A practical model of convective dynamics for stellar evolution calculations

Neil Miller[1] and Pascale Garaud[2]

[1]Department of Astronomy & Astrophysics, [2]Department of Applied Mathematics and Statistics

University of California, Santa Cruz

1. Introduction

Turbulent motions in the interior of a star play an important role in its evolution, since they transport chemical species, thermal energy and angular momentum. Our overall goal is to construct a practical turbulent closure model for convective transport including the effects of rotation, shear and magnetic fields, that can be used in multi-dimensional stellar evolution calculations. Here, we focus on the first step of this task which involves capturing the well-known transition from radiative heat transport to turbulent convection with and without rotation, as well as the asymptotic relationship between turbulent and radiative transport in the limit of large Rayleigh number. The closure model is developed following the formalism of Ogilvie (2003) and Ogilvie & Garaud (2005). In this poster, we extend the closure model to include heat transport and compare it with experimental results of Rayleigh-Benard convection.

2. Derivation of the Closure Model

We consider the Boussinesq equations for thermal convection:

$$\partial_i u_i = 0$$

$$\rho_0(\partial_t + u_j \partial_j) u_i + 2\epsilon_{ijk}\Omega_j u_k = \rho g_i - \partial_i p + \rho_0 \nu \partial_{jj} u_i$$

$$\rho = \rho_0[1 - \alpha(T - T_0)]$$

$$(\partial_t + u_i \partial_i) T = \kappa \partial_{ii} T$$

Each quantity is defined as the sum of mean and perturbation terms

$$u_j = \bar{u}_j + u_j' \qquad \Theta = \bar{\Theta} + \Theta' \qquad \psi = \bar{\psi} + \psi'$$

where $\Theta = T - T_0$ and $\psi = \dfrac{p - (p_0 + \rho_0 g_i x_i)}{\rho_0}$

so that the mean equations are

$$\partial_i \bar{u}_i = 0$$

$$(\partial_t + \bar{u}_j \partial_j)\bar{u}_i + 2\epsilon_{ijk}\Omega_j \bar{u}_k = -\alpha \bar{\Theta} g_i - \partial_i \bar{\psi} + \nu \partial_{jj} \bar{u}_i - \partial_j \bar{R}_{ij}$$

$$(\partial_t + \bar{u}_i \partial_i)\bar{\Theta} = \kappa \partial_{ii}\bar{\Theta} - \partial_i \bar{F}_i$$

where

$$R_{ij} = u_i' u_j' \quad F_i = \Theta' u_i' \quad Q = \Theta'^2 \quad \rightarrow \quad \bar{R}_{ij} = \langle u_i' u_j' \rangle \quad \bar{F}_i = \langle \Theta' u_i' \rangle \quad \bar{Q} = \langle \Theta'^2 \rangle$$

and the correlations evolve following the exact equations

$$(\partial_t + \bar{u}_k \partial_k) R_{ij} + R_{jk}\partial_k \bar{u}_i + R_{ik}\partial_k \bar{u}_j + 2\epsilon_{ilm}\Omega_l R_{jm} + 2\epsilon_{jlm}\Omega_l R_{im} + \alpha F_j g_i + \alpha F_i g_j$$
$$= -\langle u_j'\partial_i \psi' + u_i' \partial_j \psi' \rangle - \nu \langle u_j' \partial_{kk} u_i' + u_i' \partial_{kk} u_j' \rangle - \langle u_k' \partial_k R_{ij} \rangle$$

$$\partial_t \bar{F}_i + \bar{u}_k \partial_k \bar{F}_i + \bar{R}_{ik}\partial_k \bar{\Theta} + \bar{F}_k \partial_k \bar{u}_i + 2\epsilon_{ijl}\Omega_j \bar{F}_k + \alpha \bar{Q} g_i$$
$$= -\langle u_k' \partial_k F_i \rangle + \kappa \langle u_i' \partial_{kk}\Theta' \rangle + \nu \langle \Theta' \partial_{kk} u_i' \rangle - \langle \Theta' \partial_i \psi' \rangle$$

$$\partial_t \bar{Q} + u_j \partial_i \bar{Q} + 2 F_i \partial_i \Theta = 2\kappa \langle \Theta' \partial_{kk} \Theta' \rangle - 2\langle \Theta' u_i' \partial_i \Theta' \rangle$$

