Homework 3 Solutions

1) Liquid drop model

Dropping terms in symmetry energy, spin, and shell corrections

$$BE = a_1 A - a_2 A^{2/3} - a_3 \left(\frac{Z^2}{A^{1/3}}\right)$$

a) the first term is the volume term. Given the short range of the nuclear force, binding energy is linear in the total number of nucleons. This is the dominant term

- $a_2 A^{2/3}$ is a correction for "sufzce energy". Nucleons near the surface don't bind to as many neighbors. Analog to surface tension in a liquid drop. Depends on $4\pi R^2$ of the nucleus and $R \propto A^{1/3}$

- $a_3 (Z^2/A^{1/3})$ is the Coulomb energy that accounts for the mutual repulsion of the protons. Electrical potential energy goes like $Z^2/R \propto Z^2/A^{1/3}$. Reduces the overall binding energy.

b) $$BE/A = a_1 - \frac{a_2}{A^{1/3}} - \frac{a_3 Z^2}{A^{4/3}}$$

$a_2 = 18.56$

$a_3 = 0.717$

assume $Z = A/2$

$$BE/A = a_1 - \frac{a_2}{A^{1/3}} - \frac{a_3}{4} A^{2/3}$$

$$\frac{d(BE/A)}{dA} = 0 = \frac{1}{3} a_2 \frac{1}{A^{4/3}} - \frac{2}{3}\frac{a_3}{4} A^{-1/3}$$

$$\frac{a_2}{3A} = \frac{a_3}{6}$$

$$\boxed{A = \frac{2a_2}{a_3} = \frac{2(18.56)}{.717} = 52}$$

c) Nuclei heavier then $A = 52$ can release energy by fission and α-decay until they reach $A = 52$ (subject to conserving the total number of nucleons). Eg $A = 104$, energetically at least, could decay to $2 * A = 52$. In practice, this may take a very long time ($\gg \tau_{Hubble}$)

Nuclei lighter then 52 can release energy by combining with one another by fusion, again subject to conserving A.

For the premises stated - no symmetry energy, ignore shell & pairing, $A = 52$ is the most tightly bound nucleus in terms of energy per nucleon. Heavier nuclei can and do have greater BE but not greater BE/A

d) ^{56}Ni is actually the most tightly bound nucleus for $Z = N$ nuclei. We did not get that because we ignored shell effects and ^{56}Ni is "double magic" $Z = N = 28$ and 28 is a closed shell

2) Nuclear energy release $\quad 2(^{12}C) \rightarrow {}^{24}Mg$

$$q_{nuc} = 9.65 \times 10^{17} \sum (\delta Y_i)(BE_i) \quad \text{erg gm}^{-1}$$

$$= 9.65 \times 10^{17} \left(\frac{1}{24} (198.258) - \frac{1}{12} (92.162) \right)$$

$q_{nuc} = 9.65 \times 10^{17} \; (0.5806) = \boxed{5.60 \times 10^{17} \text{ erg gm}^{-1}}$

initially the ^{12}C mass fraction was assumed to be 100% so $Y_{initial}(^{12}C) = 1/12$ $\quad Y_{initial}(^{24}Mg) = 0$

It was assumed to burn entirely to ^{24}Mg so

$$Y_{final}(^{12}C) = 0 \qquad Y_{final}(^{24}Mg) = \frac{1}{24}$$

In nature the mass fraction of carbon initially would have been < 1, maybe ≈ 0.20.

3) $^{12}C(\alpha,\gamma)^{16}O$ at 1.5×10^{8} K $\qquad \alpha = {}^{4}He$

S factor $= 0.17$ or 0.27

a) Most reactions occur at the Gamow energy

$$E_0 = 0.122 \left(Z_1^2 Z_2^2 \hat{A} \, T_9^2 \right)^{1/3} \text{ MeV}$$

$$\hat{A} = \frac{12 \cdot 4}{12+4} = \frac{48}{16} = 3$$

$$E_0 = 0.122 \left(6^2 \, 2^2 \, 3 \, (\underset{0.15}{\cancel{1.5}})^2 \right)^{1/3}$$

$$= 0.260 \text{ MeV} = \boxed{260 \text{ KeV}}$$

$$KT = \left(0.08617 \frac{\text{MeV}}{T_9}\right)(0.15) = \boxed{12.9 \text{ KeV}}$$

not asked for

$$\Delta = 0.237 \left(6^2 \, 2^2 \, 3 \, (0.15)^5 \right)^{1/6} \text{ MeV}$$

$$= 0.134 \text{ MeV} = 134 \text{ KeV}$$

3b) $S(E_0) = 0.17$ MeV b

$$\lambda_{\alpha\gamma}(^{12}C) = N_A \langle \sigma v \rangle_{\alpha\gamma}$$

$$= \frac{4.34 \times 10^8}{\hat{A} Z_1^2 Z_2^2} S(E_0) \tau^2 e^{-\tau} \frac{cm^3}{\text{Mole s}}$$

