Lecture 7:
Wavefront Sensing

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Outline of lecture

• General discussion: Types of wavefront sensors

• Three types in more detail:
  - Shack-Hartmann wavefront sensors
  - Curvature sensing
  - Pyramid sensing
At longer wavelengths, one can measure phase directly

- FM radios, radar, radio interferometers like the VLA, ALMA
- All work on a narrow-band signal that gets mixed with a very precise “intermediate frequency” from a local oscillator
- Very hard to do this at visible and near-infrared wavelengths
  - Could use a laser as the intermediate frequency, but would need tiny bandwidth of visible or IR light

Thanks to Laird Close’s lectures for making this point
At visible and near-IR wavelengths, measure phase via intensity variations

- Difference between various wavefront sensor schemes is the way in which phase differences are turned into intensity differences

- General box diagram:

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Guide star --> Telescope --> Optics --> Detector of Intensity --> Reconstructor
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Wavefront sensor

Transforms aberrations into intensity variations

Computer
How to use intensity to measure phase?

- Irradiance transport equation: $A$ is complex field amplitude, $z$ is propagation direction. (Teague, 1982, JOSA 72, 1199)

\[
A(x, y, z) = \left[ I(x, y, z) \right]^{1/2} \exp[ik\phi(x, y, z)]
\]

- Follow $I(x, y, z)$ as it propagates along the $z$ axis (paraxial ray approximation: small angle w.r.t. $z$)

\[
\frac{\partial I}{\partial z} = -\nabla I \cdot \nabla \phi - I \nabla^2 \phi
\]

Wavefront curvature: Curvature Sensors

Wavefront tilt: Hartmann sensors
Types of wavefront sensors

• **“Direct” in pupil plane:** split pupil up into subapertures in some way, then use intensity in each subaperture to deduce phase of wavefront. Sub-categories:
  - Slope sensing: Shack-Hartmann, lateral shear interferometer, pyramid sensing
  - Curvature sensing

• **“Indirect” in focal plane:** wavefront properties are deduced from whole-aperture intensity measurements made at or near the focal plane. Iterative methods - calculations take longer to do.
  - Image sharpening, multi-dither
  - Phase diversity, phase retrieval, Gerchberg-Saxton (these are used, for example, in JWST)
How to reconstruct wavefront from measurements of local “tilt”

Figure 7  (a) Local tilt as a function of sampling location in pupil; (b) reconstructed wavefront estimate.
Shack-Hartmann wavefront sensor concept - measure subaperture tilts

Credit: A. Tokovinin
Example: Shack-Hartmann Wavefront Signals

Credit: Cyril Cavadore
**Displacement of centroids**

- **Definition of centroid**
  
  \[
  \bar{x} \equiv \frac{\iint I(x,y) \, x \, dx \, dy}{\iint I(x,y) \, dx \, dy}
  \]
  
  \[
  \bar{y} \equiv \frac{\iint I(x,y) \, y \, dx \, dy}{\iint I(x,y) \, dx \, dy}
  \]

- **Centroid is intensity weighted**

  Each arrow represents an offset proportional to its length

Credit: Cyril Cavador
Notional Shack-Hartmann Sensor spots

Credit: Boston Micromachines
Reminder of some optics definitions: focal length and magnification

- Focal length $f$ of a lens or mirror
  
  - Magnification $M = \frac{y'}{y} = -\frac{s'}{s}$
Displacement of Hartmann Spots

\[ m = \frac{f_{tel}}{f_{coll}} \]

\[ mf_{l} \nabla_{\perp} \phi(x, y) \]
Quantitative description of Shack-Hartmann operation

- Relation between displacement of Hartmann spots and slope of wavefront:

\[
\Delta \vec{x} \propto \nabla_{\perp} \phi(x, y)
\]

\[
k \Delta \vec{x} = M f \nabla_{\perp} \phi(x, y)
\]

where \( k = \frac{2\pi}{\lambda} \), \( \Delta x \) is the lateral displacement of a subaperture image, \( M \) is the (de)magnification of the system, \( f \) is the focal length of the lenslets in front of the Shack-Hartmann sensor.
Example: Keck adaptive optics system

- Telescope diameter $D = 10$ m, $M = 2800 \Rightarrow$ size of whole lenslet array
  $= 10/2800$ m $= 3.57 \times 10^{-3}$ m $= 3.57$ mm

- Lenslet array is approx. 18 x 18 lenslets $\Rightarrow$ each lenslet is $\sim 200$ microns in diameter

✓ Sanity check: size of subaperture on telescope mirror $= \text{lenslet diameter} \times \text{magnification} = 200 \text{ microns} \times 2800 = 56$ cm $\sim r_0$ for wavelength $\lambda$ between 1 and 2 microns
Keck AO example, continued

• Now look at scale of pixels on CCD detector:
  - Lenslet array size (200 microns) is larger than size of the CCD detector, so must put a focal reducer lens between the lenslets and the CCD: scale factor 3.15

• Each subaperture is then mapped to a size of 200 microns ÷ 3.15 = 63 microns on the CCD detector

• Choose to make this correspond to 3 CCD pixels (two to measure spot position, one for “guard pixel” to keep light from spilling over between adjacent subapertures)
  - So each pixel is 63/3 = 21 microns across.