Unfortunately, a number of the terms in the right-hand-side of these equations include triple-correlations that cannot be straightforwardly related to the known large-scale quantities, hence the need for closure. The physically motivated closure model proposed by Ogilvie (2003) and Garaud & Ogilvie (2005) is used here and modified slightly in an attempt to account for the effects of rotation on the eddy turnover timescale.

$$-\langle u_j' \partial_i \psi' + u_i' \partial_j \psi' \rangle \sim ?$$

$$-\nu \langle u_j' \partial_{kk} u_i' + u_i' \partial_{kk} u_j' \rangle \sim -\nu C_\nu \frac{\bar{R}_{ij}}{d^2}$$

$$-\langle u_k' \partial_k R_{ij} \rangle \sim \frac{-C_1 \bar{R}_{ij}}{\tau} - \frac{C_2\left(\bar{R}_{ij} - \frac{1}{3}\bar{R}\delta_{ij}\right)}{\tau}$$

$$-\langle u_k' \partial_k F_i \rangle \sim -C_6 \frac{F_i}{\tau}$$

$$\kappa \langle u_i' \partial_{kk} \Theta' \rangle \sim \frac{\kappa C_\kappa F_i}{d^2}$$

$$\nu \langle \Theta' \partial_{kk} u_i' \rangle \sim \frac{\nu C_\nu F_i}{d^2}$$

$$-\langle \Theta' \partial_i \psi' \rangle \sim ?$$

$$-2\langle \Theta' u_i' \partial_i \Theta' \rangle \sim -C_7 \frac{\bar{Q}}{\tau}$$

$$2\kappa \langle \Theta' \partial_{kk} \Theta' \rangle \sim \kappa \partial_{ii} \bar{Q} - \frac{\kappa C_\kappa \bar{Q}}{d^2}$$

with

$$\tau = \left(\Omega^2 + \frac{\bar{R}}{d^2}\right)^{-1/2}$$

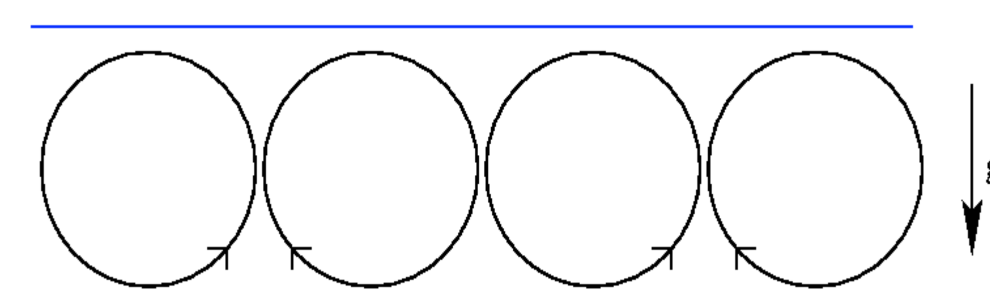
so that the eddy turnover time satisfies

$\tau \rightarrow \dfrac{1}{\Omega}$ In the limit of strong rotation

$\tau \rightarrow \dfrac{d}{\bar{R}^{1/2}}$ In the limit of weak rotation

3. The rotating Rayleigh-Benard problem

The standard Rayleigh-Benard experimental setup is shown in the following cartoon:



The experimental parameters are the distance between the two plates D, the imposed temperature difference between the two plates ΔT, the characteristics of the fluid (the viscosity ν, thermal diffusivity κ, and the coefficient of thermal expansion α) as well as constants such as gravity g. When rotation is applied, it is assumed to be parallel to the rotation axis. This yields the dimensionless parameters

$$Ta = \frac{\text{centrifugal forces}}{\text{viscous forces}} = \frac{D^4(2\Omega)^2}{\nu^2} \qquad Ra \equiv \frac{\text{Buoyancy}}{\text{Viscosity}} = \frac{\alpha g \Delta T D^3}{\kappa \nu}$$

$$Pr = \frac{\text{viscous diffusion rate}}{\text{thermal diffusion rate}} = \frac{\nu}{\kappa} \qquad Nu \equiv \frac{\text{actual heat transfer}}{\text{conductive heat transfer if there was no convection}}$$

The boundary conditions applied to the model are show on the left, and a typical solution of the model in the case of fully developed turbulence in the absence of rotation is shown on the right.

