FORMATION OF THE GIANT PLANETS BY NUCLEATED INSTABILITY

(STEVENSON, 1982)

CORE OF MASS M_c, SURROUNDED BY A GASEOUS ENVELOPE, WITH $P(r)$, $\rho(r)$, $T(r)$ AT POSITION r IN THE ENVELOPE. M_T = TOTAL MASS

$$M_T = M_{ENV} + M_c$$

IF RADIATIVE TRANSPORT DOMINATES:

ROSSELAND MEAN

$$\frac{dT}{dr} = \frac{-3\kappa}{16\sigma T^3}\frac{L(r)}{4\pi r^2} \quad \boxed{A}$$

ASSUME HSE:

$$\frac{dP}{dr} = \frac{-GM_r}{r^2}\rho \quad \boxed{B}$$

DIVIDE $\boxed{A}$ & $\boxed{B}$ AND ASSUME T & P AT BOUNDARY T_0 & P_0

$T_0 \ll T \quad P_0 \ll P$

GIVES:

$$T^4 \approx \frac{3\kappa L}{16\pi\sigma GM} P$$

ASSUME IDEAL GAS: $P = \frac{\rho}{\mu} kT$ | LET $M_T \approx M_r$

APPROX. HSE + IDEAL GAS

$$\frac{1}{\rho}\frac{P}{r} = \frac{-GM_T}{r^2}$$

$$\boxed{T \approx \frac{\mu G M_T}{4 k_B r}}$$

?

$$\rho \approx \frac{64\pi\sigma}{3\kappa L}\left(\frac{\mu G M_T}{4 k_B}\right)^4 \frac{1}{r^3}$$

FORMATION II

$$M_{ENV} = \int_{R_c}^{R_{PHOTOS}} 4\pi \rho(r) r^2 dr = \frac{64\pi\sigma}{3\kappa L}\left(\frac{\mu G M_T}{4k}\right)^4 4\pi\, \omega\left(\frac{R_{PHOTOS}}{R_{CORE}}\right)$$

ASSUME THAT L IS DUE TO THE ACCRETION OF PLANETESIMALS

$$L \approx \frac{GM_c}{R_c} \cdot \underbrace{\frac{dM_c}{dt}}_{\text{CONSTANT}}$$

$$\frac{4\pi}{3} R^3 \rho = M$$

$$L \propto \frac{GM_c}{M_c^{1/3}} \propto M_c^{2/3}$$

$$M_T = M_c + M_{ENV}$$

$$\boxed{M_T \approx M_c + \left(\frac{\alpha}{\kappa}\right)\frac{M_T^4}{M_c^{2/3}}}$$

SLOWLY VARYING CONSTANT THAT IS ~ CONSTANT WITH CHANGING M_c OR M_T, BUT HAS A $\alpha = \mu^{1/4}$ DEPENDENCE

IF M_c BECOMES TO LARGE NO SOLUTION FOR M_T IS POSSIBLE. HSE EQUILIBRIUM CANNOT BE MAINTAINED.

SET $\frac{dM_c}{dM_T} = 0$ GIVES: $M_T = \left(\frac{\kappa}{4\alpha}\right)^{1/3} M_c^{2/9}$

$$M_c = 0.38\left(\frac{\kappa}{\alpha}\right)^{3/4}$$

FORMATION III

SOLUTION RELATIVELY INSENSITIVE TO OUTER BOUNDARY CONDITION BECAUSE OF THE LN (R_{PHOTOS}) DEPENDENCE

$M_c \propto \mu^{-12/7}$ → DISSOLUTION OF PLANETESIMALS INTO THE ENVELOPE MAKES FORMING THE GIANT PLANETS EASIER, BUT WOULD ALSO TEND TO DECREASE THE CORE MASS — BELOW SOME CURRENT ESTIMATES

DYNAMO SCALINGS

PARAMATERIZED TWO WAYS:

SURFACE FIELD STRENGTH, B_p — 400 nT, MERCURY; 5×10^5 nT, JUPITER — EARTH $\sim 3 \times 10^4$ nT

DIPOLE MOMENT $M \approx B_p R_p^3$ — 4×10^{12} Tm^3, MERCURY; 1.5×10^{20} Tm^3, JUPITER

PLANETS HAVE MAGNETIC FIELDS AND WE'D LIKE TO UNDERSTAND WHAT CAUSES, MAINTAINS, AND ENDS THEM, AND WHY THEY HAVE THE STRENGTHS THAT THEY DO.

