Error Budgets, and Introduction to Class Projects

Lecture 7, ASTR 289

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What is “residual wavefront error”? 

Telescope

Very distorted wavefront

AO System

Less distorted wavefront (but still not perfect)

Science Instrument
**Units of wavefront error**

- Electromagnetic wave propagation

  \[ E = E_0 \exp(i\phi) = E_0 \exp(i(kx - \omega t)) = E_0 \exp i\left(\frac{2\pi nx}{\lambda} - \omega t\right) \]

- Change in phase due to variation in index of refraction \( n \)

- Can express as:
  - Phase \( \Phi \sim k\Delta x = k_0 n \Delta x \) (units: radians)
  - Optical path difference \( \Phi/k = \Delta x \) (units: length)
    - Frequently microns or nanometers
  - Waves: \( \Delta x / \lambda \) (units: dimensionless)
How to calculate residual wavefront error

- Optical path difference = $\Delta z$ where $k \Delta z$ is the phase change due to turbulence
- Phase variance is $\sigma^2 = \langle (k \Delta z)^2 \rangle$
- If several independent effects cause changes in the phase,

$$
\sigma_{tot}^2 = k^2 \left\langle (\Delta z_1 + \Delta z_2 + \Delta z_3 + \ldots)^2 \right\rangle \\
= k^2 \left\langle (\Delta z_1)^2 + (\Delta z_2)^2 + (\Delta z_3)^2 + \ldots \right\rangle
$$

- Sum up the contributions from individual physical effects independently
An error budget can describe wavefront phase or optical path difference

\[ \sigma_{tot}^2 = k^2 \left\langle \left( \Delta z_1 + \Delta z_2 + \Delta z_3 + \ldots \right)^2 \right\rangle \]

\[ = k^2 \left\langle \left( \Delta z_1 \right)^2 + \left( \Delta z_2 \right)^2 + \left( \Delta z_3 \right)^2 + \ldots \right\rangle \]

- Be careful of units (Hardy and I will both use a variety of units in error budgets):
  - For tip-tilt residual errors: variance of tilt angle \( \langle \alpha^2 \rangle \)
  - Optical path difference \( n\Delta z \) in meters: \( \text{OPD}_m \)
  - Optical path difference \( n\Delta z \) in waves: \( \text{OPD}_\lambda = n\Delta z / \lambda \)
  - Optical path difference in radians of phase:
    \( \approx \phi = 2\pi \text{OPD}_\lambda = (2\pi/\lambda) \ \text{OPD}_m = k \ \text{OPD}_m \)
Question

If the total wavefront error is

$$\sigma_{\text{tot}}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + ...$$

- List as many physical effects as you can think of that might contribute to the overall wavefront error $\sigma_{\text{tot}}^2$
Elements of an adaptive optics system
Elements of an adaptive optics system

Deformable mirror fitting error

Temporal delay, noise propagation

Light From Telescope

Distorted Wavefront

Adaptive Mirror

Control System

Not shown: tip-tilt error, anisoplanatism error

Non-common path errors

Corrected Wavefront

Beam splitter

Wavefront Sensor

High-resolution Camera

Measurement error
Figure 2.32 Main sources of wavefront error in adaptive optics.
What is an “error budget”? 

1. The allocation of statistical variations and/or error rates to individual components of a system, in order to satisfy the full system's end-to-end performance specifications.

2. The term “error budget” is a bit misleading: it doesn’t mean “error” as in “mistake” - it means performance uncertainties, and/or the imperfect, “real life” performance of each component in the system.

3. In a new project: Start with “top down” performance requirements from the science that will be done. Allocate “errors” to each component to satisfy overall requirements. As design proceeds, replace “allocations” with real performance of each part of system. Iterate.
Wavefront errors due to time lags, $\tau_0$

- Wavefront phase variance due to $\tau_0$
  
  If an AO system corrects turbulence “perfectly” but with a phase lag characterized by a time $\tau$, then

  \[ \sigma^2_\tau = 28.4 \left( \frac{\tau}{\tau_0} \right)^{5/3} \]  

  Hardy Eqn 9.57

- The factor of 28.4 out front is a significant penalty: have to run AO system a lot faster than $\tau = \tau_0$

- For $\sigma^2_\tau < 1$, $\tau < 0.13 \tau_0$

- In addition, closed-loop bandwidth is usually $\sim 10x$ sampling frequency ⇒ have to run even faster
Wavefront variance due to isoplanatic angle $\theta_0$

- If an AO system corrects turbulence “perfectly” but uses a guide star at an angle $\theta$ away from the science target,

$$\sigma_{angle}^2 = \left( \frac{\theta}{\theta_0} \right)^{5/3}$$

Hardy Eqn 3.104

- Typical values of $\theta_0$ are a few arc sec at $\lambda = 0.5 \, \mu m$, 15-20 arc sec at $\lambda = 2 \, \mu m$
**Deformable mirror fitting error**

- Accuracy with which a deformable mirror with subaperture diameter $d$ can remove wavefront aberrations.

