BEHIND WHEN MOST OF THE ATMOS. WAS LOST

- GREATER FRACTION OF NOBLE GASES, MORE ATMOSPHERE WAS LOST

TITAN: A SPECIAL CASE

$90\%\ N_2$

$7\%\ CH_4$

OUTER SOLAR SYSTEM "ROCKS" INCLUDED

$H_2O / NH_3 / CH_4$

$H_2O \downarrow$ SOLID

$NH_3 \downarrow$ $2NH_3 + h\nu \longrightarrow N_2 + 3H_2$

$N_2 \downarrow$ STAYS

$H_2 \downarrow$ LOST VIA ESCAPE

ATMOSPHERIC ESCAPE

INHERENTLY TIED IN TO CURRENT ATMOSPHERIC COMPOSITION

- YOU ONLY HAVE TO SHOW THE ATMOSPHERE THAT YOU DIDN'T LOSE!
- YOU MAY LOSE EVERYTHING

3 STAGES OF ESCAPE

1) TRANSPORT FROM LOWER ATMOSPHERE TO UPPER ATMOSPHERE
2) CONVERSION OF GAS FROM MOLECULAR TO ATOMIC/IONIC FORM
3) THE ACTUAL ESCAPE PROCESS

ANY ONE CAN BE THE BOTTLENECK STAGE

THREE GENERAL PROCESSES

1) THERMAL HYDROSTATIC ESCAPE
2) THERMAL HYDRODYNAMIC ESCAPE
3) NONTHERMAL ESCAPE

DOMINANT PROCESS DEPENDS ON

M_p, ATM. COMPOSITION, a,
EXOSPHERE TEMPERATURE, B-FIELD,
STELLAR TYPE, XUV FLUX,
STELLAR WIND

- NOT ALL OF THESE ARE MEASURABLE QUANTITIES FOR EXOPLANETS
- (OR KNOWN AS A FUNCTION OF TIME)

ESCAPE II

SEAGER FIG. 4.13

1) HYDROSTATIC THERMAL ESCAPE

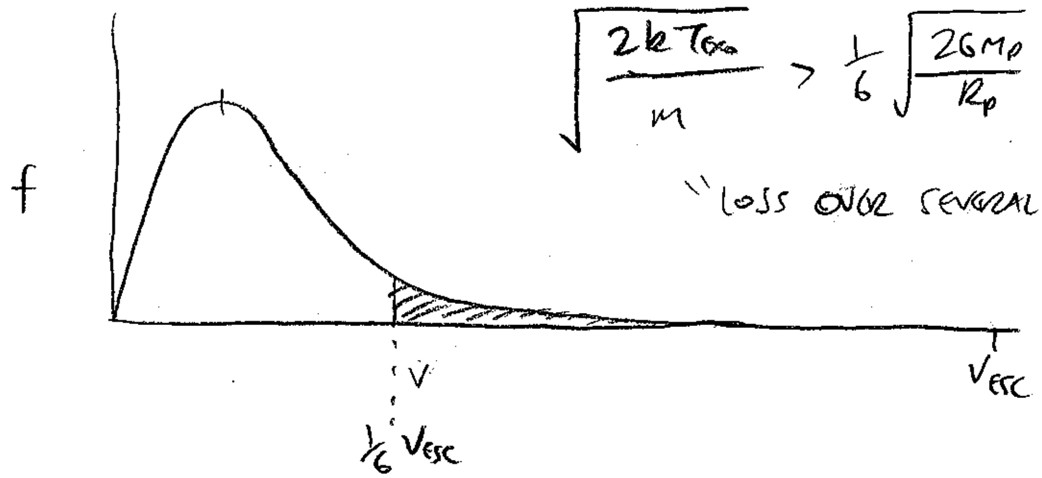
$$V_{\text{molecule}} = \sqrt{\frac{2kT_{exo}}{m}} > \sqrt{\frac{2GM_p}{R_p}}$$

IF THAT WERE TRUE YOU'D LOSE YOUR ATMOSPHERE VERY FAST

REALLY, THOUGH, YOU LOSE YOUR ATMOSPHERE FROM THE HIGH-VELOCITY TAIL OF THE M-B DISTRIBUTION

$$\sqrt{\frac{2kT_{exo}}{m}} > \frac{1}{6}\sqrt{\frac{2GM_p}{R_p}}$$

"LOSS OVER SEVERAL Gyr"



