Moments of the collisionless Boltzmann equation...

"The Jeans Equations"

The First Moment

integrate the cbe over all velocities $\int dv_1 dv_2 dv_3$ at a point $x, t$:

define space density $v = \int f d^3v$

- Bulk flow or mean streaming motion $\vec{V}_i$

$$\vec{V}_i = \frac{\int f v_i d^3v}{\int f d^3v}$$

Integrate each term of the cbe to get:

and use the summation convention over $i$

$$\frac{d}{dt} \int f d^3v + \int v_i \frac{df}{dx_i} d^3v - \frac{d}{dx_i} \int \frac{df}{dv_i} d^3v = 0$$

$$\frac{2}{\tau} \int f d^3v + \frac{2}{\tau} \int v_i f d^3v - \frac{2}{\tau} \int df d^3v \int dv_1 dv_2 \int df + 2$$
\[ v_3 = +\infty \]

Note: \[ \int df = f(+\infty) - f(-\infty) = 0 - 0 = 0 \]

\[ v_3 = -\infty \]

because there are no stars moving infinitely fast.

The other 2 terms involving \( V_1 \) and \( V_2 \) go away for the same reason.

so using our definitions for the stellar density \( \nu \) and the bulk motion \( \vec{V} \), we get that

\[
\frac{\partial \nu}{\partial t} + \frac{\partial}{\partial x_i} (\nu \vec{V}_i) = 0
\]

or

\[
\frac{\partial \nu}{\partial t} + \text{div}_3 (\nu \vec{V}) = 0
\]

The first moment of the CBE is thus a scalar equation of continuity in real space.

it expresses the conservation of mass, or more precisely, the conservation of stars, since we are studying the point motion of test point particles in a fixed potential.

The Second Moments

multiply the CBE by \( \nu_i \) and integrate the equation over all velocities.

\[ \lim_{|\vec{v}| \to 0} f \nu_i = 0 \]
Define \[ \vec{V}_i \cdot \vec{V}_j = \frac{\int \vec{V}_i \vec{V}_j f d^3\vec{V}}{\int f d^3\vec{V}} \]

and the velocity dispersion around the mean streaming motion \( \vec{V} \):

\[ \sigma_{ij}^2 = (\vec{V}_i - \vec{V}_i)(\vec{V}_j - \vec{V}_j) = \vec{V}_i \cdot \vec{V}_j - \vec{V}_i \cdot \vec{V}_j \]

"Total Dispersion"

"Streaming motion"

Doing the integration, we get

\[ \frac{2}{\partial t} \int f \vec{V}_j d^3\vec{V} + \int \vec{V}_i \vec{V}_j \frac{\partial f}{\partial x_i} d^3\vec{V} - \frac{\partial \phi}{\partial x_i} \int \vec{V}_j \frac{\partial f}{\partial \vec{V}_i} d^3\vec{V} = 0 \]

We can kill the last term by using some trickery involving the divergence theorem and the vector analog of integration by parts.

\[ \int g \vec{V} \cdot \vec{F} d^3\vec{x} = \int g \vec{F} \cdot d^2\vec{s} - \int \left( \vec{F} \cdot \vec{n} \right) g d3\vec{x} \]

(see homework)

The last term on the right hand side turns into

\[ \oint \vec{n} \frac{\partial \vec{F}}{\partial \vec{V}_i} \cdot d\vec{s} = -\int \delta_{ij} f d^3\vec{V} = -\delta_{ij} \]
so, using the definitions for \( \vec{V}_j \) and \( \vec{V}_i \cdot \vec{V}_j \), eqn (*) becomes

\[
\frac{\partial}{\partial t} (\nu \vec{V}_j) + \frac{\partial}{\partial x_i} (\nu \vec{V}_i \cdot \vec{V}_j) + \nu \frac{\partial \Phi}{\partial x_j} = 0
\]

This equation can be put into a remarkable form. Subtract \( \vec{V}_j \cdot (\text{Equation of continuity}) \), i.e.