• Now calculate angular displacement corresponding to one pixel, using

\[ k \Delta \hat{x} = M f \nabla_{\perp} \phi(x, y) \]
Keck AO example, concluded

- Angle corresponding to one pixel = $\Delta z / \Delta x$ where the phase difference $\Delta \varphi = k \Delta z$.
- $\Delta z / \Delta x = (\text{pixel size} \times 3.15) \div (2800 \times 200 \times 10)$
- Pixel size is 21 microns.
- $\Delta z / \Delta x = (21 \times 3.15) \div (2800 \times 2000) = 11.8$ microradians
- **Now use factoid: 1 arc sec = 4.8 microradians**
- $\Delta z / \Delta x = 2.4$ arc seconds.
- So when a subaperture has 2.4 arc seconds of slope across it, the corresponding spot on the CCD moves sideways by 1 pixel.
How to measure distance a spot has moved on CCD? “Quad cell formula”

\[ \delta_x \equiv \frac{b}{2} \left[ \frac{(I_2 + I_1) - (I_3 + I_4)}{(I_1 + I_2 + I_3 + I_4)} \right] \]

\[ \delta_y \equiv \frac{b}{2} \left[ \frac{(I_3 + I_2) - (I_4 + I_1)}{(I_1 + I_2 + I_3 + I_4)} \right] \]
Disadvantage: “gain” depends on spot size $b$ which can vary during the night.

\[
\delta_{x,y} = \frac{b \text{ (difference of } I\text{'s})}{2 \text{ (sum of } I\text{'s})}
\]
Question

- What might happen if the displacement of the spot is > radius of spot? Why?
Signal becomes nonlinear and saturates for large angular deviations

“Rollover” corresponds to spot being entirely outside of 2 quadrants
Measurement error from Shack-Hartmann sensing

- Measurement error depends on size of spot as seen in a subaperture, $\theta_b$, wavelength $\lambda$, subaperture size $d$, and signal-to-noise ratio $SNR$:

$$
\sigma_{S-H} = \frac{\pi^2}{2\sqrt{2}} \frac{1}{SNR} \left[ \left( \frac{3d}{2r_0} \right)^2 + \left( \frac{\theta_b d}{\lambda} \right)^2 \right]^{1/2} \text{ rad for } r_0 \leq d
$$

$$
\sigma_{S-H} \approx \frac{6.3}{SNR} \text{ rad of phase for } r_0 = d \text{ and } \theta_b = \frac{\lambda}{d}
$$

(Hardy equation 5.16)
• If we want the wavefront error to be $< \lambda/20$, we need

$$\Delta z \equiv \frac{\sigma}{k} < \frac{\lambda}{20} \quad \text{or} \quad \sigma \equiv \frac{6.3}{SNR} < \frac{2\pi}{20} \quad \text{so that} \quad SNR > 20$$
General expression for signal to noise ratio of a pixelated detector

- \( S \) = flux of detected photoelectrons / subap
- \( n_{pix} \) = number of detector pixels per subaperture
- \( R \) = read noise in electrons per pixel per pixel

The signal to noise ratio in a subaperture for fast CCD cameras is dominated by read noise, and

\[
SNR \approx \frac{St_{int}}{(n_{pix}R^2/t_{int})^{1/2}} = \frac{S\sqrt{t_{int}}}{\sqrt{n_{pix}R}}
\]

See McLean, “Electronic Imaging in Astronomy”, Wiley

We will discuss SNR in much more detail in a later lecture.
Trade-off between dynamic range and sensitivity of Shack-Hartmann WFS

- If spot is diffraction limited in a subaperture $d$, linear range of quad cell (2x2 pixels) is limited to $\pm \frac{\lambda_{\text{ref}}}{2d}$.

- Can increase dynamic range by enlarging the spot (e.g. by defocusing it).

- But uncertainty in calculating centroid $\propto \text{width} \times \sqrt{N_{\text{ph}}}$ so centroid calculation will be less accurate.

