Control of Adaptive Optics

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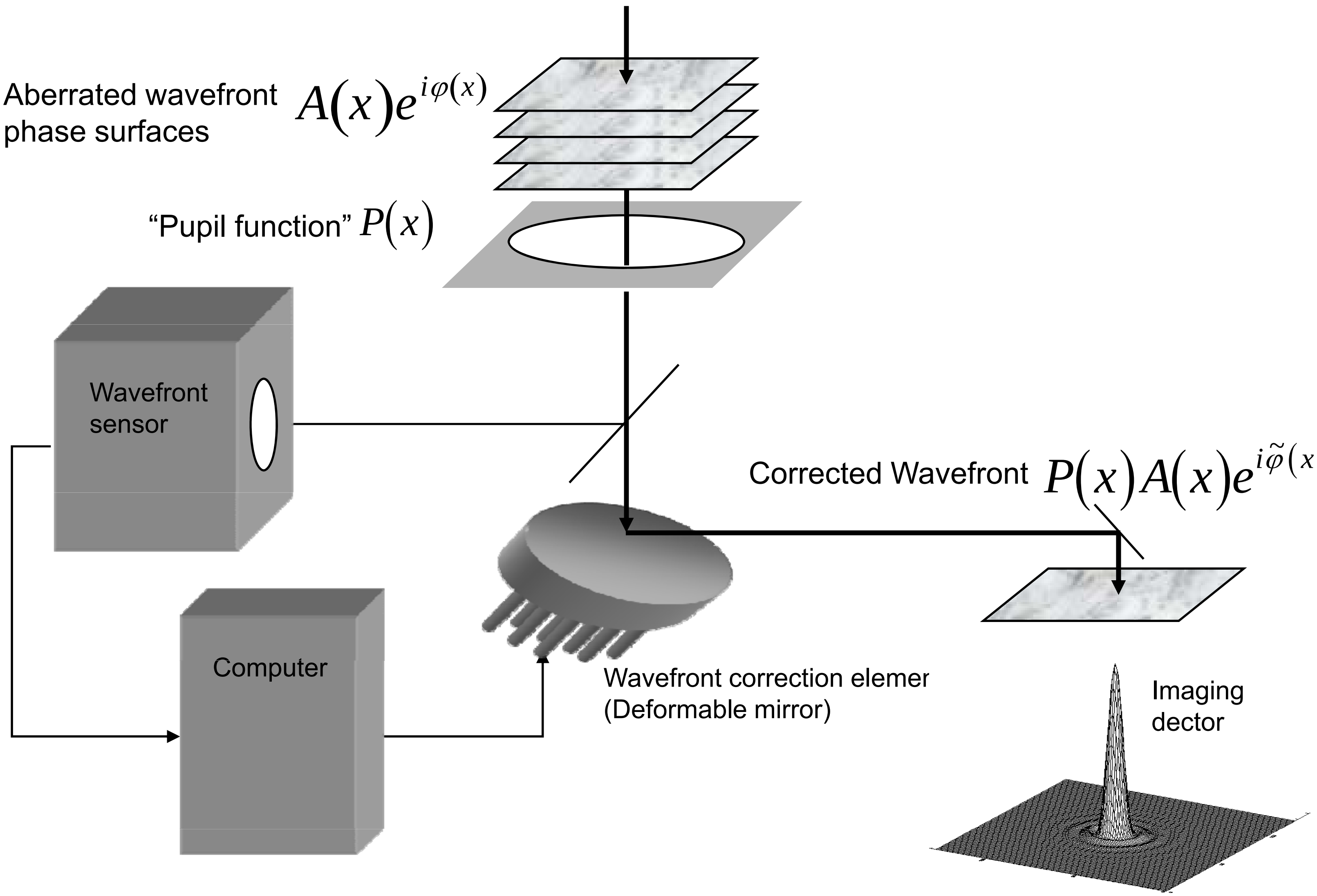
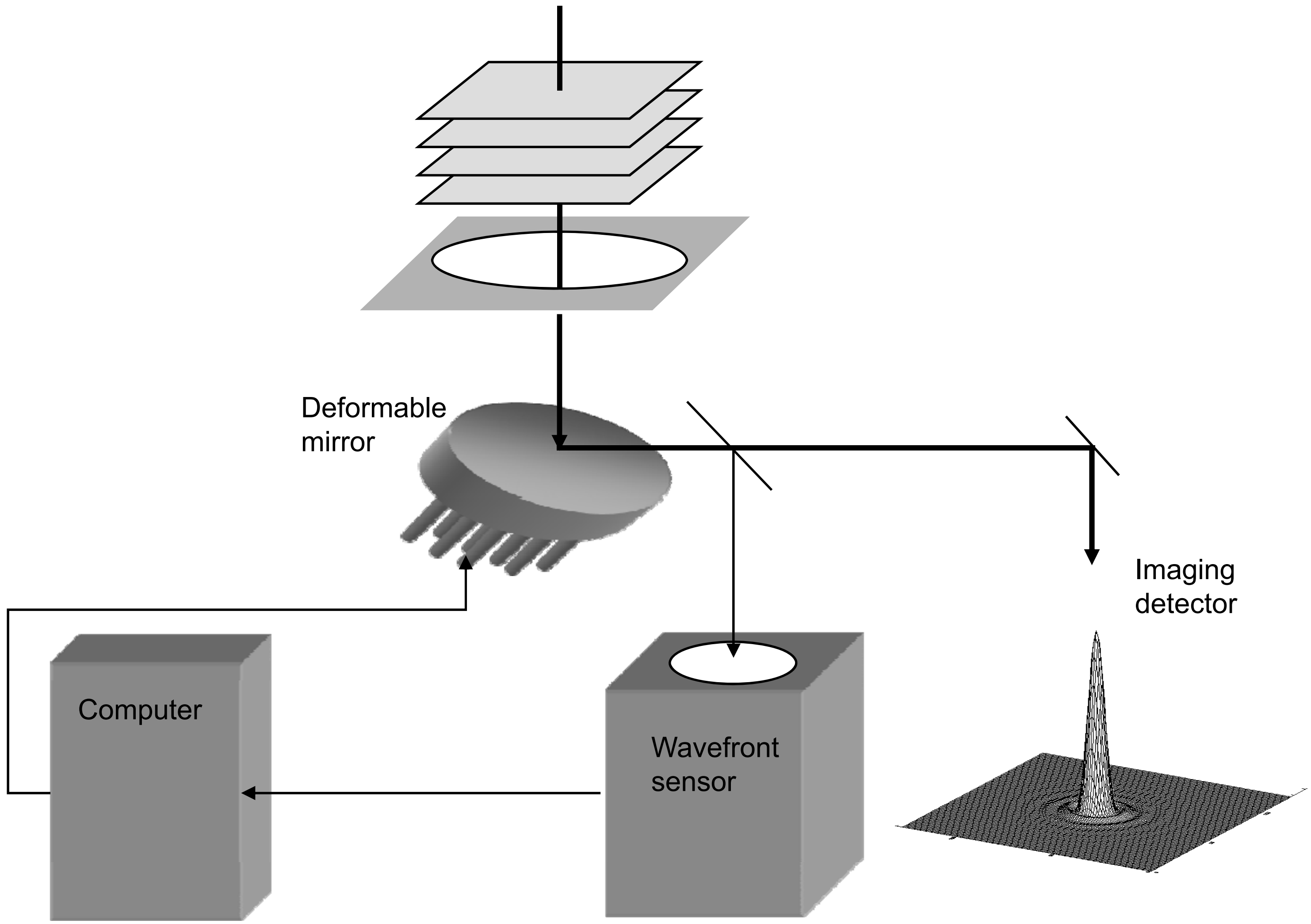
In this class you will learn:

- What is control?
  - The concept and architecture of closed loop feedback control
- About a basic tool: the Laplace transform
  - Using the Laplace transform to characterize the time and frequency domain behavior of a system
  - Manipulating *Transfer functions* in the system analysis
- How to predict statistical performance of the controller, including
  - Calculating how much atmospheric turbulence is rejected and how much remains in the “controlled” wavefront
  - Calculating the degree to which measurement noise corrupts the controlled wavefront
- How to optimize the controller for overall noise performance

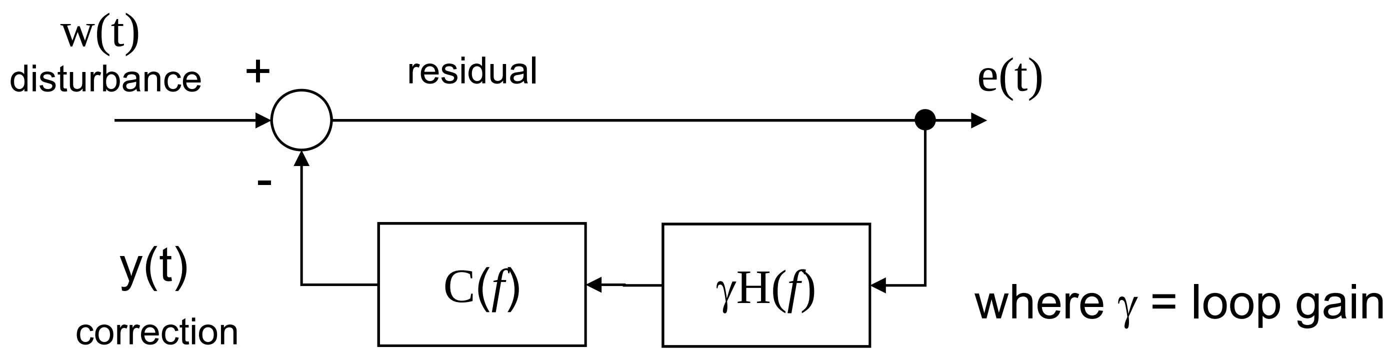
Adaptive Optics Control - the basic idea
measure the wavefront ... and correct it optically



Feedback control architecture:
Measure the wavefront *after* correction...and “null-seek”



Closed loop control



H(*f*) = Hartmann sensor, sampling dynamics, computational delay,
dynamics of the DM and drive electronics.

Our goal will be to suppress e(t) (residual) so that y(t)~w(t)

$$e(t) = w(t) - \gamma CHe(t)$$

We can design a filter, C(*f*), into the feedback loop to:

a) Stabilize the feedback (i.e. keep it from oscillating)
b) Optimize performance

The Laplace Transform Pair

$$H(s) = \int_0^{\infty} h(t) e^{-st}\, dt$$

$$h(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} H(s) e^{st}\, ds$$

- Example 1: decaying exponential

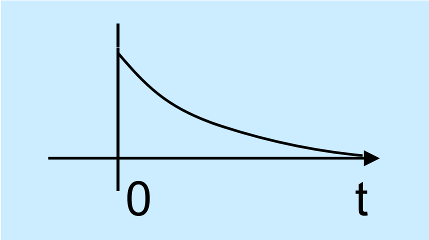
Transform:

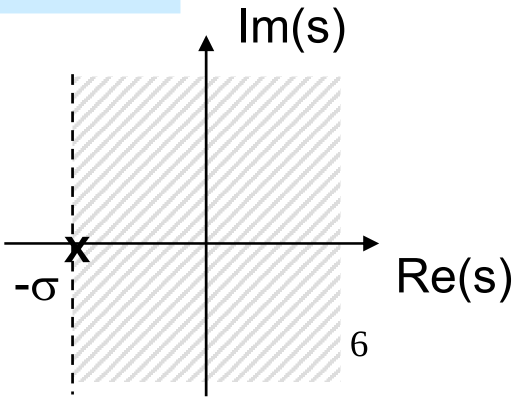
$$h(t) = e^{-\sigma t}$$

$$H(s) = \int_0^{\infty} e^{-(s+\sigma)t}\, dt$$

$$= \frac{-1}{s+\sigma} e^{-(s+\sigma)t} \Big|_0^{\infty}$$

$$= \frac{1}{s+\sigma}; \qquad \mathrm{Re}(s) > -\sigma$$





Example 1 (continued), decaying exponential

Inverse Transform:

$$h(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{st} \frac{1}{s+\sigma} ds$$

$$= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \rho^{-1} e^{-\sigma t} \, i\rho \, d\theta$$

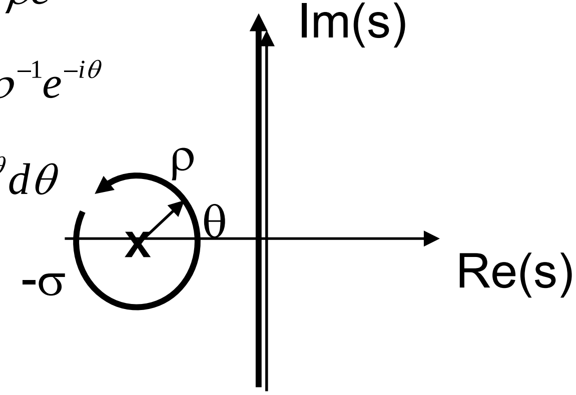
$$= e^{-\sigma t} \frac{1}{2\pi i} \int_{-\pi}^{\pi} i \, d\theta$$

$$= e^{-\sigma t}$$

$$s = -\sigma + \rho e^{i\theta}$$

$$\frac{1}{s+\sigma} = \rho^{-1} e^{-i\theta}$$

$$ds = i\rho e^{i\theta} d\theta$$

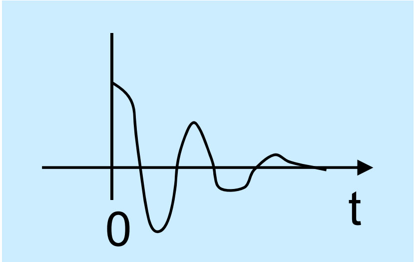


The above integration makes use of the Cauchy Principal Value Theorem:

If F(s) is *analytic* then $\oint F(s) \frac{1}{s-a} ds = 2\pi i F(a)$

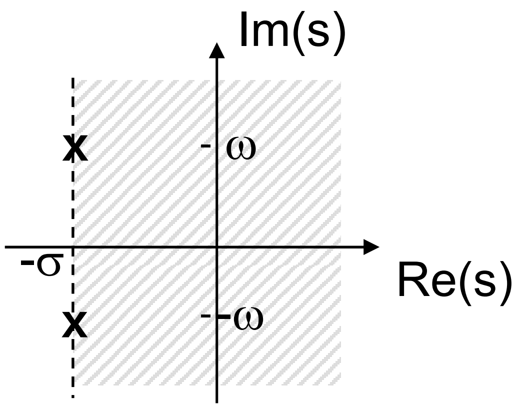
The Laplace Transform Pair

Example 2: Damped sinusoid

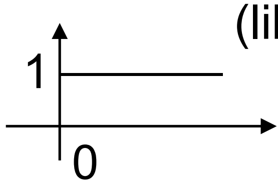
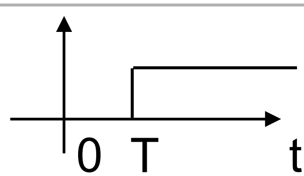
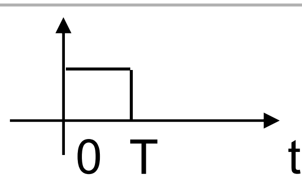


$$h(t) = e^{-\sigma t}\cos(\omega t)$$
$$= e^{-\sigma t}\frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right)$$
$$= \frac{1}{2}\left(e^{-(\sigma - i\omega)t} + e^{-(\sigma + i\omega)t}\right)$$

$$H(s) = \frac{1}{2}\left(\frac{1}{s+\sigma-i\omega} + \frac{1}{s+\sigma+i\omega}\right); \quad \mathrm{Re}(s) > -\sigma$$



Laplace Transform Pairs

	h(t)	H(s)
unit step	 (like $\lim_{\sigma \to 0} e^{-\sigma t}$)	$\frac{1}{s}$
	$e^{-\sigma t}$	$\frac{1}{s+\sigma}$
	$e^{-\sigma t}\cos(\omega t)$	$\frac{1}{2}\left(\frac{1}{s+\sigma-i\omega}+\frac{1}{s+\sigma+i\omega}\right)$
	$e^{-\sigma t}\sin(\omega t)$	$\frac{1}{2i}\left(\frac{1}{s+\sigma-i\omega}-\frac{1}{s+\sigma+i\omega}\right)$
delayed step		$\frac{e^{-sT}}{s}$
unit pulse		$\frac{1-e^{-sT}}{s}$

Laplace Transform Properties (1)

$$L\{\ \alpha h(t)+\beta g(t)\ \}=\alpha H(s)+\beta G(s)$$ Linearity

$$L\{\ h(t+T)\ \}=e^{sT}H(s)$$ Time-shift $(T \le 0)$

$$L\{\ \delta(t)\ \}=1$$ Dirac delta function transform ("sifting" property)

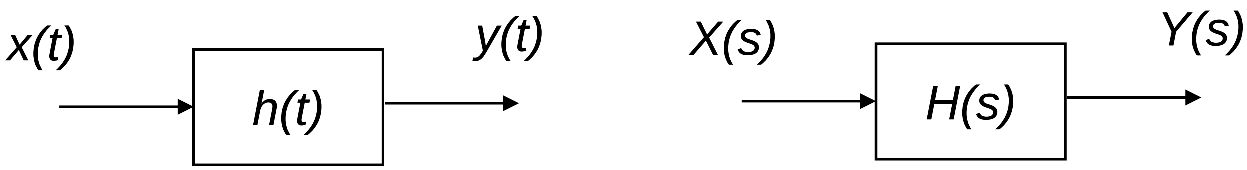
$$L\left\{\int_0^t h(t')g(t-t')dt'\right\}=H(s)\,G(s)$$ Convolution

$$L\left\{\int_0^t h(t')\,\delta(t-t')dt'\right\}=H(s)$$ Impulse response

$$\int_0^t h(t-t')\,e^{i\omega t'}dt'=H(i\omega)\,e^{i\omega t}$$ Frequency response

Laplace Transform Properties (2)

System Block Diagrams



convolution of input *x(t)* with impulse response *h(t)* ⟷ product of input spectrum *X(s)* with frequency response *H(s)* (*H(s)* in this role is called the *transfer function*)

$$L\left\{ \int_0^t h(t')x(t-t')dt' \right\} = H(s)X(s) = Y(s)$$

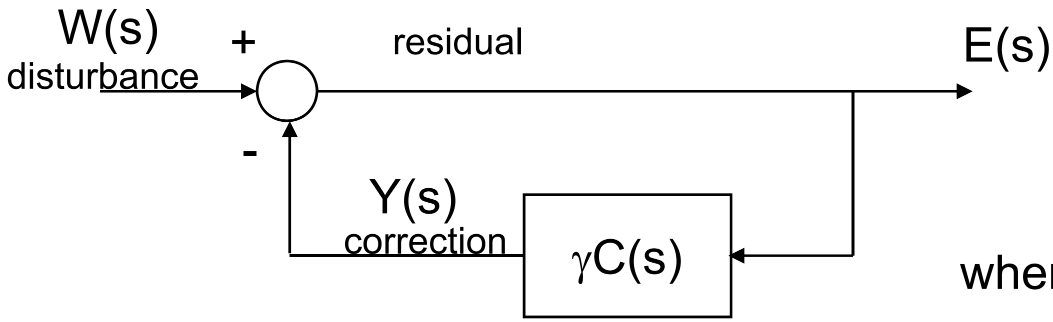
Power or Energy

$$\int_0^\infty h(t)^2 dt = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \left|H(s)\right|^2 ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|H(i\omega)\right|^2 d\omega$$

"Parseval's Theorem"

Closed loop control (simple example, H(s)=1)



where γ = loop gain

Our goal will be to suppress X(s) (residual) by high-gain feedback so that Y(s)~W(s)

$$E(s) = W(s) - \gamma C(s) E(s)$$

solving for E(s),

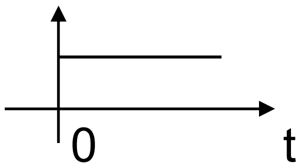
$$E(s) = \frac{W(s)}{1+\gamma C(s)}$$

Note: for consistency "around the loop," the units of the gain γ must be the inverse of the units of C(s).

[Q1. What would happen if $\gamma C(i\omega)$ = -1 at some ω?]

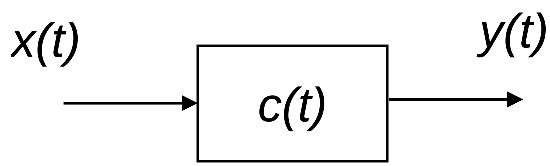
What is a good choice for C(s)?...

