Problem set 1 solutions

1. **Magnitudes: 15 points**
   The absolute bolometric magnitude, M, of the Sun is 4.755.

   (a) Show that that the absolute magnitude of a star with luminosity L is given by

$$M = 4.755 - 2.5 \log \left(\frac{\mathrm{L}}{\mathrm{L}_\odot} \right).$$

Solution:

The relation between magnitudes and flux is given by Hershel's calibration of 5 magnitudes as the equivalent, on a log scale, of a factor of 100 in flux. Defining flux as $L/(4\pi d^2$ and evaluating for two stars, 1 and 2, both at 10 pc and both bolometrically corrected:

$$M_{bol}(1) - M_{bol}(2) = 2.5\, log \left(\frac{L_1/4\pi(10^2)}{L_2/4\pi(10^2)} \right)$$

Taking star 1 to be some arbitrary star with absolute magnitude, M, and star 2 to be the sun:

$$\begin{aligned} M_{bol}(sun) - M_{bol}(star) &= 2.5\, log \left(\tfrac{L}{\mathrm{L}_\odot} \right) \\ 4.755 - M &= 2.5\, log \left(\tfrac{L}{\mathrm{L}_\odot} \right) \\ log \frac{L}{\mathrm{L}_\odot} &= \tfrac{1}{2.5} (4.755 - M) \end{aligned}$$

   (b) Now solve this equation for $\mathrm{L}/\mathrm{L}_\odot$ given M.

Solution:

This is given simply by taking the antilog of both sides

$$L = 10^{1.9-0.4M}\, \mathrm{L}_\odot = 79.8\, \mathrm{L}_\odot\, 10^{-0.4M}$$

(c) Hipparcos measures a parallax of a star of 0.01 arc sec. Its apparent magnitude is 8.0. Ignoring bolometric corrections, and using equations in your notes, what is the absolute bolometric magnitude of the star?

Solution:

The distance to the star in pc is 1/p, the parallax angle in arc sec, or 100 pc. From the notes M - m = 5 - 5 log (d_{pc}) so that M = 3.0

(d) What is the luminosity of the star in units of solar luminosities?

Solution:

$$L = 79.8\,10^{(-0.4)(3)}\,\mathrm{L}_\odot = 5.0\,\mathrm{L}_\odot$$

2. **Blackbody radiation: 20 points**
A spherical planet orbits an F-v star with luminosity 1.5×10^{34} erg s^{-1}. The orbit is circular with a radius from the star of 2 AU. Assume that the planet is rapidly rotating, has an atmosphere and reflects 20% of the light that falls on it, but absorbs the other 80% and, in the assumed steady state, radiates it as a blackbody.

(a) Ignoring any greenhouse effect, what is the temperature of the planet?

Solution:

The solution to the planetary tempearture comes from assuming a state of "balanced power". The energy received per second by the planet from its host star, $\dot{E}_{in}$ is balanced by the energy it radiates per second as a blackbody, $\dot{E}_{out}$. If this were not so, then the temperature of the planet would rise or fall until the balance was achieved. This would take something like the heat capacity of the atmosphere divided by E_{in} or in the case of the earth a few weeks.

The energy in is the cross sectional area of the planet as viewed from the star, a disk with radius R_p intersects the radiation that would have passed througha disk of area πR_p^2 (not $2\pi R_p^2$, then one would have to integrate $\cos\theta$ for the incident radiation, which adds work). The flux passing through each square cm of the disk is $L_*/(4\pi d^2)$ where d is the distance from the star to the planet. Additionally it was specified that only 80% of the flux gets through to the planets surface and contributes to its warming, so

$$\dot{E}_{in} = 0.8\frac{L_*}{4\pi d^2}(\pi R_p^2)$$

The energy radiated depends on the temperature of the surface of the planet, T_p - to be solved for - and the emitting area. If a rapid rotator with an atmosphere the temperature is pretty much the same wordwide so the emitting area is $4\pi R_p^2$ and

$$\dot{E}_{out} = (4\pi R_p^2)(\sigma T_p^4)$$

Setting $\dot{E}_{in} = \dot{E}_{out}$, we see that the planet's radius drops out. It doesn't matter if its a rapidly spinning baseball or an earth-sized planet, and

$$\begin{aligned} 0.8\frac{L_*}{4\pi d^2} &= (4)(\sigma T_p^4) \\ T_p &= \left(\tfrac{0.8\,L_*}{16\pi\sigma d^2}\right)^{1/4} \\ &= 262\,\mathrm{K} = -11\,\mathrm{C} \end{aligned}$$

(b) Does the radius of the planet matter? Why or why not?