THE "EXOSPHERE" IS YOUR COLLISIONLESS ATMOSPHERE

- EXOBASE IS THE BOTTOM OF THIS, WHERE MFP = H_p

ESCAPE III

A MORE ACCURATE VIEW IS FROM JEAN'S ESCAPE:

- LOSS OF YOUR M-B DISTRIBUTION ATMOSPHERE FROM THE EXOBASE

$$f(v)dv = \frac{4n}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2} v^2 e^{-mv^2/2kT}\, dv$$

$$n(r) = n(r_0)\, e^{-r/H(r_0)} \qquad \left(H = \frac{kT}{\mu m_H\, g(r)}\right)$$

JEAN'S ESCAPE FLUX:

$$\Phi_{JEANS} = \int_{v_{esc}}^{\infty}\int_0^{2\pi}\int_{\pi/2}^{0} f(v)\cos\theta \sin\theta\, d\theta\, d\phi\, dv$$

$$\Phi_{JEANS} = \frac{n_c}{2\sqrt{\pi}}\, \beta \sqrt{\frac{2kT}{m}}\,(1+\lambda_c)\, e^{-\lambda_c} = \frac{\#}{cm^2 \cdot s}$$

β = FUDGE FACTOR
= 0.5 - 0.8
⟹ REPOPULATION OF FAST PARTICLES IS DIFFUSION-LIMITED

$$\boxed{\lambda_c = \text{ESCAPE PARAMETER} = \frac{GM_p m}{kT_c r_c} = \frac{r_c}{H}}$$

"c" AT EXOBASE.

r_c: DISTANCE FROM PLANETARY CENTER

$$\boxed{\lambda_c = \frac{\frac{1}{2} m v_{esc}^2}{kT_c}}$$

ESCAPE IV

HYDRODYNAMIC ESCAPE

- GRAVITY IS NOT ENOUGH TO STOP OUTWARD FLUID FLOW OF THE EXPANDING ATMOSPHERE

- HIGH XUV FLUX DRIVES HIGH EXOBASE TEMPERATURES

  - HEAVIER ELEMENTS MAY BECOME ENTRAINED IN THE FLOW

HYDRODYNAMIC VS. HYDROSTATIC REGIME:

$$\lambda_c \sim 1$$

$$\lambda_c \gg 1 \implies \text{JEANS}$$

- A SIMPLE MODEL OF HYDRODYNAMIC ESCAPE IS "ENERGY LIMITED ESCAPE"

$$E_{GPE} = \frac{-G M_p m_{ATM}}{\underbrace{\beta R_p}_{\text{RADIUS OF THE ATMOSPHERE}}} \qquad \beta \approx 1$$

INCIDENT XUV POWER $\left(\frac{\text{erg}}{\text{s}}\right) = P_{XUV} = \pi R_p^2 F_{XUV}$

$$\frac{E_{GPE}}{P_{XUV}} = \tau \approx \left[\frac{G}{\pi}\right] \underbrace{\left(\frac{M_p}{R_p^3}\right)}_{\text{KNOWN FOR TRANSITING PLANETS}} \frac{m_{ATM}}{\beta \, \eta \, F_{XUV}}$$

β: $(\gtrsim 1)$ η: FRACTION ABSORBED $(\lesssim 1)$

ESCAPE V

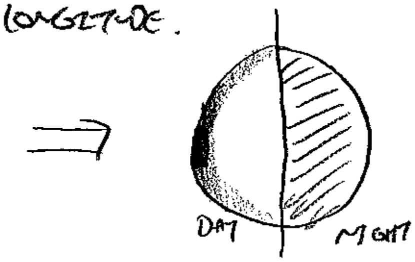
NONTHERMAL ESCAPE

COLLISIONAL PROCESSES THAT CAN LEAVE AN ATOM
WITH A HIGH VELOCITY

PROCESS	EXAMPLES
CHARGE EXCHANGE	

ATMOSPHERIC DYNAMICS

- ATMOSPHERIC CIRCULATION IS DRIVEN BY UNEVEN STELLAR HEATING, AS A FUNCTION OF LATITUDE & LONGITUDE.