A *system* who's impulse response is the unit step

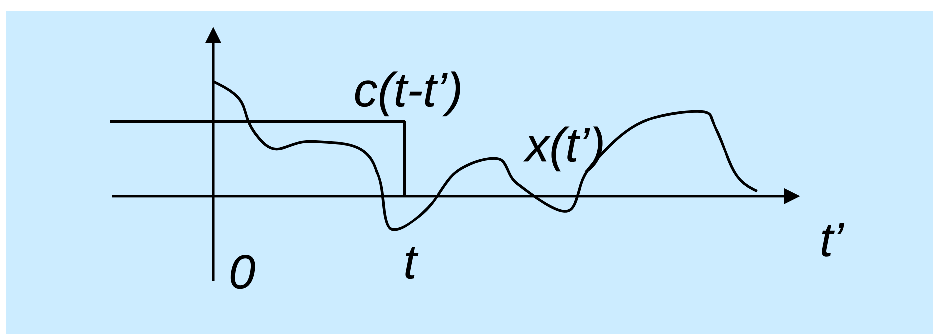


$$c(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad \leftrightarrow \quad C(s) = \frac{1}{s}$$

acts as an integrator to the input signal:



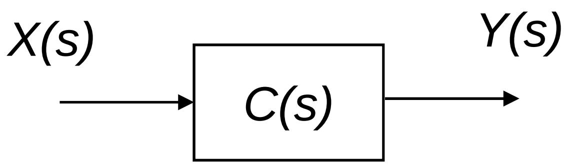
$$y(t) = \int_0^t c(t - t')x(t')dt' = \int_0^t x(t')dt'$$



that is, *C(s)* integrates the past history of inputs, *x(t)*

The Integrator (2)

In Laplace terminology:

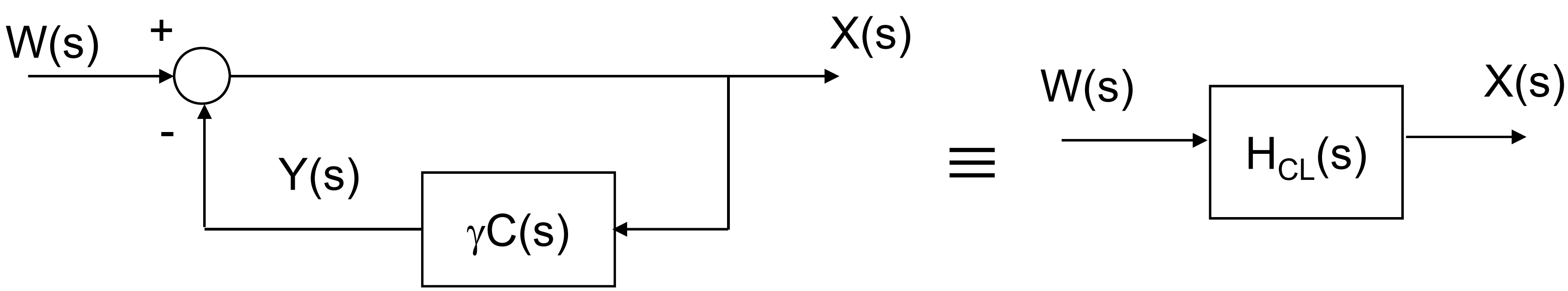


$$Y(s) = \frac{X(s)}{s}$$

An integrator has high gain at low frequencies, low gain at high frequencies.

Write the input/output transfer function for an integrator in closed loop:

The closed loop transfer function with the integrator in the feedback loop is:



$$C(s) = \frac{1}{s} \Rightarrow X(s) = \frac{W(s)}{1+\gamma/s} = \frac{sW(s)}{s+\gamma} = H_{CL}(s)W(s)$$

output (e.g. residual wavefront to science camera)

closed loop transfer function

input disturbance (e.g. atmospheric wavefront)

The integrator in closed loop (2)

where

$$H_{CL}(s) = \frac{s}{\gamma + s}$$

$H_{CL}(s)$, viewed as a sinusoidal response filter:

$$H_{CL}(i\omega) \to 0 \quad \text{as} \quad \omega \to 0$$

DC response = 0
("Type-0" behavior)

$$H_{CL}(i\omega) \to 1 \quad \text{as} \quad \omega \to \infty$$

High-pass behavior

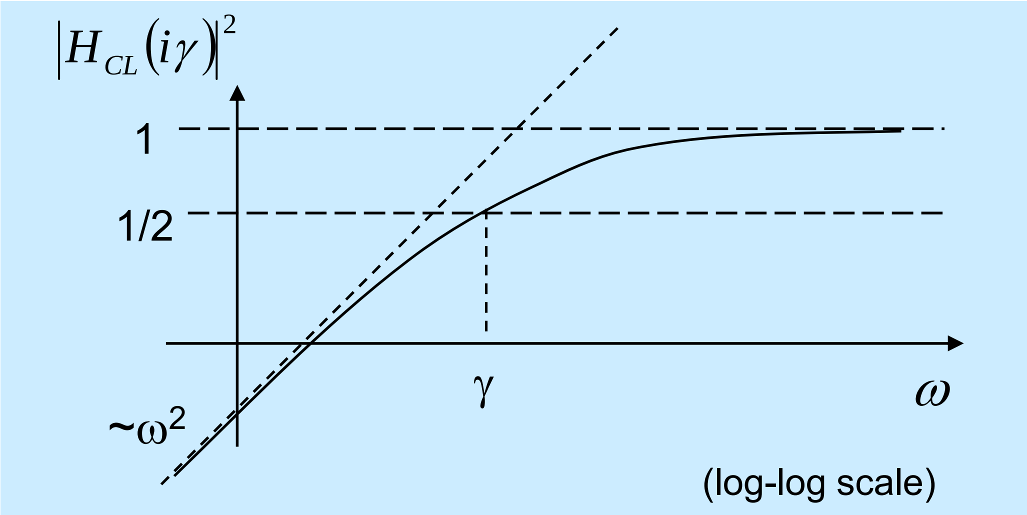
and the "break" frequency (transition from low freq to high freq behavior) is around $\omega \sim \gamma$

The integrator in closed loop (3)

At the break frequency, $\omega = \gamma$, the *power rejection* is:

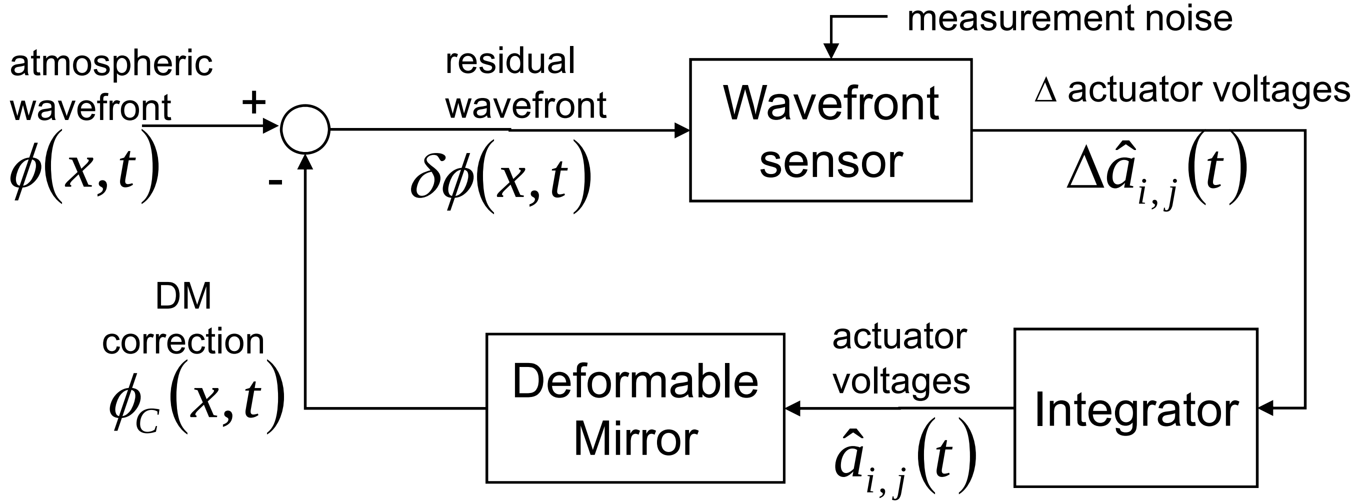
$$\left|H_{CL}(i\gamma)\right|^2 = \frac{i\gamma}{(\gamma + i\gamma)} \frac{-i\gamma}{(\gamma - i\gamma)} = \frac{\gamma^2}{2\gamma^2} = \frac{1}{2}$$

Hence the break frequency is often called the "half-power" frequency

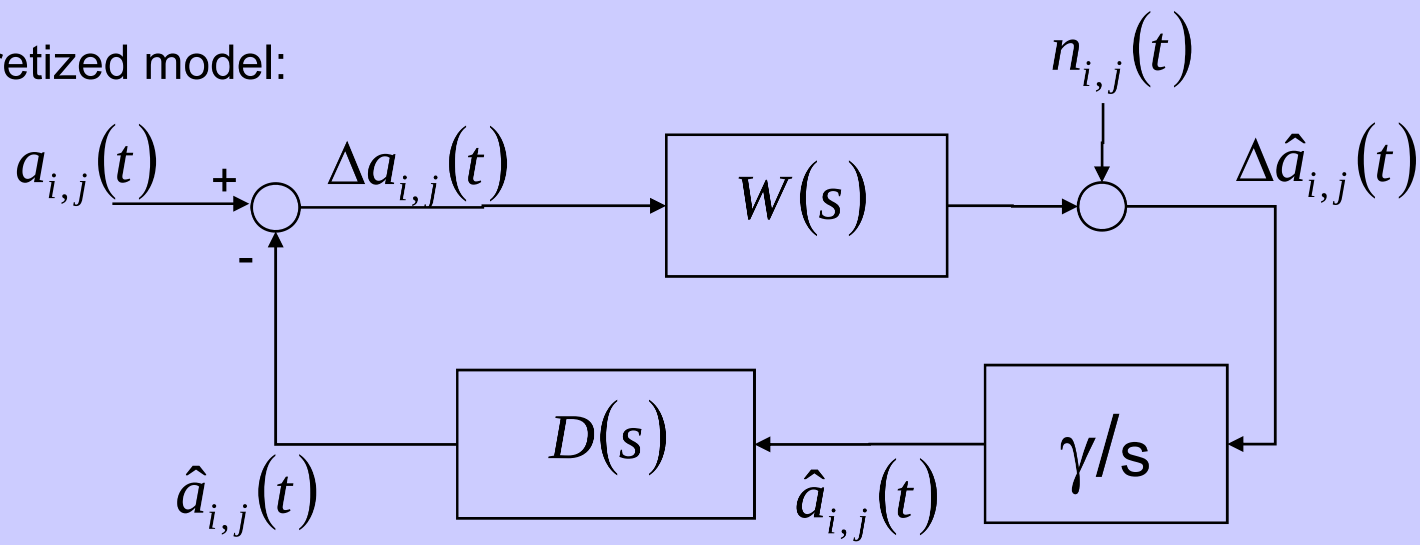


☞ Note also that the loop gain, γ, is the *bandwidth* of the controller. Frequencies below γ are rejected, frequencies above γ are passed. By convention, γ is thus known as the gain-bandwidth product.

Now let's apply these concepts to the adaptive optics controller

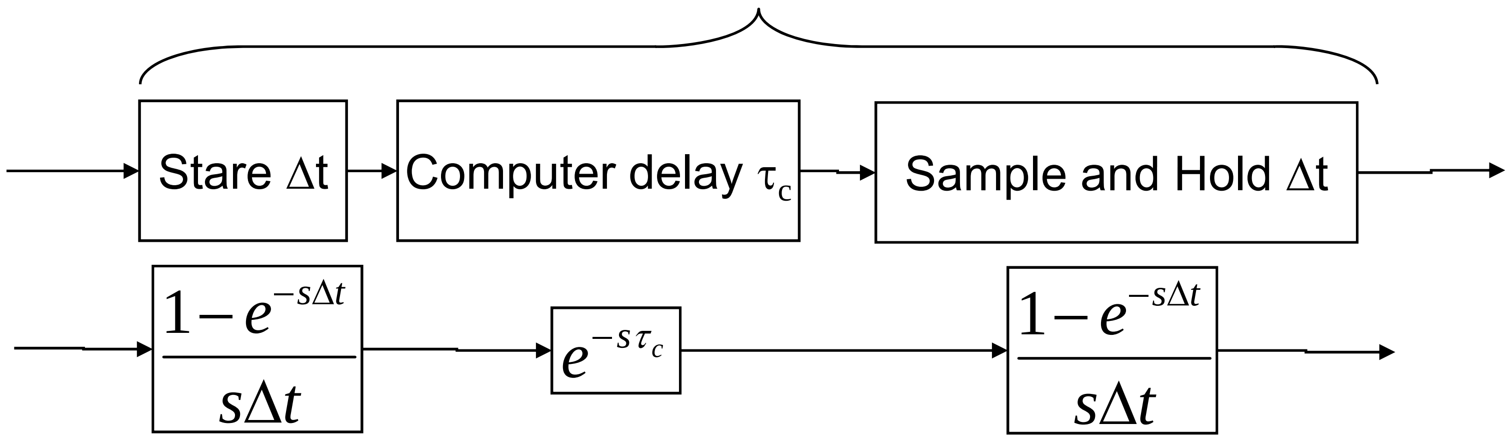


Discretized model:



Note sloppy misuse of notation in the block diagram, mixing time domain signals and Laplace domain system transfer functions - this is common; live with it. But don't do it in equations.
(γ/s means integrate from 0 to t and multiply by γ)

Wavefront sensor transfer function, $W(\omega)$



Deformable mirror transfer function, $D(\omega)$:

http://www.okotech.com/publications/sid.pdf

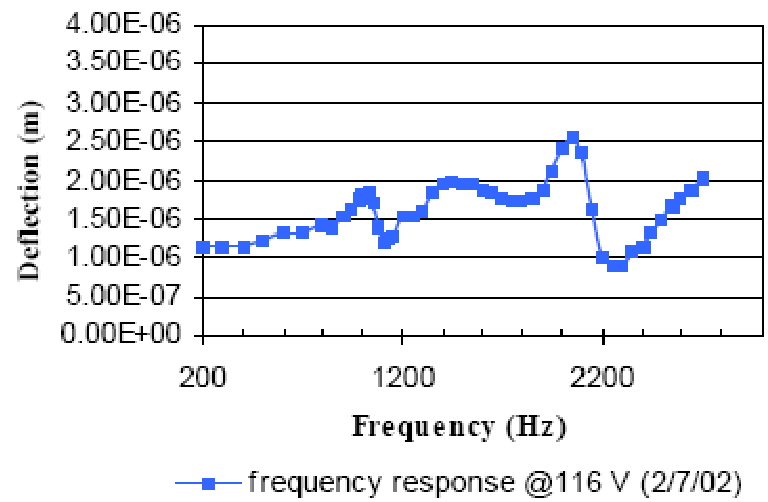


Figure 8. Frequency response curve of OKO deformable mirror. First resonant frequency peak at 1 kHz.