- Alternative: use more than 2x2 pixels per subaperture. Decreases SNR if read noise per pixel is large (spreading given amount of light over more pixels, hence more read noise).
Correlating Shack-Hartmann wavefront sensor uses images in each subaperture

- Solar adaptive optics: Rimmele and Marino
  http://solarphysics.livingreviews.org/Articles/lrsp-2011-2/

- Cross-correlation is used to track low contrast granulation

- Left: Subaperture images, Right: cross-correlation functions
Curvature wavefront sensing


\[
\frac{I_+ - I_-}{I_+ + I_-} \propto \nabla^2 \phi - \frac{\partial \phi}{\partial \vec{r}} \delta_R
\]

Laplacian (curvature)

More intense

Less intense

Normal derivative at boundary
Wavefront sensor lenslet shapes are different for edge, middle of pupil

- Example: This is what wavefront tilt (which produces image motion) looks like on a curvature wavefront sensor
  - Constant $I$ on inside
  - Excess $I$ on right edge
  - Deficit on left edge
Simulation of curvature sensor response

Wavefront: pure tilt  
Curvature sensor signal

$Z_{1,-1}$  
Difference Image

Credit: G. Chanan
Curvature sensor signal for astigmatism

$Z_{2,-2}$

Difference Image

Credit: G. Chanan
Third order spherical aberration

\[ Z_{4,0} \]

Difference Image

Credit: G. Chanan
Practical implementation of curvature sensing

- Use oscillating membrane mirror (2 kHz!) to vibrate rapidly between $I_+$ and $I_-$ extrafocal positions

- Measure intensity in each subaperture with an “avalanche photodiode” (only need one per subaperture!)
  - Detects individual photons, no read noise, QE ~ 60%
  - Can read out very fast with no noise penalty
Measurement error from curvature sensing

- Error of a single set of measurements is determined by photon statistics, since detector has NO read noise!

\[ \sigma_{cs}^2 = \pi^2 \frac{1}{N_{ph}} \left( \frac{\theta_b d}{\lambda} \right)^2 \]

where \( d \) = subaperture diameter and \( N_{ph} \) is no. of photoelectrons per subaperture per sample period

- Error propagation when the wavefront is reconstructed numerically using a computer scales poorly with no. of subapertures \( N \):

\( (\text{Error})_{\text{curvature}} \propto N \), whereas \( (\text{Error})_{\text{Shack-Hartmann}} \propto \log N \)
Question

• Think of as many pros and cons as you can for
  - Shack-Hartmann sensing
  - Curvature sensing
Advantages and disadvantages of curvature sensing

• Advantages:
  - Lower noise ⇒ can use fainter guide stars than S-H
  - Fast readout ⇒ can run AO system faster
  - Can adjust amplitude of membrane mirror excursion as “seeing” conditions change. Affects sensitivity.
  - Well matched to bimorph deformable mirror (both solve Laplace’s equation), so less computation.
  - Curvature systems appear to be less expensive.

• Disadvantages:
  - Avalanche photodiodes can fail if too much light falls on them. They are bulky and expensive.
  - Hard to use a large number of avalanche photodiodes.
  - BUT - recently available in arrays
Review of Shack-Hartmann geometry

- Pupil plane
- Image plane
Pyramid sensing

- From Andrei Tokovinin’s tutorial
Pyramid for the William Herschel Telescope’s AO system
Schematic of pyramid sensor

Credit: Iuliia Shatokhina et al.
Figure 3- 4: Organization of SH wavefront data (left) versus pyramid wavefront data (right). The circle indicates the beam footprint on the WFS. The heavily-weighted squares on the left indicate the various subapertures (8x8 grid of subapertures). Each subaperture has 4 pixels (a quad cell). In a pyramid wavefront sensing scheme, each pixel represents a subaperture; the 4 images of the pupil correspond to the quadrants of the quad cell.
Here’s what a pyramid-sensor measures looks like

- Courtesy of Jess Johnson
**Potential advantages of pyramid wavefront sensors**

- Wavefront measurement error can be much lower
  - Shack-Hartmann: size of spot limited to $\lambda / d$, where $d$ is size of a sub-aperture and usually $d \sim r_0$  
  - Pyramid: size of spot can be as small as $\lambda / D$, where $D$ is size of whole telescope. So spot can be $D/r_0 = 20 - 100$ times smaller than for Shack-Hartmann
- Measurement error (e.g. centroiding) is proportional to spot size/SNR. Smaller spot = lower error.

- Avoids bad effects of charge diffusion in CCD detectors
  - Fuzzes out edges of pixels. Pyramid doesn’t mind as much as S-H.
Potential pyramid sensor advantages, continued

- Linear response over a larger dynamic range
- Naturally filters out high spatial frequency information that you can’t correct anyway
Summary of main points

• Wavefront sensors in common use for astronomy measure intensity variations, deduce phase. Complementary.
  - Shack-Hartmann
  - Curvature sensors

• Curvature systems: cheaper, fewer degrees of freedom, scale more poorly to high no. of degrees of freedom, but can use fainter guide stars

• Shack-Hartmann systems excel at very large no. of degrees of freedom

• New kid on the block: pyramid sensors
  - Very successful for fainter natural guide stars