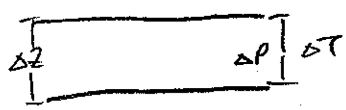


- 1ST ORDER: IS YOUR ATMOSPHERE DOMINATED BY THE RADIATIVE-EQUILIBRIUM SOLUTION OR IS IT DOMINATED BY CIRCULARIZATION?

$$\tau_{ADV} = \frac{L}{U} \approx \frac{R_p}{U} = \frac{10^5\ \mathrm{km}}{1\ \mathrm{km/s}} \approx 10^5\ \mathrm{sec}$$

ADVECTIVE TIMESCALE

τ_{RAD}:



OUT OF RADIATIVE EQUILIBRIUM BY AN AMOUNT ΔT, LAYER THICKNESS IS Δz & ΔP

LAYER: $T + \Delta T$

$$\frac{\text{EXCESS } E}{\text{AREA}} = \boxed{\rho c_p \Delta T \Delta z}$$

RADIATE = $\sigma (T+\Delta T)^4 \approx \sigma T^4 \left(1 + \frac{4\Delta T}{T}\right)$

AS B.B.

$$\approx \sigma T^4 + \boxed{4\sigma T^3 \Delta T}$$

$$\tau_{RAD} = \frac{\rho c_p \cancel{\Delta T} \Delta z}{4\sigma T^3 \cancel{\Delta T}}$$

$$\frac{dP}{dz} = -\rho g$$

$$\Delta z \rho = \frac{\Delta P}{g}$$

$$\boxed{\tau_{RAD} \approx \frac{\Delta P\, c_p}{g\, 4\sigma T^3}}$$

RADIATIVE TIMESCALE

DYNAMICS II

- PHOTOSPHERIC P IS A FUNCTION OF λ.
- WHETHER OR NOT ATMOSPHERE IS CLOSE TO RADIATIVE EQUILIBRIUM DEPENDS ON THE PRESSURE OF INTEREST

UPPER ATMOSPHERE : P low, $T \sim$ "HIGH"

SHORT $\tau_{RAD} \rightarrow$ RADIATIVE EQUIL. HOLDS

DEEP ATMOSPHERE : P HIGH, $T \sim$ "HIGH"

LONG $\tau_{RAD} \longrightarrow$ PLANET WELL HOMOGENIZED

(ALSO, WHEN OPTICALLY THICK, $\tau_{RAD} \propto P^2$

PLANET	τ_{RAD}	$\tau_{ADV,\,LON}$	ΔT_{LON}	$\tau_{ADV,\,LAT}$	ΔT_{LAT}
EARTH	WEEKS	1 DAY	~ 10 K	WEEKS	20–30 K
~~JUPITER~~	~~$\gtrsim$ DECADES~~	~~DECADES~~	~~$\gtrsim$ FEW K~~	~~$\gtrsim$ DECADES~~	~~$\gtrsim$ FEW K~~
VENUS	YEARS	DAYS	FEW K	WEEKS	FEW K

WELL HOMOGENIZED ($\tau_{RAD} \gg \tau_{ADV}$)

OK HOMOGENIZED ($\tau_{RAD} \gtrsim \tau_{ADV}$)

~~WELL HOMOGENIZED (τ_{RAD}~~

$\boxed{\text{SKIP TO R20}}$

THERE ARE SOME COMPLEXITIES HERE, THOUGH

$\rightarrow$ FOR JUPITER, WE HAVE LONG τ_{RAD}, BUT ALSO

$\rightarrow$ RAPID ROTATION (SMALL DAY-NIGHT)

$\rightarrow$ ABSORPTION OF SUNLIGHT INTO THE CONVECTIVE INTERIOR (SMALL EQUATOR-TO-POLE)

SIMPLE ESTIMATE OF DAY-NIGHT CONTRAST

$$R_{DN} = \frac{\Delta T_{DAY-NIGHT}}{\Delta T_{RAD-EQ}} \approx 1 - e^{-\tau_{ADV}/\tau_{RAD}}$$

$$\frac{\tau_{ADV}}{\tau_{RAD}} \gg 1 \Rightarrow R_{DN} = 1 \rightarrow \Delta T_{DAY-NIGHT} = \Delta T_{RAD\ EQ}$$