No, as was shown

(c) At what wavelength does the planet emit most of its radiation?

Solution:

Use Wiens law as given in the notes.

$$\begin{aligned} \lambda_{max} &= \tfrac{0.28978\ \mathrm{cm}}{T} = \tfrac{0.28978\ \mathrm{cm}}{262} \\ &= 1.1 \times 10^{-3}\ \mathrm{cm} = 11\ \text{micrometers or}\ \mu\mathrm{m} \end{aligned}$$

(d) What would the temperature on the bright side be if the planet was tidally locked and kept the same face pointed at the star, had no atmosphere, and absorbed 100% of the incident light? Express your answer in Centigrade (C).

Solution

Now there is a bright side and a dark side. The planet only emits over one half of its area which we assume - approximately - to have the same temperature. Also assuming 100% of the incident radiation.

$$\begin{aligned} (1.0)\frac{L_*}{4\pi d^2} &= (2)(\sigma T_p^4) \\ T_p &= \left(\tfrac{L_*}{8\pi\sigma d^2}\right)^{0.25} \\ &= 329K = 56C \end{aligned}$$

3. **Measuring Stellar Masses Using Spectroscopic Binaries - 25 points**
For spectroscopic binaries, we can directly observe the maximum line-of-sight velocities $v_{1,\mathrm{LOS}}$ and $v_{2,\mathrm{LOS}}$ of the two stars, and their orbital period P. Given this information, we want to calculate the masses of the two stars, M_1 and M_2. For simplicity we will assume that the orbit is circular, with semi-major axis a. The orbital plane of the binary is inclined at an unknown angle i relative to the plane of the sky, where $i = 0$ corresponds to an orbit that is perfectly face-on and $i = 90°$ to one that is perfectly edge-on.

(a) In terms of M_1, M_2, and a, calculate the velocities v_1 and v_2 of the two stars about their common center of mass.

Solution:

Let the stars be at distances r_1 and r_2 from the center of mass; clearly we have

$$\begin{aligned} M_1 r_1 &= M_2 r_2 \\ r_1 + r_2 &= a \\ \frac{Gm_1}{a^2} &= \frac{v_2^2}{r_2} \\ \frac{Gm_2}{a^2} &= \frac{v_1^2}{r_1}. \end{aligned}$$

The first two equations give

$$\begin{aligned} r_1 &= \frac{M_2}{M_1 + M_2} a \\ r_2 &= \frac{M_1}{M_1 + M_2} a. \end{aligned}$$

Since the stars are always separated by a distance a, star 1 feels a gravitational acceleration $\ddot{r}_1 = GM_2/a^2$ and star 2 feels an acceleration $\ddot{r}_2 = GM_1/a^2$. Equating the gravitational acceleration of star 1 with the centripetal acceleration required to keep it in circular motion gives

$$\begin{aligned} \frac{GM_2}{a^2} &= \frac{v_1^2}{r_1} = \frac{v_1^2}{a}\left(\frac{M_1 + M_2}{M_2}\right) \\ v_1 &= \sqrt{\frac{GM_2}{a}}\sqrt{\frac{M_2}{M_1 + M_2}}. \end{aligned}$$

Using the exact same argument for star 2 gives

$$v_2 = \sqrt{\frac{GM_1}{a}}\sqrt{\frac{M_1}{M_1 + M_2}}.$$

(b) Calculate the orbital period P in terms of M_1, M_2, and a.

Solution:

The period is the circumference of the orbit divided by the velocity. For star 1, this is

$$P = \frac{2\pi r_1}{v_1} = 2\pi\sqrt{\frac{a^3}{G(M_1 + M_2)}}.$$

Of course repeating the calculation for star 2 gives the same value. This is just Kepler's third law.