\[
\vec{V}_j \cdot \left( \frac{\partial \nu}{\partial t} + \frac{\partial (\nu \vec{V}_i)}{\partial x_i} \right)
\]

so we're just subtracting zero

\[
= 0
\]

\[
\nu \frac{\partial \vec{V}_j}{\partial t} - \vec{V}_j \cdot \frac{\partial (\nu \vec{V}_i)}{\partial x_i} + \frac{\partial (\nu \vec{V}_i \cdot \vec{V}_j)}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j}
\]

using \( \sigma_{ij} \equiv \vec{V}_i \cdot \vec{V}_j - \vec{V}_i \cdot \vec{V}_j \) in

get:

\[
\left\{ \begin{align*}
\nu \frac{\partial \vec{V}_j}{\partial t} + \nu \vec{V}_i \cdot \frac{\partial \vec{V}_j}{\partial x_i} &= -\nu \frac{\partial \Phi}{\partial x_j} - \frac{\partial (\nu \sigma_{ij}^2)}{\partial x_i}.
\end{align*} \right.
\]

The second moment of the CBE.
What does this equation mean?

\[ \begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial x_i} &= -\frac{\partial \Phi}{\partial x_i} - \frac{\rho}{\partial x_j} \frac{\partial^2 \sigma_{ij}}{\partial x_i} \\
\end{align*} \]

It looks exactly the same as Euler's Equation for the velocity of a fluid.

\[ \begin{align*}
\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\frac{1}{\rho} \nabla p - \nabla \Phi \\
\end{align*} \]

TOTAL change in velocity of a fluid element as it moves along.

\[ \Delta \mathbf{v} \] change in velocity at a particular point (fixed) over a tiny interval of time.

Imagine that there is no pressure.

\[ \frac{\Delta \mathbf{v}}{\Delta t} = -\frac{1}{\rho} \nabla p - \nabla \Phi \]

\[ \frac{\Delta \mathbf{v}}{\Delta t} = -\nabla \Phi \]

So the second moment of the CBE is a nasty way of saying \( F = ma \).
The velocity ellipsoid $\sigma_{ij}^2 = \frac{1}{V} (V_i - \bar{V_i})^2$ is symmetric. Hence it can be diagonalized.

That is at any given point, we can choose the same principle axes $\sigma_1$, $\sigma_2$, $\sigma_3$ so that $\sigma_{ij} = \delta_{ij}$.

No sum can.

The pressure provided by the velocity dispersion is anisotropic.

Spilling coherent motion into velocity dispersion leads to a decrease associated with streaming motion.

The stress tensor $\sigma_{ij}$ plays the role of gravity acceleration.

$\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_j \frac{\partial \bar{v}_i}{\partial x_j} = -\frac{\partial}{\partial x_i} \left( \frac{\partial \bar{v}_j}{\partial x_j} \right)$
This gaussian distribution function has the same stress tensor $\sigma_{ij}$.

Such a gaussian form is not required by the CBE, but in practice is found in many stellar systems because Gaussian distribution functions maximize entropy and so are a natural equilibrium state for systems in which any processes can redistribute energy and momentum.

The value and limitations of the Jean's Equations

- can relate observables like $\nabla$, $\nu$, and $\sigma_{ij}$ to the gravitational potential $(\frac{\partial \Phi}{\partial x_i})$. "Weighs galaxies."

But:

The Jeans equations describe a massless "tracer population" in an external potential. Need to add Poisson's equation to get $\Phi$ from $\rho$

$$\nabla^2 \Phi = 4\pi G \rho$$

no feedback for self consistency between the potential and the density that creates it.

- No equation of state to relate $\sigma$ to $\nu$, as ideal gas equation would relate $T, p$ to $\sigma, p$ for ideal gas. So you have to assume $\sigma_{ij}$, that is $f(\nu)$. Every different assumption leads to a different solution.

→ solutions to Jean's equations depend on $f(\nu)$ and are thus non-unique.
Higher moments can be taken \( \Rightarrow \) BBGKY hierarchy. However, each moment introduces a higher-order tensor (like \( T_{0} = (v_i - \bar{v}_i)(v_j - \bar{v}_j)(v_k - \bar{v}_k) \)) for which you need to make assumptions. The equations never close. To close, you have to assume some form for the \( N \)-th velocity tensor. That is, you have to assume some form for \( F(\bar{v}) \).