LARGE, SINCE T_{RAD-EQ} NIGHT IS 100–300 K

$\Downarrow$

1500 − 300 ≈ 1200

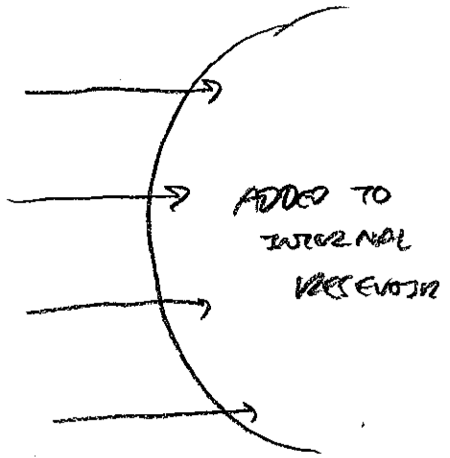
2000 − 300 ≈ 1700

$$\frac{\tau_{ADV}}{\tau_{RAD}} \ll 1 \Rightarrow R_{DN} \Rightarrow 0, \Delta T_{DAY-NIGHT} \longrightarrow 0 \quad \text{(HOMOGENIZED)}$$

$$\frac{\tau_{ADV}}{\tau_{RAD}} = 0.3 \Rightarrow \Delta T_{DAY-NIGHT} \approx \frac{1}{4} \Delta T_{RAD-EQ}$$

CAN USE OBSERVATIONS TO ESTIMATE RADIATIVE + ADVECTIVE TIMESCALES → OR AT LEAST THEIR RELATIVE IMPORTANCE

DYNAMICS III



ALL LESS-IRRADIATED GAS GIANTS SHOULD BE EXTREMELY WELL-HOMOGENIZED.

- BUT FOR THE HOT JUPITERS, A DEEP RADIATIVE ZONE IS FORCED, AND ABSORBED ENERGY IS REDISTRIBUTED ⟶ SMALL EQUATOR TO POLE DIFFERENCE (INGERSOLL & PORCO, 1978)

INTERESTING NUMBERS

RHINES SCALE = WIDTH OF JETS

Ω = ANGULAR VELOCITY

$f = 2\Omega \sin\theta$ (θ = LATITUDE)

$$L_\beta = \pi \sqrt{\left(\frac{U}{\beta}\right)} \qquad U = \text{WIND SPEED}$$

$$\beta = \frac{df}{dy} = 2\Omega \cos\theta / R_p$$

$$N \sim \frac{\pi R_p}{L_\beta} = \text{NUMBER OF BANDS}$$

$N \sim 10\text{-}20$ FOR JUPITER & SATURN

$N \simeq 1$ FOR HD209458b

INTERESTING NUMBERS II

ROSSBY DEFORMATION RADIUS

$$L_D \approx \frac{N_{BF}\, H}{f}$$

N_{BF} = BRUNT-VÄISÄLÄ FREQUENCY

= OSCILLATION FREQ. IN RADIATIVE REGIONS

$\approx$ 40,000 km FOR HD209 $\Rightarrow$ 1-2 JETS

$\approx$ 2,000 km FOR JUPITER $\Rightarrow$ 30 JETS

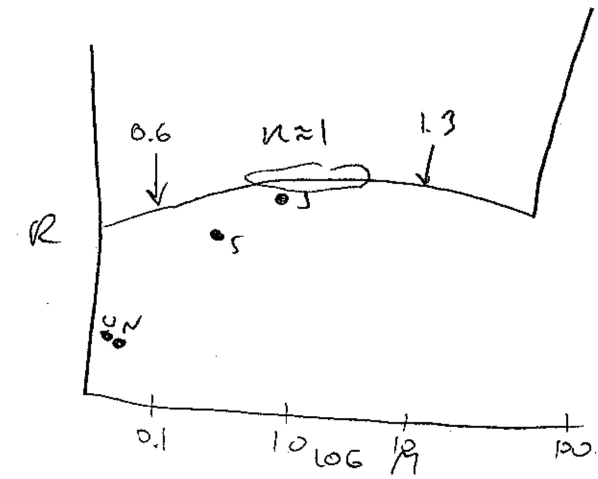
RHINES SCALE + ROSSBY DEF. RADIUS IMPLY FLOW ON HOT JUPITERS SHOULD BE PLANET-WIDE IN SCOPE $\rightarrow$ MUCH DIFFERENT THAN THAT FOUND FOR OUR GAS GIANTS