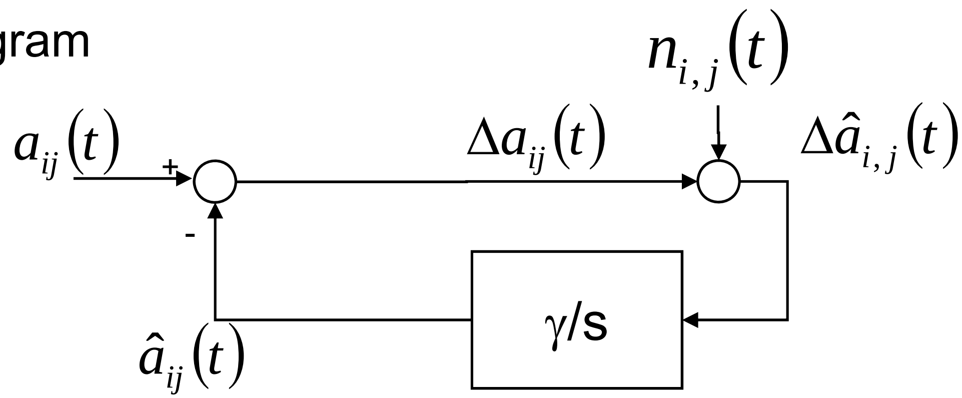
Optimizing the accuracy of the controller

There are two sources for residual wavefront:

1) atmosphere (a_{ij})
2) measurement noise (n_{ij})

We'll use a simple model ignoring the dynamics of the wavefront sensor and deformable mirror: $W(s) = 1$ and $D(s) = 1$

Block diagram



Laplace Domain Equation

$$\Delta A_{ij}(s) = \underbrace{\left(\frac{s}{\gamma+s}\right)}_{H_{CL}(s)} A_{ij}(s) - \underbrace{\left(\frac{\gamma}{\gamma+s}\right)}_{H_N(s)} N_{ij}(s)$$

residual wavefront — atmospheric wavefront — measurement noise

Optimizing the accuracy of the controller

We can now solve for a control gain that optimizes the accuracy of the controller by trading off atmosphere rejection against sensitivity to noise.

First, calculate the variance of the residual atmospheric wavefront without noise

$$\Delta A(s) = H_{CL}(s) A(s)$$

$$\Delta a(t) = \int_0^\infty h_{CL}(t') a(t-t') dt'$$

convolution property

$$\sigma_{BW}^2 = \left\langle \lim_{T\to\infty} \frac{1}{T} \int_0^T \Delta a(t)^2 dt \right\rangle$$

Time-average, *and*, formally, statistical average

$$= \left\langle \lim_{\Delta\omega\to 0} \frac{\Delta\omega}{(2\pi)^2} \int_{-\infty}^{\infty} \left|\Delta A(i\omega)\right|^2 d\omega \right\rangle$$

Parseval's theorem, and $\Delta\omega \equiv \frac{2\pi}{T}$

$$= \left\langle \lim_{\Delta\omega\to 0} \frac{\Delta\omega}{(2\pi)^2} \int_{-\infty}^{\infty} \left|H_{CL}(i\omega)\right|^2 \left|A(i\omega)\right|^2 d\omega \right\rangle$$

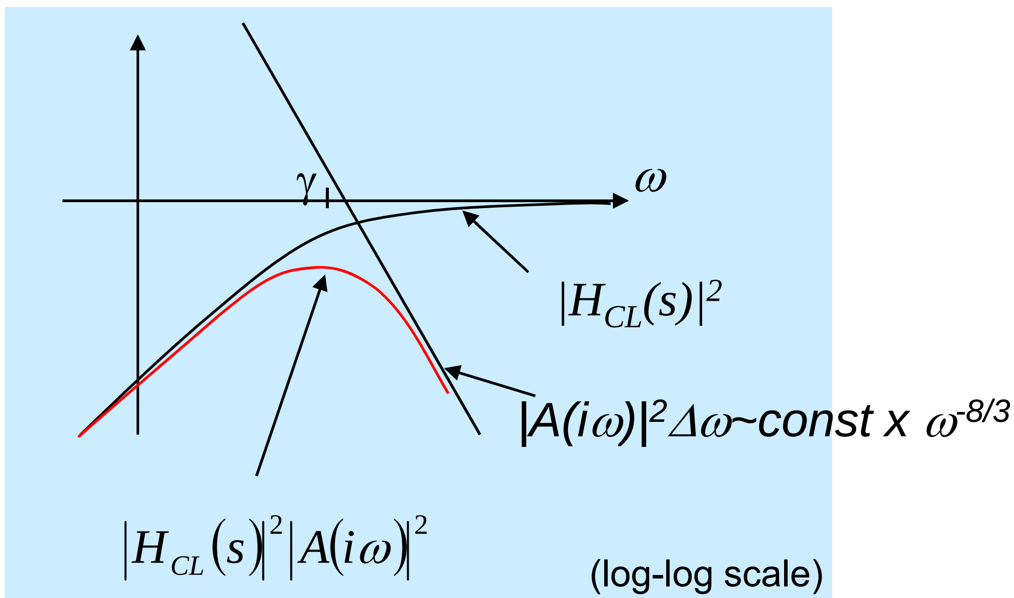
Optimizing the accuracy of the controller (2)

Calculating the variance of the residual atmospheric wavefront without noise

Example, substitute power spectrum of *Kolmogorov* turbulence

$$\left\langle \lim_{\Delta\omega \to 0} \frac{\Delta\omega}{(2\pi)^2} \left|A(i\omega)\right|^2 \right\rangle = \text{const} \times \left|\omega\right|^{-8/3}$$

const depends on wind velocity and turbulence strength



Optimizing the accuracy of the controller (3)

Calculating the variance of the residual atmospheric wavefront without noise

$$\sigma_{BW}^2 = \int_{-\infty}^{\infty} \frac{\omega^2}{\gamma^2 + \omega^2} const \times |\omega|^{-8/3} d\omega$$

change of variables $\omega' = \omega / \gamma$

$$= \gamma^{-5/3} \int_{-\infty}^{\infty} \frac{\omega'^2}{1+\omega'^2} const \times |\omega'|^{-8/3} d\omega'$$

$$= \gamma^{-5/3} \omega_g^{5/3}$$

By definition of Greenwood frequency, σ_{BW},has units of radians of wavefront phase.

$$\sigma_{BW}^2 = \left(\omega_g / \gamma\right)^{5/3}$$

where

$$\omega_g = \left(const \times \int_{-\infty}^{\infty} \frac{\omega^2}{1+\omega^2} |\omega|^{-8/3} d\omega \right)^{3/5}$$

ω_g is a characteristic bandwidth of the atmosphere (Greenwood's frequency).
$\gamma = \omega_c$ is the gain-bandwidth product of the controller.