$$\tau = \frac{3E_0}{kT} = 4.248 \left(\frac{Z_1^2 Z_2^2 \hat{A}}{T_9} \right)^{1/3}$$

$$= 4.248 \left(\frac{6^2 2^2 3}{0.15} \right)^{1/3} = 60.44$$

$$\lambda_{\alpha\gamma} = \frac{4.34 \times 10^8}{(3)(6)(2)} (0.17)(60.44)^2 e^{-60.44}$$

$$\boxed{\lambda_{\alpha\gamma} = 4.22 \times 10^{-17}}$$

c) Lifetime

$$\tau_{\alpha\gamma} = \left(\frac{1}{Y_{12}} \frac{dY_{12}}{dt} \right)^{-1} \qquad \rho = 10^4 \text{ g cm}^{-3}$$

$$Y_\alpha = 1/4$$

$$\frac{dY_{12}}{dt} = \cdots - \lambda_{\alpha\gamma} Y_\alpha \rho Y_{12}$$

$$\tau_{\alpha\gamma} = \left(\frac{1}{\rho Y_\alpha \lambda_{\alpha\gamma}} \right) = \frac{1}{(0.25)(10^4)(4.22 \times 10^{-17})}$$

$$= 9.5 \times 10^{12} \text{ s} = \boxed{300{,}000 \text{ years}}$$

d) $\lambda \propto T^n$

$$n = \frac{\tau - 2}{3} = \frac{60.44 - 2}{3} = 19.5$$

$$\boxed{\lambda_{\alpha\gamma} \propto T^{19.5}}$$

4) Star formation

$\mu = 2.33$ $\quad n = 10^4$ $\quad T = 20\ K$

a) Jeans mass

$$M_J = \frac{44\ M_\odot}{\mu^2}\ \frac{T^{3/2}}{n^{1/2}} = \frac{44}{(2.33)^2}\ \frac{20^{3/2}}{10^2}$$

$$\boxed{M_J = 7.2\ M_\odot}$$

b) The relevant scale is hydrodynamic. Pressure offers little resistence

$$\tau_{HD} = \frac{2680.}{\sqrt{\rho}}$$ or some variant thereon

but ρ is gm/cm^3

$$\rho = \mu\ {}^{n}/N_A \qquad n = \rho N_A/\mu$$

$$= (2.33)(10^4/6.02 \times 10^{23}) = 3.9 \times 10^{-20}$$

$$\tau_{HD} = \frac{2680.}{(3.9 \times 10^{-20})^{1/2}} = 1.4 \times 10^{13}\ sec$$

$$= \boxed{4.3 \times 10^5\ years}$$

c) Accretion rate + luminosity (your numbers may vary if you changed ρ)

$$\frac{dM}{dt} = \frac{0.5\ M_\odot}{\tau_{HD}}$$

$$= \boxed{7.31 \times 10^{19}\ gm/s} = 1.2 \times 10^{-6}\ M_\odot/yr$$

$$L = \frac{GM}{R}\ \frac{dM}{dt} \qquad R = 5\ R_\odot$$

$$M = 0.5\ M_\odot$$

$$L_{acc} = \frac{(6.67\times10^{-8})(1\times10^{33})}{(6.96\times10^{10})(5)} \quad (7.31\times10^{19})$$

$$= 1.4\times10^{34} \text{ erg/s} \quad = \boxed{3.6\ L_\odot}$$

d) $L = 4\pi R^2 \sigma T^4 \qquad T = \left(\frac{L}{4\pi\sigma R^2}\right)^{1/4}$

$$T = \left[\frac{1.4\times10^{34}}{(4\pi)(25)(6.96\times10^{10})^2(5.67\times10^{-5})}\right]^{1/4}$$

$$= \boxed{3570\ K}$$

e) $\lambda = \frac{2.89\times10^7 \text{ Å}}{3570} = \boxed{8100\text{ Å}}$

red to infrared

f) Obviously it is τ_{KH} (I said so!)