(c) In terms of v_1, v_2, and i, what is the largest component of each star's velocity that will lie along our line of sight? We will call these $v_{1,\mathrm{LOS}}$ and $v_{2,\mathrm{LOS}}$. You may neglect the constant offset to $v_{1,\mathrm{LOS}}$ and $v_{2,\mathrm{LOS}}$ that comes from the motion of the binary's center of mass relative to Earth.

Solution:

This is just vector geometry. If the magnitude of the velocity vector for star 1 is v_1, and the orbit is rotated relative to the line of sight by angle $90° - i$ (since $i = 0$ corresponds to the vector being perfectly perpendicular to the line of sight), then the component along the line of sight is $v_{1,\mathrm{LOS}} = v_1 \cos(90° - i) = v_1 \sin i$. The same argument shows that $v_{2,\mathrm{LOS}} = v_2 \sin i$.

(d) Use your answers to the previous parts to calculate M_1 and M_2 in terms of the observed quantities and i.

Solution:

In this problem we have three unknowns, M_1, M_2, and a, constrained by three equations that we derived in previous parts of the problem:

$$\begin{aligned}
\frac{v_{1,\mathrm{LOS}}}{\sin i} &= \sqrt{\frac{GM_2}{a}}\sqrt{\frac{M_2}{M_1+M_2}} \\
\frac{v_{2,\mathrm{LOS}}}{\sin i} &= \sqrt{\frac{GM_1}{a}}\sqrt{\frac{M_1}{M_1+M_2}} \\
P &= 2\pi\sqrt{\frac{a^3}{G(M_1+M_2)}}.
\end{aligned}$$

It is easiest to solve for the unknowns if we let $M = M_1 + M_2$ be the total system mass. Taking the ratio of the first two equations immediately shows that $v_{1,\mathrm{LOS}}/v_{2,\mathrm{LOS}} = M_2/M_1$, so we can write

$$M = M_1 + M_2 = \left(\frac{v_{1,\mathrm{LOS}} + v_{2,\mathrm{LOS}}}{v_{2,\mathrm{LOS}}}\right) M_1 = \left(\frac{v_{1,\mathrm{LOS}} + v_{2,\mathrm{LOS}}}{v_{1,\mathrm{LOS}}}\right) M_2.$$

Solving for M_2 in this equation and substituting into the equation for $v_{1,\mathrm{LOS}}$ gives

$$\begin{aligned}
\frac{v_{1,\mathrm{LOS}}}{\sin i} &= \sqrt{\frac{GM}{a}}\left(\frac{v_{1,\mathrm{LOS}}}{v_{1,\mathrm{LOS}} + v_{2,\mathrm{LOS}}}\right) \\
\frac{v_{1,\mathrm{LOS}} + v_{2,\mathrm{LOS}}}{\sin i} &= \sqrt{\frac{GM}{a}}
\end{aligned}$$

If we now substitute this expression for $\sqrt{GM/a}$ into our equation for the period, we can re-arrange to get

$$a = \frac{P}{2\pi}\left(\frac{v_{1,\mathrm{LOS}} + v_{2,\mathrm{LOS}}}{\sin i}\right),$$

which gives the semi-major axis a solely in terms of observables and $\sin i$. Solving the equation for the period to isolate M and substituting in this value for a gives

$$M = \frac{4\pi^2 a^3}{GP^2} = \frac{P}{2\pi G}\left(\frac{v_{1,\mathrm{LOS}} + v_{2,\mathrm{LOS}}}{\sin i}\right)^3.$$

Finally, we can write the masses of each of the two stars using our relationship $v_{1,\mathrm{LOS}}/v_{2,\mathrm{LOS}} = M_2/M_1$:

$$\begin{aligned}
M_1 &= \frac{P}{2\pi G}\left[\frac{v_{2,\mathrm{LOS}}(v_{1,\mathrm{LOS}} + v_{2,\mathrm{LOS}})^2}{\sin^3 i}\right] \\
M_2 &= \frac{P}{2\pi G}\left[\frac{v_{1,\mathrm{LOS}}(v_{1,\mathrm{LOS}} + v_{2,\mathrm{LOS}})^2}{\sin^3 i}\right]
\end{aligned}$$

(e) Some spectroscopic binary star systems have eclipses, where one stars blocks the light of the other. Why would these systems particularly useful for measuring stellar masses?