Homework problem #1

- For a disturbance wavefront with a Greenwood frequency of 50 Hz, and zero measurement noise, calculate the residual closed loop error in an AO system in the case where closed loop gain-bandwidth product equals:
  - 10 Hz
  - 50 Hz
  - 100 Hz

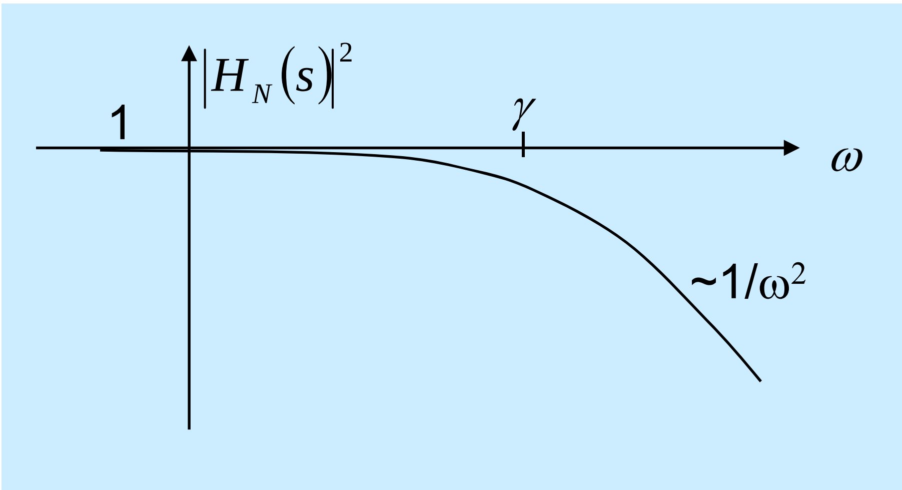
  Calculate this error in nanometers, rms, assuming that the Greenwood frequency is specified for λ = 0.5 microns wavelength.

Optimizing the accuracy of the controller (4)

Now calculate the contribution to the variance of the residual wavefront due to the measurement noise

residual wavefront due to meas. noise → $\Delta A(s)$; Noise transfer function → $H_N(s)$; Noise → $N_a(s)$

$$\Delta A(s) = \frac{\gamma/s}{1+\gamma/s} N_a(s) = \underbrace{\left(\frac{\gamma}{\gamma+s}\right)}_{H_N(s)} N_a(s)$$



Optimizing the accuracy of the controller (5)

Calculating the contribution to the variance of the residual wavefront due to the measurement noise

Use Parseval's theorem to calculate the power:

$$\sigma_{noise}^2 = \lim_{T\to\infty} \frac{1}{2\pi T} \int_0^\infty \left|\Delta a(t)\right|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^\infty \left|\Delta A(i\omega)\right|^2 d\omega$$

$$= \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^\infty \left|H_N(i\omega)\right|^2 \left|N(i\omega)\right|^2 d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{\gamma^2}{\gamma^2+\omega^2} \sigma_N^2 d\omega$$

assuming white measurement noise

$$\sigma_{noise}^2 = \gamma \sigma_N^2 \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1}{1+\omega'^2} d\omega'$$

Note σ_{noise}, has units of radians of wavefront phase. σ_N has units of radians of phase per square-root Hz.

Summary of contributors to residual wavefront variance:

Due to atmosphere: $\sigma_{BW}^2 = \left(\omega_g / \gamma\right)^{5/3}$

Due to measurement noise:

$$\sigma_{noise}^2 = \alpha\gamma\sigma_N^2$$ **where**

$$\omega_g = \frac{const}{2\pi} \int_{-\infty}^{\infty} \frac{\omega^2}{1+\omega^2} |\omega|^{-8/3} d\omega = \text{Greenwood frequency}$$

$$\alpha = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1+\omega^2} d\omega = \frac{1}{2}$$

$const \propto$ atmospheric turbulence strength and wind velocity

$\gamma =$ loop gain, rad/sec (= controller gain - bandwidth product)

$\sigma_N^2 =$ measurement noise variance, $\text{rad}^2/(\text{rad/sec})$

Homework problem #2

- For white measurement noise of 20 nm/$\sqrt{Hz}$ (σ_N = 0.1), and zero wavefront disturbance, calculate the residual wavefront error in closed loop in the case where closed loop gain-bandwidth product equals:
  - 10 Hz
  - 50 Hz
  - 100 Hz

  Calculate this error in nanometers, rms.

Optimizing the accuracy of the controller (7)

The loop gain (= controller bandwidth) can now be optimized for minimum residual wavefront variance:

$$\sigma^2 = \sigma_{BW}^2 + \sigma_{noise}^2$$

$$\frac{\partial \sigma^2}{\partial \gamma} = -\frac{5}{3}\gamma^{-8/3}\omega_g^{5/3} + \alpha\sigma_N^2 = 0$$

$$\gamma^* = \left[\frac{3}{5}\omega_g^{-5/3}\alpha\sigma_N^2\right]^{-3/8}$$

$$\gamma^* = \omega_g^{5/8}\sigma_N^{-3/4}\left(\frac{3}{10}\right)^{-3/8}$$

Optimal gain setting γ^* is

- increased when atmospheric bandwidth (ω_g) increases
- decreased when measurement noise increases

Homework problem #3

- For the situation in the previous 2 problems (50 Hz Greenwood frequency and 20 nm/√Hz rms measurement error), determine the optimum gain-bandwidth product for a control loop design. Express the answer in Hertz.

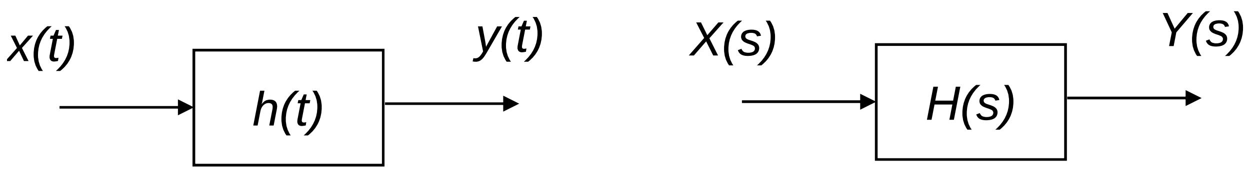
- Also compute the total rms residual wavefront error, in nanometers, at this gain and verify that this “optimum” is reasonable by comparing to the root-sum-square of error terms from each of the case studies in problems 1 and 2.