- With a finite number of actuators, you can’t do a perfect fit to an arbitrary wavefront.

\[
\sigma_{\text{Fitting}}^2 = \mu \left( \frac{d}{r_0} \right)^{5/3}
\]

- Constant $\mu$ depends on specific design of deformable mirror.

- For segmented mirror that corrects tip, tilt, and piston (3 degrees of freedom per segment) $\mu = 0.14$.

- For deformable mirror with continuous face-sheet, $\mu = 0.28$. 
Error budget concept (sum of $\sigma^2$ 's)

\[ \sigma_{tot}^2 = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \ldots \text{ radians}^2 \]

- There’s not much to be gained by making any particular term much smaller than all the others: try to roughly equalize all the terms

- Individual terms we know so far:
  - Anisoplanatism \[ \sigma_{angle}^2 = \left(\frac{\theta}{\theta_0}\right)^{5/3} \]
  - Temporal error \[ \sigma_\tau^2 = 28.4 \left(\frac{\tau}{\tau_0}\right)^{5/3} \]
  - Fitting error \[ \sigma_{Fitting}^2 = \mu \left(\frac{d}{r_0}\right)^{5/3} \]
We will discuss other wavefront error terms in coming lectures

- **Measurement error**
  - Wavefront sensor doesn’t make perfect measurements
  - Finite signal to noise ratio, optical limitations, ...

- **Non-common-path errors**
  - Calibration of different optical paths between science instrument and wavefront sensor isn’t perfect

- **Calibration errors**
  - What deformable mirror shape would correspond to a perfectly flat wavefront?

- **Tip-Tilt errors**
  - Need to rephrase these in terms of wavefront error rather than angle of arrival variance
Error budget so far

$\sigma_{tot}^2 = \sigma_{fitting}^2 + \sigma_{anisopl}^2 + \sigma_{temporal}^2 + \sigma_{meas}^2 + \sigma_{calib}^2 + \sigma_{tip-tilt}^2 + \ldots.$

- $\checkmark$
- $\checkmark$
- $\checkmark$

Still need to work these out

Try to “balance” error terms: if one is big, no point struggling to make the others tiny
Keck AO error budget example (not current)

<table>
<thead>
<tr>
<th>Error Term (nm)</th>
<th>Predicted</th>
<th>Measured</th>
</tr>
</thead>
<tbody>
<tr>
<td>DM: Atmospheric fitting error</td>
<td>110</td>
<td>139</td>
</tr>
<tr>
<td>DM: Telescope fitting error</td>
<td>66</td>
<td>60</td>
</tr>
<tr>
<td>Calibration (non-common path)</td>
<td>114</td>
<td>113</td>
</tr>
<tr>
<td>Finite Bandwidth (high order)</td>
<td>115</td>
<td>103</td>
</tr>
<tr>
<td>WFS measurement error*</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>TT bandwidth</td>
<td>91</td>
<td>75</td>
</tr>
<tr>
<td>TT measurement</td>
<td>5</td>
<td>9</td>
</tr>
<tr>
<td>Miscellaneous</td>
<td>106</td>
<td>120</td>
</tr>
<tr>
<td>Total wavefront error</td>
<td>249</td>
<td>258</td>
</tr>
<tr>
<td>K-band Strehl</td>
<td>0.60</td>
<td>0.58</td>
</tr>
</tbody>
</table>

* Very bright star

Assumptions:
- Natural guide star is very bright (no measurement error)
- 10 degree zenith angle
- Wavefront sensor bandwidth: 670 Hz

Note that uncorrectable errors in telescope itself are significant
We want to relate phase variance $<\sigma^2>$ to the “Strehl ratio”

- Two definitions of Strehl ratio (equivalent):

1. Ratio of the maximum intensity of a point spread function to what the maximum would be without any aberrations:

$$S \equiv \left( \frac{I_{\text{max}}}{I_{\text{max\_no\_aberrations}}} \right)$$

2. The “normalized volume” under the optical transfer function of an aberrated optical system

$$S \equiv \frac{\int \text{OTF}_{\text{aberrated}}(f_x, f_y) \, df_x \, df_y}{\int \text{OTF}_{\text{un\_aberrated}}(f_x, f_y) \, df_x \, df_y}$$

where $\text{OTF}(f_x, f_y) = \text{Fourier Transform}(\text{PSF})$
Relation between phase variance and Strehl ratio

- “Maréchal Approximation”

\[ \text{Strehl} \approx \exp\left(-\sigma_{\phi}^2\right) \]

where \( \sigma_{\phi}^2 \) is the total wavefront variance

- Valid when Strehl > 10% or so
- Under-estimates the Strehl for low-Strehl situations (larger values of \( \sigma_{\phi}^2 \))
High Strehl $\Rightarrow$ PSF with higher peak intensity and narrower “core”

- **High Strehl**
- **Medium Strehl**
- **Low Strehl**
Summary of topics discussed today

• Wavefront errors due to:
  - Timescale of turbulence
  - Isoplanatic angle
  - Deformable mirror fitting error
  - Other effects

• Concept of an “error budget”

• Goal: to calculate $<\sigma_\phi^2>$ and thus the Strehl ratio

\[ \text{Strehl} \equiv \exp\left(-\sigma_\phi^2\right) \]