GIVEN THAT THIS IS EXTREMELY COMPLICATED, THERE HAS BEEN A WEIRD FASCINATION IN SIMPLE LAWS THAT CAN MATCH OBSERVATIONS, AND POTENTIALLY HAVE PREDICTIVE POWER.

RELEVANT PARAMETERS

R_p

R_c

B_p

B

ρ_{core}

Ω

σ_{core}

q_c : CONVECTED ENERGY FLUX

CAN COMPARE TO OBSERVATIONS OR CAN COMPARE TO NUMERICAL RESULTS

MAGNETIC BODE'S LAW

$$B_p R_p^3 \propto \left(\rho \Omega R_p^5\right)^{3/5}$$

ELASSER NUMBER RULE

$$B^2 \propto \frac{\rho \Omega}{\sigma}$$

ENERGY FLUX SCALING

$$B^2 \propto \rho R_c^{4/3} q_c^{2/3}$$

B FIELDS

MERCURY : STRONG DIPOLE ~ $\frac{1}{100}$ EARTH STRENGTH
~ALIGNED W/ ROTATION

VENUS : NO FIELD
- SLOW ROTATION, BUT PROBABLY ENOUGH
- STAGNANT LID EFFECTS CORE COOLING

MARS : WEAK CRUSTAL FIELD, PROBABLY FROM PAST DIPOLE — LOST 3-4 Gyr AGO

EARTH : STRONG DIPOLE, ALIGNED W/ ROTATION

JUPITER : VERY STRONG DIPOLE, ~20° MISALIGNED

SATURN : VERY STRONG DIPOLE, NEARLY PERFECT ALIGNMENT

URANUS : 59° MISALIGNMENT, CENTER OF FIELD DISPLACED BY $\frac{1}{3}R_U$

NEPTUNE : 47° MISALIGNMENT, " " " " " $\frac{1}{2}R_N$

QUADRUPOLE AS STRONG AS DIPOLE, ALSO STRONG OCTUPOLE

DYNAMO THEORY

MAGNETIC REYNOLDS NUMBER, MUST EXCEED A CRITICAL VALUE

$$\underbrace{U}_{\text{VELOCITY}} \quad R_m = \frac{U\overset{\text{THICKNESS}}{D}}{\lambda} \qquad \lambda = \text{MAGNETIC DIFFUSIVITY} = \frac{1}{\mu_0\sigma}$$

ROTATING, CONVECTING, ELECTRICALLY CONDUCTING FLUID

OHMIC DECAY WOULD TURN OUR FIELD OFF IN ~20,000 YEARS

NEED THE FLUID TO KEEP MOVING → ROLE OF CONVECTION + ROTATION

IN EARTH, OUTER CORE CONVECTS, HAS BEEN, AND WILL BE, BECAUSE THE MANTLE A POOR CONDUCTOR + CONVECTOR, SO TRANSPORT IS SLOW

- MANY ENERGY SOURCES
  - CRYSTALLIZATION OF Fe (LATENT HEAT)
  - REJECTION OF LIGHT ELEMENTS FROM Fe (GPE)
  - RADIOACTIVE DECAY
  - CORE COMPRESSION (GPE)

EQUATION OF MOTION FOR AN INCOMPRESSIBLE, ROTATING, CONDUCTING, BUOYANT FLUID IS:

$$\underbrace{\underset{\text{DENSITY}}{\rho}\left(\frac{d\bar{u}}{dt}+(\underset{\text{VELOCITY}}{\bar{u}}\cdot\bar{\nabla})\bar{u}\right)}_{\text{LINEAR + NON-LINEAR INERTIAL FORCES}} + \underbrace{2\rho\overset{\text{ROTATION RATE}}{\Omega}\underset{\text{VERT. UNIT VECTOR}}{\bar{e}_z}\times\bar{u}}_{\text{CORIOLIS FORCE}} + \underbrace{\bar{\nabla}\overset{\text{NON-HSE PRESSURE}}{p}}_{\text{PRESSURE GRADIENT FORCE}} = \overbrace{\rho\underset{\text{KINEMATIC VISCOSITY}}{\nu}\bar{\nabla}^2\bar{u}}^{\text{VISCOUS FORCE}} + \underbrace{\rho\overset{\text{THERMAL EXPANSIVITY}}{\alpha}\underset{\text{GRAVITY}}{g}\overset{\text{TEMP}}{T}\underset{\text{RADIAL UNIT VECTOR}}{\bar{e}_r}}_{\text{BUOYANT FORCE}} + \overbrace{\underset{\text{CURRENT DENSITY}}{\bar{j}}\times\underset{\text{MAGN. FIELD}}{\bar{B}}}^{\text{LORENTZ FORCE}}$$