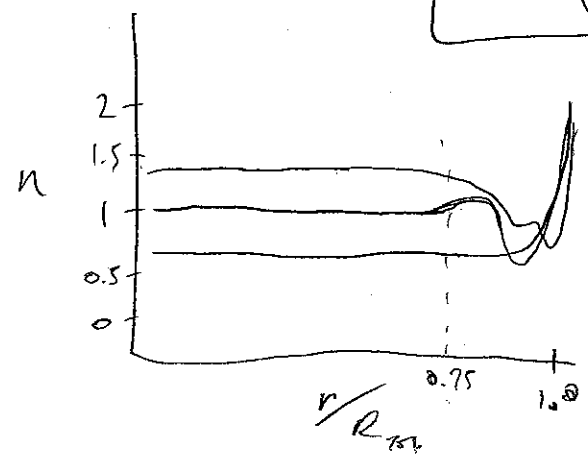
EFFECTS OF IRRADIATION ON M-R

$n=1$ IS THE CONSTANT RADIUS POLYTROPE

GOOD APPROXIMATION FOR JUPITER

$$\boxed{P = K\rho^{1+\frac{1}{n}}}$$





IF K IS NOT $K(M)$, THEN

$$R \propto M^{\frac{1-n}{3-n}}$$

IF $K = K(M)$ DUE TO THERMAL EFFECTS

$$\boxed{R \propto K^{\frac{n}{3-n}} M^{\frac{1-n}{3-n}}}$$

WHAT IS K?

- $K = P_0^{-\frac{1}{n}} \left(\frac{\mathcal{R} T_0}{\mu}\right)^{1+\frac{1}{n}}$ FOR IDEAL GAS ATMOSPHERE
- ASSUME PLANETARY INTERIOR IS TIED TO THE SURFACE BY A CONSTANT n.
- LET T_0 BE SET BY INCIDENT STELLAR FLUX, AND LET T_0 BE INDEPENDENT OF MASS

EDDINGTON ATMOSPHERIC BOUNDARY:

1) $P_0 \propto \frac{g}{\kappa}$

2) $\kappa \propto P$ IS A DECENT APPROXIMATION IN THESE ATMOSPHERES

IRRADIATED M-R RELATION II

$$K = P_0^{-\frac{1}{n}} \left(\frac{R T_0}{\mu}\right)^{1+\frac{1}{n}}$$

$$K \propto P_0^{-\frac{1}{n}}$$

$$P_0 \propto \frac{g}{\kappa} \qquad \kappa \propto P$$

$$P \propto \frac{g}{P}$$

$$P^2 \propto g$$

$$P \propto \sqrt{g}$$

$$K \propto \left[(g)^{\frac{1}{2}}\right]^{-\frac{1}{n}}$$

$$\boxed{K \propto \left(\frac{M}{R^2}\right)^{-\frac{1}{2n}}}$$

$$R \propto K^{\frac{n}{3-n}} M^{\frac{1-n}{3-n}}$$

$$\boxed{R \propto M^{\frac{\frac{1}{2}-n}{2-n}}}$$

For $n=1$, $R \propto M^{-\frac{1}{2}}$

For $n=\frac{3}{2}$, $R \propto M^{-2}$

LOWER MASS PLANETS ARE PREFERENTIALLY INFLATED DUE TO INCIDENT STELLAR FLUX

- COMPLICATED BECAUSE: $n \neq$ CONSTANT IN INTERIOR

  DEEP RADIATIVE ATMOSPHERE

SEMI-ANALYTICAL COOLING MODEL

- ADIABATIC INTERIOR
- MOSTLY LIQUID METALLIC HYDROGEN

$$T \approx C\, T_{1BAR}\, \rho^{\gamma} \qquad (1)$$

FOR JUPITER, CALCULATIONS YIELD

$C \approx 42.8$

$\gamma = 0.64$ = GRÜNEISEN PARAMETER

$$\frac{\partial L}{\partial m} = -T \frac{\partial S}{\partial t}$$

$$= \underbrace{-C_V \underbrace{\frac{\partial T}{\partial t}}_{(-)}}_{\substack{(+) \\ \text{COOLING}}} + \underbrace{C_V \underbrace{\left(\frac{\partial T}{\partial \rho}\right)_S}_{(+)} \underbrace{\frac{\partial \rho}{\partial t}}_{(+)}}_{\substack{(+) \\ \text{CONTRACTION}}} \qquad (2)$$

$$\int \frac{\partial L}{\partial m} = L = 4\pi R^2 \sigma T_{EFF}^{4} = -\int C_V \left(\frac{\partial T}{\partial t} - \gamma \frac{T}{\rho} \frac{\partial \rho}{\partial t}\right) dm \qquad (3)$$