$\tau_{KH}(\odot) \approx \boxed{30\text{ My}}$

g) Nuclear time scale / main sequence lifetime

$\boxed{10^{10}\text{ yr}}$

5) Minimum ignition mass

$$P_c = C_n G M^{2/3} \rho_c^{4/3}$$

$$\frac{\rho_c N_A k T_c}{\mu} = 0.48\ G M^{2/3} \rho_c^{4/3}$$

$$\boxed{T_c = \frac{0.48\ G M^{2/3} \mu}{N_A k} \rho_c^{1/3}}$$

5) continued

NR deg. $P = 1.00 \times 10^{13} (\rho Y_e)^{5/3}$

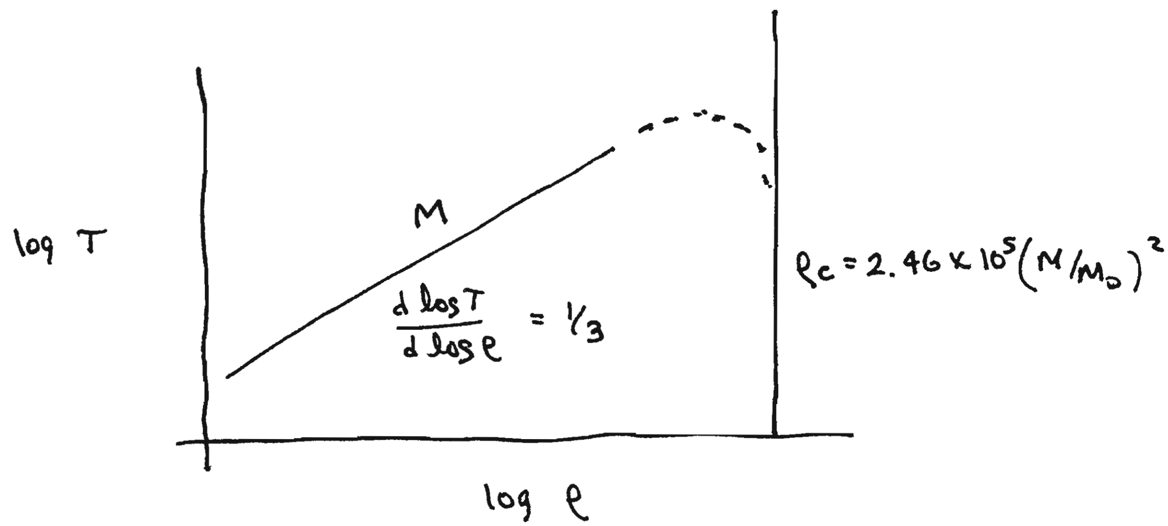
$$1.00 \times 10^{13} (\rho_c\, 0.88)^{5/3} = 0.48\, G M^{2/3} \rho_c^{4/3}$$

$$\rho_c^{1/3} = \frac{0.48\, G M^{2/3}}{(10^{13})(0.88)^{5/3}} = 3.96 \times 10^{-21} M^{2/3}$$

$$\rho_c = 6.22 \times 10^{-62} M^2$$

$$\boxed{= 2.46 \times 10^5 (M/M_\odot)^2 \text{ gm cm}^{-3}}$$

e)



d) From a) + b)

$$\rho_c^{1/3} = \frac{N_A k T_c}{0.48\, G M^{2/3} \mu} = \frac{0.48\, G M^{2/3}}{10^{13}\, 0.88^{5/3}}$$

$$\cancel{\rho_c = \frac{(N_A k T)^3}{(0.48 G \mu)^3} \frac{1}{M^2} = \left(\frac{0.48\, G}{10^{13}}\right)^3 \frac{1}{0.88^5} M^2}$$

$$T_{max} = \frac{(0.48\, G)^2 \mu}{N_A k\, 10^{13}\, 0.88^{5/3}} M^{4/3}$$

$$\boxed{= 2.3 \times 10^8 (M/M_\odot)^{4/3}}$$

class notes give a value 4 x smaller because the pressure is partitioned 50% deg + 50% ideal. Both are approximate

5) continued

For ideal gas, from homology

e) $$\boxed{T_c = 15.7 \times 10^{6} \left(\frac{\mu}{0.6}\right)\left(M/M_\odot\right)^{0.57}}$$

set equal

$$2.3 \times 10^{8} \left(M/M_\odot\right)^{4/3} = 1.57 \times 10^{7} \left(M/M_\odot\right)^{0.57}$$

$$M/M_\odot^{(1.33-0.57)} = \frac{1.57}{23}$$

$$\left(M/M_\odot\right)^{0.76} = 0.068$$

f) $$\boxed{M/M_\odot = 0.029}$$

0.08 is correct natural value

g) skip see class notes

6) $L_\odot = 3.84 \times 10^{33}$ erg/s

each $4H \rightarrow {}^4He$ releases $28.296 - 2.09 = 26.2$ MeV

1 MeV = 1.602×10^{-6} erg

each reaction releases $(26.2)(1.602 \times 10^{-6}) = 4.20 \times 10^{-5}$ erg

each reaction releases 2 neutrinos

$$L_\nu = \frac{3.84 \times 10^{33}}{4.20 \times 10^{-5}} * 2 = 1.8 \times 10^{38}\ \nu/s$$

a) $$\phi_\nu (1\ AU) = \frac{1.8 \times 10^{38}}{(4\pi)(1.5 \times 10^{13})^2} = \boxed{6.4 \times 10^{10}\ \frac{\nu}{cm^2\,s}}$$

b) day or night does not matter