DYNAMO II

SI UNITS

FOR EARTH, $U \approx 10^{-4}$

$\Omega = 7\times10^{-5}$

$\rho \approx 10^{4}$

$B = 5\times10^{-3}$

$\ell \approx 3\times10^{5}$ (LENGTH SCALE FOR SPACIAL DERIVATIVES)

VISCOUS FORCE, INERTIAL FORCE $\ll$ LORENTZ FORCE, CORIOLIS FORCE — SIMILAR ORDER, LARGEST

MOST COMMON SCALING LAW

ELASSER # RULE:

RELATE $\bar{j}$ TO $\bar{u}$ VIA OHM'S LAW: $\bar{j} = \sigma(\cancel{\bar{E}} + \bar{u}\times\bar{B})$ ← IGNORE

$J \propto \sigma U B$

LORENTZ FORCE: $\underline{JB} \propto \sigma U B^2$

CORIOLIS FORCE: $2\rho\Omega U$

$$\text{RATIO} = \Lambda = \frac{\sigma B^2}{2\rho\Omega}$$ ELASSER # (U DROPS OUT)

$\Lambda \approx 1$ — SIMULATIONS BEAR THIS OUT, + SOLAR SYSTEM DYNAMOS APPEAR TO BE AT $\Lambda \approx 0.1-10$

IMPLIES $\boxed{B^2 \propto \frac{\rho\Omega}{\sigma}}$

DYNAMO III

A CERTAIN AMOUNT OF CONVECTIVE HEAT FLUX IS NEEDED TO MAINTAIN THE DYNAMO, SO IT IS A LITTLE ODD THAT IT DOESN'T COME INTO THE ELASSER SCALING RELATION. PROBABLY WORKS IF SOME MINIMUM q_c IS MET

1) POWER/UNIT VOLUME, $P = q_c / H_T \rightarrow$ T SCALE HEIGHT

2) OHMIC DISSIPATION: $D = \frac{(\bar{\nabla} \times \bar{B})^2}{\sigma \mu_0^2} \propto \frac{\lambda B^2}{2\mu_0 l_B^2}$ — LENGTH SCALE OF THE FIELD

f_{ohm} = FRACTION OF AVAILABLE EVENTUALLY LOST TO OHMIC DISSIPATION =

~~E_m = MAGN E DENSITY $\frac{B^2}{2\mu_0} \propto f_{ohm} \frac{l_B^2}{\lambda} \frac{q_c}{H_T}$~~

$$f_{ohm} = \frac{D}{P} = \frac{\frac{\lambda B^2}{2\mu_0 l_B^2}}{q_c / H_T} \qquad E_m = \frac{B^2}{2\mu_0}$$

$$\boxed{\frac{B^2}{2\mu_0} \propto f_{ohm} \left(\frac{l_B^2}{\lambda}\right) \frac{q_c}{H_T}}$$

OHMIC DISSIPATION TIME $\tau_{\lambda} = \frac{l_B^2}{\lambda} = \frac{E_m}{D}$

~~[illegible]~~

τ_λ RELATED TO CHARACTERISTIC VELOCITY & DISTANCE

$$\begin{tikzcd} \text{VELOCITY} \arrow[d] & \text{DISTANCE} \arrow[d] \\ U & R_c = L \end{tikzcd}$$

$$\frac{l_B^2}{\lambda} = \tau_\lambda = \frac{L}{U}$$

DYNAMO IV

USE MIXING LENGTH TO GET U

$$U \propto \left(\frac{l\, q_c}{\rho\, H_T} \right)^{1/3} \qquad H_T = \frac{c_p}{\alpha g}$$

NOW YIELDS:

$$\frac{B^2}{2\mu_0} \propto f_{ohm}\, \rho^{1/3} \left(\frac{q_c L}{H_T} \right)^{2/3}$$

LET $L = R_c$

$$\frac{1}{H_T} \propto \rho R_c$$

YIELDS : $\boxed{B^2 \propto \rho R_c^{4/3} q_c^{2/3}}$

CHRISTENSEN & AUBERT (2006)

INDEPENDENT OF Ω AND σ

CAN BE COMPARED TO 1) MODELS

2) REAL PLANETS

3) FULLY CONVECTIVE STARS