SURFACE BOUNDARY CONDITION:

$$T_{1BAR} = K\, T_{EFF}^{a}\, g^{-b} \qquad (4)$$

$K = 1.5$

$a = 1.243$

$b = 0.167$

$\frac{\partial T}{\partial t}$ CAN BE WRITTEN AS:

$$\frac{\partial T}{\partial t} = T\left(\underbrace{\overbrace{\cancel{-b \frac{d \ln g}{dt}}}^{\text{SMALL}}}_{\approx 0} + a \frac{d \ln T_{EFF}}{dt} + \gamma \frac{d \ln \rho}{dt}\right) \qquad (5)$$

① + ⑤ YIELD:

$$dt = -\alpha(T_{EFF})\, T_{EFF}^{a-5}\, dT_{EFF} \quad (6)$$

WHERE $\alpha(T_{EFF}) = \frac{a C K}{4\pi R^2 \sigma g^b} \int C_V \rho^\gamma dm$

α IS SOME KIND OF STRUCTURAL NUMBER

$C_V \approx 1.66\ k_B/m_H$

$\alpha(\text{NOW} = T_{EFF} = 124.4\,K) = 2.8 \times 10^{23}$ ERG

IF WE ASSUME α = CONSTANT ...

(NO STRUCTURAL CHANGES W/ TIME)

... ONE CAN COMPARE $T_{EFF,0}$ TO $T_{EFF,1}$

$$\Downarrow$$

$$\Delta t = \frac{\alpha}{4-a}\left(T_{EFF,1}^{a-4} - T_{EFF,0}^{a-4}\right)$$

$a \approx 1.24$

$$\frac{\Delta t_1}{\Delta t_2} = \frac{T_{EFF} = \infty \text{ TO } T_{EFF,1}}{T_{EFF,1} \text{ TO } T_{EFF,1}/4} \approx \frac{1}{50}$$

IMPLIES EVOLUTIONARY HISTORY IS VERY WEEKLY DEPENDENT ON THE INITIAL CONDITIONS

JUPITER: $T_{EFF} = \infty$ TO $124.4\,K \approx 5.4$ Gyr

SATURN: $T_{EFF} = \infty$ TO $95\,K \approx 2$ Gyr

CORRECTION DUE TO ABSORBING STELLAR FLUX:

$$4\pi R^2 \sigma T_{EFF}^4 \equiv L_{INT} + L_{EQ}$$

REWRITE (3)

$$4\pi R^2 \sigma \underbrace{\left(T_{EFF}^4 - T_{EQ}^4\right)}_{T_{INT}^4} = -\int c_V \left(\frac{dT}{dt} - \gamma \frac{T}{\rho} \frac{d\rho}{dt}\right) dm$$

EQUATION (6), $dt = -\alpha \left(T_{EFF}\right) \dfrac{T_{EFF}^{a-5}}{1 - \left(\frac{T_{EQ}}{T_{EFF}}\right)^4} dT_{EFF}$

EXPANSION IN POWERS OF $\left(\frac{T_{EQ}}{T_{EFF}}\right)^4$ GIVES:

$$t = \frac{\alpha}{4-a} T_{EFF}^{a-4} \left[1 + \frac{4-a}{8-a}\left(\frac{T_{EQ}}{T_{EFF}}\right)^4 + \frac{4-a}{12-a}\left(\frac{T_{EQ}}{T_{EFF}}\right)^8 + \cdots\right]$$

$\frac{T_{EQ}}{T_{EFF}}$ JUP = 0.60

$\frac{T_{EQ}}{T_{EFF}}$ SAT = 0.56 — $[\,] \approx 1.3$ NOT NEGLIGIBLE

t_{COOL} (∞ TO 124.4 K) JUPITER $\approx$ 7 Gyr

$\boxed{\alpha \text{ NOT REALLY A CONSTANT}}$