Solution:

If one star passes in front of the other then, unless one of the stars has a radius comparable to the semi-major axis of the orbit, we know that the inclination must be close to $i = 90°$, i.e. the system is close to edge-on. In this case we know that $\sin^3 i \sim 1$, and we can measure the masses almost exactly, rather than up to an unknown inclination.

4. **Hydrostatic Equilibrium and the Virial Theorem - 25 points**

Suppose that a star of mass M and radius R has a density distribution $\rho(r) = \rho_c(1 - r/R)$, where ρ_c is the density at the center of the star. (This isn't a particularly realistic density distribution, but for this calculation that doesn't matter.)

(a) Calculate ρ_c in terms of M and R. For all the remaining parts of the problem, express your answer in terms of M and R rather than ρ_c.

Solution:

The mass and density are related by

$$M = \int_0^R 4\pi r^2 \rho(r)\, dr = \int_0^R 4\pi r^2 \rho_c \left(1 - \frac{r}{R}\right) dr = 4\pi\rho_c \left(\frac{R^3}{3} - \frac{R^3}{4}\right) = \frac{1}{3}\pi R^3 \rho_c$$

Inverting this relationship,

$$\rho_c = \frac{3M}{\pi R^3}$$

and

$$\rho(r) = \frac{3M}{\pi R^3}\left(1 - \frac{r}{R}\right).$$

(b) Calculate the mass $m(r)$ interior to radius r.

Solution:

This is the same as the integral for the previous part, but integrating to some radius $r < R$:

$$m(r) = m_r = \int_0^r 4\pi r'^2 \rho_c \left(1 - \frac{r'}{R}\right) dr' = 4\pi\rho_c \left(\frac{r^3}{3} - \frac{r^4}{4R}\right) = \frac{4}{3}\pi\rho_c r^3 \left(1 - \frac{3r}{4R}\right).$$

Substituting in the expression for ρ_c from part (a) gives

$$m_r = M\left(\frac{r}{R}\right)^3\left(4-3\frac{r}{R}\right).$$

(c) Calculate the total gravitational binding energy of the star.

Solution: Substitute m_r and dm_r into the definition of the gravitational potential energy (i.e., binding energy) and define $x = r/R$.

$$\begin{aligned}
\Omega &= -\int_0^M \frac{Gm_r}{r}dm_r \\
&= -12\frac{GM^2}{R}\int_0^1 x^4(4-3x)(1-x)\,dx \\
&= -12\frac{GM^2}{R}\int_0^1 (4x^4-7x^5+3x^6)\,dx \\
&= -12\frac{GM^2}{R}\left(4\frac{x^5}{5}-7\frac{x^6}{6}+3\frac{x^7}{7}\right) \\
&= -12\frac{GM^2}{R}\left(\frac{168-245+90}{210}\right) \\
&= -\left(\frac{26}{35}\right)\frac{GM^2}{R}
\end{aligned}$$

(d) Using hydrostatic equilibrium, calculate the pressure $P(r)$ at radius r. You may assume that the $P(R) = 0$.

Solution:

Hydrostatic equilibrium requires that

$$\begin{aligned}
\frac{dP}{dr} &= -\rho\frac{Gm_r}{r^2} \\
&= -\left(\frac{3}{\pi}\right)\frac{GM^2}{R^5}\left(1-\frac{r}{R}\right)\left(\frac{r}{R}\right)\left(4-3\frac{r}{R}\right)
\end{aligned}$$