At z=0	At z=1
$F_z = 0$	$F_z = 0$
$\bar{R} = 0$	$\bar{R} = 0$
$\bar{R}_{zz} = 0$	$\bar{R}_{zz} = 0$
$\bar{Q} = 0$	$\bar{Q} = 0$
$\bar{\Theta} = 0$	
$\partial_z \bar{\Theta}_z = \bar{F}_0$	

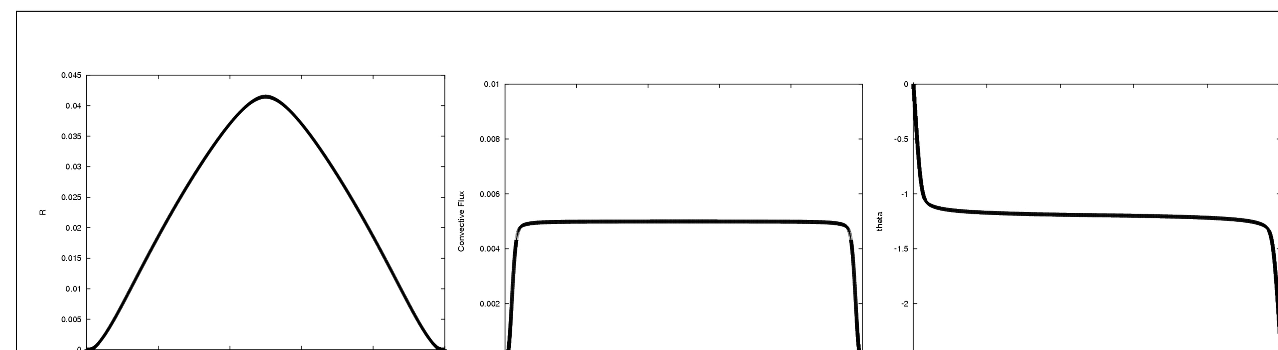


Figure 1. Solutions to closure model for Benard problem. Notice that in the convective region, the temperature is essentially constant. In this run, Pr = 10.