In conclusion, you have learned:

- The architecture and problems associated with feedback control systems
- The use of the Laplace transform to help characterize closed loop behavior
- How to predict the performance of the adaptive optics under various conditions of atmospheric seeing and measurement signal-to-noise
- How to optimize the controller for overall noise performance, if we know or measure the second order statistics of the atmospheric turbulence and the measurement noise

Quick Review

System Block Diagrams



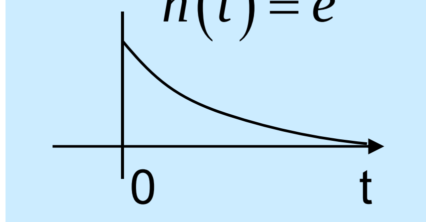
convolution of input *x(t)* with impulse response *h(t)* ⟷ product of input spectrum *X(s)* with frequency response *H(s)* (*H(s)* in this role is called the *transfer function*)

$$L\left\{ \int_0^t h(t')x(t-t')dt' \right\} = H(s)X(s) = Y(s)$$

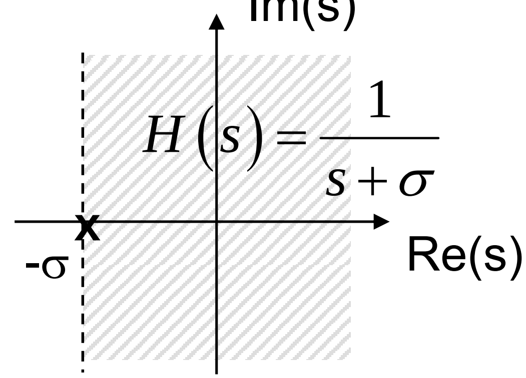
Stable input-output property: system response consists of a series of decaying exponentials

Time Domain | **Transform Domain**

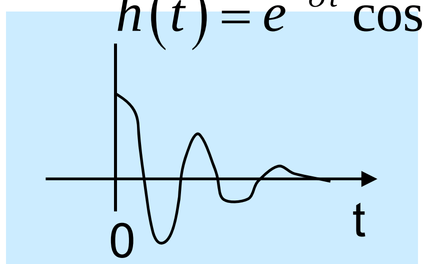
$$h(t) = e^{-\sigma t}$$

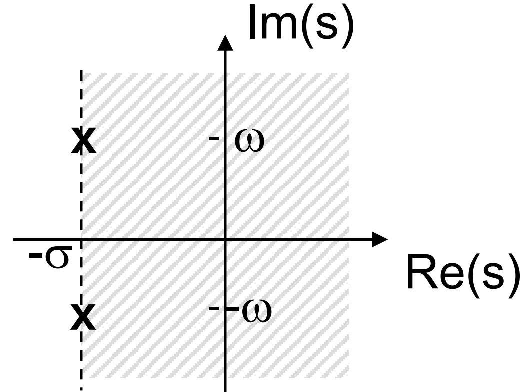


$$H(s) = \frac{1}{s+\sigma}$$



$$h(t) = e^{-\sigma t} \cos(\omega t)$$





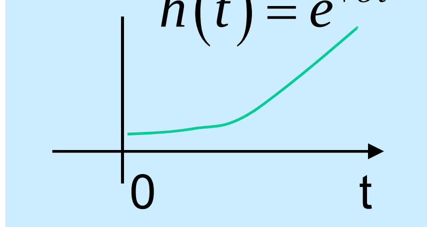
$$H(s) = \frac{1}{2}\left(\frac{1}{s+\sigma-i\omega} + \frac{1}{s+\sigma+i\omega}\right)$$

What Instability Looks Like

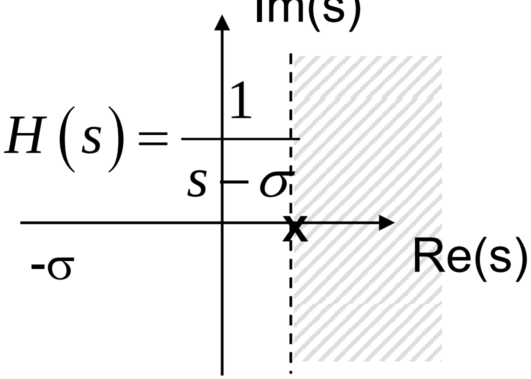
Time Domain

Transform Domain

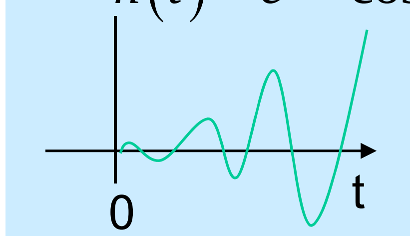
$$h(t) = e^{+\sigma t}$$

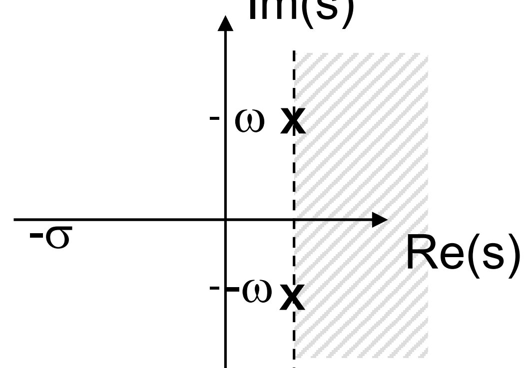


$$H(s) = \frac{1}{s - \sigma}$$



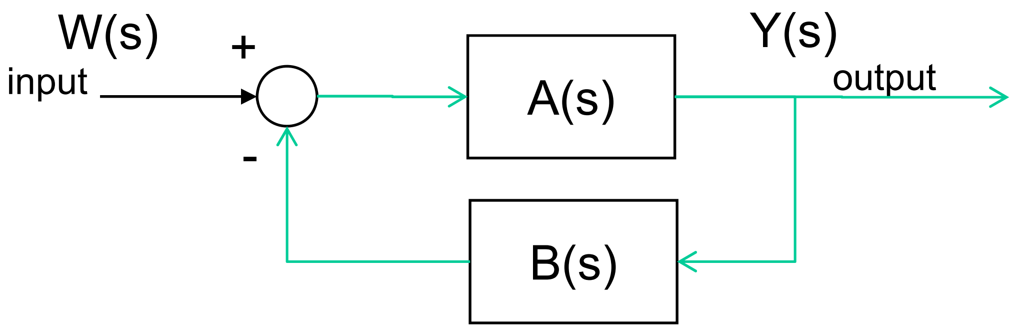
$$h(t) = e^{+\sigma t} \cos(\omega t)$$





$$H(s) = \frac{1}{2}\left(\frac{1}{s - \sigma - i\omega} + \frac{1}{s - \sigma + i\omega}\right)$$

Control Loop Arithmetic



$$Y(s) = A(s)W(s) - A(s)B(s)Y(s)$$

solving for Y(s),

$$Y(s) = \frac{A(s)W(s)}{1 + A(s)B(s)}$$

Instability if any of the roots of the polynomial 1+A(s)B(s) are located in the right-half of the s-plane