Integrating both sides from $r' = r$ to $r' = R$, we obtain

$$
\begin{aligned}
\int_{P(r)}^{P(R)} dP &= -\left(\frac{3}{\pi}\right)\frac{GM^2}{R^5}\int_r^R \left(1-\frac{r'}{R}\right)\left(\frac{r'}{R}\right)\left(4-3\frac{r'}{R}\right) dr' \\
P(R) - P(r) &= -\left(\frac{3}{\pi}\right)\frac{GM^2}{R^4}\int_x^1 x(4-3x)(1-x)dx \\
P(r) &= -\left(\frac{3}{\pi}\right)\frac{GM^2}{R^4}\left(\frac{4x^2}{2}-\frac{7x^3}{3}+3\frac{x^4}{4}\right)_x^1, \\
P(r) &= -\left(\frac{3}{\pi}\right)\frac{GM^2}{R^4}\left(2-\frac{7}{3}+\frac{3}{4}-\frac{4x^2}{2}+\frac{7x^3}{3}-3\frac{x^4}{4}\right),
\end{aligned}
$$

where again we have defined $x = r/R$. Since $P(R) = 0$ and it is important to note that both extremes of the integral on the right must be evaluated and the difference. Since $P(R) = 0$, we obtain

$$
\begin{aligned}
P(r) &= \left(\frac{3}{\pi}\right)\frac{GM^2}{R^4}\left(\frac{5}{12}-2x^2+7\frac{x^3}{3}-3\frac{x^4}{4}\right) \\
P(r) &= \left(\frac{1}{4\pi}\right)\frac{GM^2}{R^4}\left(5-24x^2+28x^3-9x^4\right).
\end{aligned}
$$

You could have solved this portion by integrating from 0 to r but then you would have the central pressure as an additive constant that you would need to evaluate by doing an integral from 0 to R.

(e) Assume that the material in the star is a monatomic ideal gas. Calculate the total internal energy of the star from $P(r)$, and show that the virial theorem is satisfied.

Solution:

The internal energy per unit mass is

$$u = \frac{3}{2}\frac{P}{\rho},$$

so the total internal energy is

$$\begin{aligned}
U &= \int_0^R 4\pi r^2 \rho u \, dr = 6\pi \int_0^R r^2 P(r) \, dr \\
&= \left(\frac{3}{2}\right) \frac{GM^2}{R} \int_0^1 x^2 \left(5 - 24x^2 + 28x^3 - 9x^4\right) \, dx \\
&= \left(\frac{3}{2}\right) \frac{GM^2}{R} \left(\frac{5}{3} - \frac{24}{5} + \frac{28}{6} - \frac{9}{7}\right) \\
&= \left(\frac{3}{2}\right) \frac{GM^2}{R} \left(\frac{52}{210}\right) \\
&= \left(\frac{13}{35}\right) \frac{GM^2}{R}.
\end{aligned}$$

This is clearly $-1/2$ of $\Omega = -(26/35)GM^2/R$, so the virial theorem is satisfied.

5. **Powering Jupiter by Gravity - 15 points**
The Sun is in thermal equilibrium because its thermal timescale is short compared to its age. However, smaller objects need not be in thermal equilibrium, and their radiation can be powered entirely by gravity.

   (a) Jupiter radiates more energy than it receives from the Sun by 8.7×10^{-10} $L_\odot$. Jupiter's radius is 7.0×10^4 km and its mass is 1.9×10^{30} g. Compute its Kelvin Helmholz timescale. Could gravitational contraction power this luminosity for Jupiter's entire lifetime of 4.5 Gyr?

   Solution:
   The Kelvin-Helmholz timescale is

   $$t_{\rm KH} = \frac{GM^2}{RL} = 1.0 \times 10^{19} \text{ s} = 320 \text{ Gyr}.$$

   This is vastly longer than 4.5 Gyr, so gravitational contraction could easily power this luminosity for 4.5 Gyr.

   (b) Use conservation of energy to estimate the rate at which Jupiter's radius is shrinking to power this radiation. You may ignore the factor of order unity that arises from Jupiter's unknown density distribution.