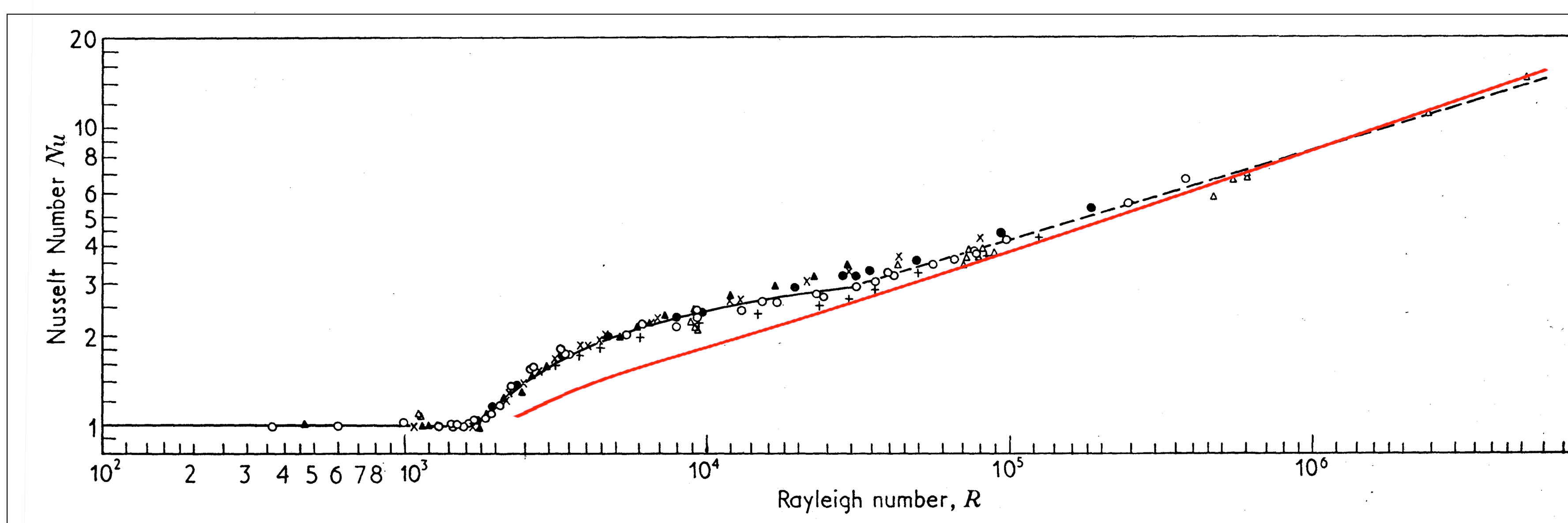


FIG. 13. Silveston's experimental results on the heat transfer in various liquids (○ water; + heptane; × ethylene glycol; ● silicone oil AK 3; ▲ silicone oil AK 350; △ air data of Mull and Reiher). The Nusselt number is plotted against the Rayleigh number.

Figure 2. Plot of Nusselt number as a function of Rayleigh number. A Nusselt number of 1 indicates that all of the actual heat transfer is by conduction. Black points and lines are experimental and theoretical results from Chandrasekhar respectively. The red line shows results from the closure model. These are all run in the non rotating case with Prandtl number 10. Notice there is good agreement of the onset position as well as the asymptotic behaviour for large Rayleigh number.

Figure 3. Plot of Critical Rayleigh number as a function of Taylor number. The black points and lines are experimental results and theoretical predictions respectively from Chandrasekhar, 1961. The theoretical prediction is based on determining when the first oscillatory mode will be unstable to growth. In the asymptotic regime it has been found experimentally and theoretically that $Ra_c \sim T^{2/3}$. The red line is the prediction using our closure model. This indicates that further work needs to be done modelling the effects of rotation even though the qualitative behaviour is similar.

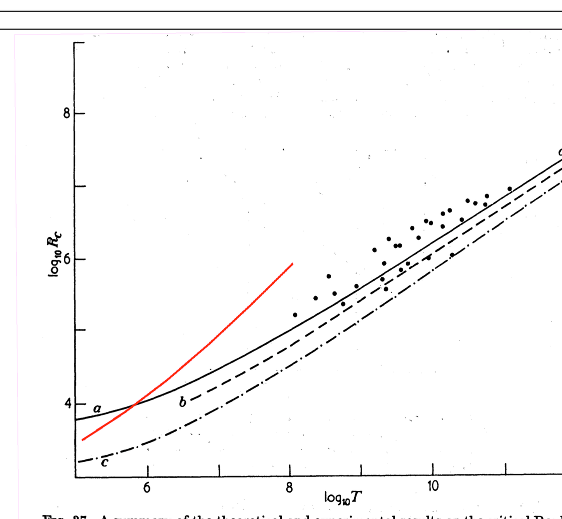


FIG. 37. A summary of the theoretical and experimental results on the critical Rayleigh numbers for the onset of instability in mercury. The curves aa, bb, and cc are the theoretical (R_c, T)-relations for the three cases (i) both bounding surfaces rigid, (ii) one bounding surface rigid and the other free, and (iii) both bounding surfaces free for p = 0·025. The solid circles are the experimentally determined points for mercury (p = 0·025).

4. Conclusions

The non-rotating case is well-described by the closure model but further progress needs to be made in modeling the effects of rotation on the onset of convection.

References

1. Chandrasekhar. Hydrodynamic and Magnetic Stability. 1961
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3. Garaud and Ogilvie. 2005, JFM, vol 530, pp. 145-176