We've seen that the 2nd moment of the CBE is equivalent to Euler's momentum equation for a fluid. This is conventionally derived assuming a collisional system with pressure. Yet the equations are identical save the anisotropic "pressure", \( v_{ij} \). How can the equations for collisional and collisionless systems be so similar when the microphysics is so different?
Understanding the Jean's Equations

\[ \frac{\partial \vec{v}_x}{\partial t} = -\vec{v}_x \frac{\partial \vec{v}_x}{\partial x} - \vec{v}_y \frac{\partial \vec{v}_x}{\partial y} - \vec{v}_3 \frac{\partial \vec{v}_x}{\partial z} - \frac{\partial \phi}{\partial x} - \frac{1}{\nu} \frac{\partial}{\partial x} \nu \sigma_{xx}^2 - \frac{1}{\nu} \frac{\partial}{\partial y} \nu \sigma_{yx}^2 - \frac{1}{\nu} \frac{\partial}{\partial z} \nu \sigma_{zx}^2 \]

Bulk velocity being blown in from 3 orthogonal directions.

now:

\[ \sigma_{xx}^2 = \vec{v}_x \vec{v}_x - \vec{v}_x \vec{v}_x = \frac{1}{V} \int \vec{v}_x \vec{v}_x f d^3\vec{v} - \frac{1}{V} \int f v_x d^3\vec{v} \]

\[ \sigma_{yx}^2 = \vec{v}_y \vec{v}_x - \vec{v}_y \vec{v}_x = \frac{1}{V} \int \vec{v}_y \vec{v}_x f d^3\vec{v} - \frac{1}{V} \int f v_y d^3\vec{v} \]

\[ \sigma_{zx}^2 = \vec{v}_3 \vec{v}_x - \vec{v}_3 \vec{v}_x = \frac{1}{V} \int \vec{v}_3 \vec{v}_x f d^3\vec{v} - \frac{1}{V} \int f v_3 d^3\vec{v} \]

so the equation in its fully expanded form is:

\[ \frac{\partial \vec{v}_x}{\partial t} = -\vec{v}_x \frac{\partial \vec{v}_x}{\partial x} - \vec{v}_y \frac{\partial \vec{v}_x}{\partial y} - \vec{v}_3 \frac{\partial \vec{v}_x}{\partial z} - \frac{\partial \phi}{\partial x} - \frac{1}{\nu} \frac{\partial}{\partial x} \nu \sigma_{xx}^2 - \frac{1}{\nu} \frac{\partial}{\partial y} \nu \sigma_{yx}^2 - \frac{1}{\nu} \frac{\partial}{\partial z} \nu \sigma_{zx}^2 \]

\[ -\frac{1}{\nu} \frac{\partial}{\partial y} \left[ \int \vec{v}_y \vec{v}_x f d^3\vec{v} \right] - \frac{1}{V} \left[ \int f v_y d^3\vec{v} \right]^2 \]

\[ -\frac{1}{\nu} \frac{\partial}{\partial z} \left[ \int \vec{v}_3 \vec{v}_x f d^3\vec{v} \right] - \frac{1}{V} \left[ \int f v_3 d^3\vec{v} \right]^2 \]
Consider the following chunk of phase space: $x, y, v_x, v_y$ (a 4-Dimensional plot!)

Principal Axes of the velocity ellipsoid

$\sigma_{ij}$ Tensor is diagonalized when these are the $x$ axes

$\Delta v_x = 1, \Delta v_y = 1$

$\Delta v_x = \Delta t \cdot \left[ -v_x \cdot \frac{\Delta v_x}{\Delta x} - v_y \cdot \frac{\Delta v_y}{\Delta y} - \frac{\Delta \phi}{\Delta x} - \frac{1}{v} \frac{\Delta (v \sigma_{xx})}{\Delta x} - \frac{1}{v} \frac{\Delta (v \sigma_{xy})}{\Delta y} \right]$
Diagonalizing the dispersion tensor:

\[
\begin{bmatrix}
\sigma_{xx} & \sigma_{xy} \\
\sigma_{xy} & \sigma_{yy}
\end{bmatrix}
\begin{bmatrix}
V_{1x} \\
V_{1y}
\end{bmatrix} = \lambda
\begin{bmatrix}
V_{1x} \\
V_{1y}
\end{bmatrix}
\]

Eigenvalue

\[
\det \begin{bmatrix}
\sigma_{xx} - \lambda & \sigma_{xy} \\
\sigma_{xy} & \sigma_{yy} - \lambda
\end{bmatrix} = 0
\]

\[
\lambda = \sigma_{xx} + \sigma_{yy} \pm \sqrt{(\sigma_{xx} + \sigma_{yy})^2 - 4(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2)}
\]

\[
\frac{1}{2}
\]

\[
\lambda_+ = \sigma_{xx}'
\]
\[
\lambda_- = \sigma_{yy}'
\]

These are the lengths of the axes of velocity ellipsoid.

The eigenvectors are the columns of the rotation matrix corresponding to the angle which causes \( \sigma_{xy} \to 0 \).