   Solution:

Jupiter is in hydrostatic balance; so using the virial theorem and conservation of energy we showed that when, as is the case for Jupiter, kinetic energy changes more slowly than internal and gravitational potential energy and no nuclear reactions are occurring,

$$L \simeq -\frac{\dot{\Omega}}{2} .$$

Then, since

$$\Omega \simeq -\frac{GM^2}{R}$$

and

$$\dot{M} \simeq 0 ,$$

$$\dot{\Omega} \simeq \frac{GM^2}{R^2}\dot{R} .$$

Therefore, dropping the factor of one-half,

$$\dot{R} \simeq -\frac{R^2 L}{GM^2} = -6.9 \times 10^{-10} \text{ cm s}^{-1} = -0.022 \text{ cm yr}^{-1} .$$

6. **The thirsty professor - not required, but could be substituted, at the time of submission, for one of the above**

The professor likes to drink wine and prefers it chilled to 10 degrees C (283 K). The wine and bottle have a mass, m, of 1000 gm and a surface area of 700 cm^2. The heat capacity of the wine and bottle together is $C_P = 4 \times 10^7$ erg gm^{-1} K^{-1}. The total heat content of the bottle of wine at temperature, T, is assumed to be C_Pm T. The wine and bottle are good conductors and maintain a uniform temperature throughout. Initially the wine is at room temperature, 20 degrees C (293 K).

(a) The professor dreams he is an astronaut in space and puts the wine out on the dark side of the space craft where it is in a vacuum at nearly 0 K. Calculate how long it takes the wine to cool to 10 C. (an integral will be necessary for an accurate answer).

Solution:

Because the wine radiates into a vacuum with close to zero temperature we can beglect any "back radiation". The energy loss, which is area times blackbody flux, is powered by a decrease in the internal heat content which is carried by (assumed instantaneous) conduction to the bottle's edge.

$$\left(\frac{dE}{dt}\right)_{\text{loss}} = (A)(\sigma\, T^4)$$

This is provide by decreasing the internal energy so

$$\begin{aligned}
(C_P)(m)\frac{dT}{dt} &= -A(\sigma\ T^4) \\
\int_{293}^{283}\frac{dT}{T^4} &= -\tfrac{A\sigma}{C_P m}\int_0^t dt \\
-\frac{1}{3}\left(\frac{1}{283^3}-\frac{1}{293^3}\right) &= -\tfrac{A\sigma}{C_P m}t \\
4.37\times10^{-9}\frac{C_P m}{A\sigma} &= t \\
t &= \tfrac{1.45\times10^{-9}(4\times10^7)(1000)}{(700)(5.67\times10^{-5})} \\
t &= 1460 \text{ sec}
\end{aligned}$$

or 24 minutes.

(b) The professor wakes up, but is still thirsty, so he puts the same wine, initially at 20 C, in a freezer where the temperature is -10 C (263 K). How how long does it take the wine to cool down to +10 C (283 K)? An approximate answer will get substantial credit, but an accurate one requires a quartic integral (dx/(x^4 - a^4)) that can be evaluated at mathportal.org (http://www.mathportal.org/calculators/calculus/integral-calculator.php). If evaluating an arctan on your calculator, be sure it is set to radians mode and not degrees.

Solution:

The solution is very similar to before, but now radiation is being absorbed from the "freezer", as well as emitted

$$\left(\frac{dE}{dt}\right)_{loss} = -A\sigma\ (T^4_{\text{bottle}} - T^4_{\text{freezer}})$$

This time the integral is more complicated

$$\begin{aligned}
(C_P)(m)\frac{dT}{dt} &= -A\sigma(T^4-(273)^4) \\
\int_{293}^{283}\frac{dT}{T^4-273^4} &= -\left[(A\sigma)/(C_P m)\right]t
\end{aligned}$$

Fortunately the integral can be evaluated at mathportal (use class link) or www.integral-calculator.com. Both give the indefinite integral

$$\int dx/(x^4-a^4) = \frac{1}{4a^3}\left(-log(x+a)+log(x-a)-2\text{Arctan}(x/a)\right)$$

which could be evaluated - perilously - by hand or do the definite integral at the same sites

$$\int_{293}^{283} \frac{dx}{x^4 - 273^4} = -7.87 \times 10^{-9}$$

So

$$t = 7.87 \times 10^{-9} \frac{C_P m}{A\sigma} = 7900 \text{ sec}$$

or 2.2 hours. [postscript: the professor has done many such experiments and finds that the cooling time is less than 30 minutes in the freezer. Apparently conduction and convection, in the air at the bottle's edge, neglected here, are important.]