Ay 112 Midterm review

In general, read thoroughly all the notes on the web. The questions will be based on the material covered in the notes, not so much from the accompanying reading, except as it complements the notes. This is not a guarantee, but a guideline. Pay close attention to all equations in the notes that are highlighted in yellow. For the open notes part of the exam, review the two HW solutions, though again not all questions will have counterparts in the homework.

The closed notes test will cover terminology, concepts, definitions, simple plots. You need to know "simple" equations like flux, Wiens law, parallax, magnitude, Kepler's third law, Virial theorem, etc, but not to solve problems like in the homework. The open notes part is for that.

Brief incomplete summary of the course so far:

Lecture 1

Age of the earth and sun 4.55 Gy and that radioactive decay is the best way of getting it. Two isotopes of uranium are involved. You don't need to remember their numbers of lifetimes.

Most stars are less massive and less luminous than the sun. On the main sequence L is proportional to M**3. The lifetime thus goes as 1/M**2. The leftime of the sun is 10 Gy.

Open clusters and globular clusters. The former are groups of stars recently born. The latter are very old and contain red, low mass stars, plus red giants, horizontal branch stars, and white dwarfs.

The sun is not at the center of the Milky Way but about 8.5 kpc out. The total mass of the galaxy is about 10**12 Msun, but only about 2 x 10**11 is within the solar orbit. The mass is obtained using Kepler's third law.

There are two kinds of stars – Population 1 and 2. Pop 2 is older. Properties

Pop 1	Pop 2
Low and high M	low M
Young	old
Red and blue	red
Faint and bright	faint
Found in disk	Often found above disk in halo
Metal content like the sun	lower metal content, typically by 10

Low v perpendicular to disk   High v

Distances by parallax – will be good with Gaia to most stars in our galaxy. Presently Hipparcos parallaxes are the best, about a miliarcsecond or more (7 mas is typical).

d = 1/p   the parallax angle in arc sec; distance in parsecs
Measuring extragalactic distances and distances over 1 kpc (for now) requires standard candles. Flux = L/(4 pi d**2), know L, measure flux get d.

Magnitudes – 5 magnitudes = factor of 100 in flux. Each magnitude is a 100**0.2 change in flux. Absolute magnitude, M, is magnitude,m, if the star is at 10 pc.
M-m = 5 -5 log d with d in pc. The bigger m, the farther away the star.

A useful standard candle for distances to nearby galaxies is Cepheid variables. Cepheids have a relation between their period of variation and their average luminosity. They can be used out to about 20 Mpc (the Virgo cluster). There are Several varieties of Cepheids and the P-L relation is different for Population I and II. This caused historical errors in the estimated size and age of the universe. Cepheids vary because of a changing transparency of their outer layers. The envelope pulses, the core doesn't. The instability involves the ionization of He+ to He++. Cepheid variables are not main sequence stars.

For still greater distances, Type Ia supernovae can be used. They are very bright and reach almost the same brightness in each explosion. Small variations can be corrected for using the "width luminosity relation" – broader light curves are brighter at peak.

Lecture 2

The Hertzsprung Russel diagram began as a plot of color measured by B-V vs absolute magnitude (increasing B-V on the bottom axis and decrasing M on the vertical). There is a relation between B-V and T that can be obtained from the Planck function. Smaller B-V is higher T. Absolute magnitude can also be corrected to give bolometric absolute magnitude, again using the Planck function. Then the HR diagram can be plotted as luminosity (increasing) vs T_eff (decreasing). Patterns appear – the main sequence, red giants, horizonal branch stars, Cepheid variable, white dwarfs – know these systematics.

Stars are blackbody emitters. The light takes a long time to diffuse out and comes into equilibrium with the temperature of its surroundings. For the sun it takes 150,000 years to diffuse out. The continuous spectrum can be described by the Planck function. This function can be integrated to give the flux at all frequencies of light, σT^4 erg/(cm^2 s). The wavelength where the emission is a maximum is given by Wiens law, 0.289 Angstroms/T_{eff}. The luminosity of the star is given by its area times σ T^4 , that is 4 πR^2 σ T^4. This gives us a way of measuring stellar radii if we know L and T. It also describes lines of constant R in the HR diagram which is useful for understanding the distinction between red giants, white dwarfs, etc.

We get stellar masses directly from measurements of binary star systems. Once we understand things well enough, we can use the mass luminosity relation to get the masses of solitary main sequence stars.

Most stars are found in multiple systems. Only two star systems have analytic solutions. When 2 stars orbit each other they follow Kepler's laws. The orbits are ellipses with the center of mass located at a common focal point for the two ellipses. The product of mass times distance to the center of mass is the same for both stars. This defines center of mass. A line connecting the two stars (cenetrs) always passes through the center of mass. The product of mass times speed is also the same for the two stars. The square of the period is proportional to the cube of the sum of the semimajor axes of the two orbits cubed.

For circular orbits $m_1r_1= m_2r_2$; $m_1v_1= m_2v_2$; $a = r_1+r_2$; $P^2 = (4\ pi^2/(G(m_1+m_2))\ \ a^3$
Also $P = 2\ pi\ r_1/ v_1=2\ pi\ r_2/ v_2$.

When the orbit is not viewed edge on the masses derived are lower limits to the actual masses which are actually larger by a factor of $1/Sin^3(i)$ with i the inclination angle at which the orbit is observed.

Masses can be determined from the periodic Dopler shifts observed in spectroscopic binaries (those which cannot be visibly resolved but exhibit periodic time variable) spectra. If the spectroscopic binary is an eclipsing spectroscopic binary we know we are observing it in the plane of the orbit. An example was given of how to derive individual masses from observations of the period and v_1 and v_2.

The masses so determined give greater insight into the HR diagram. The main sequence is a sequence of masses with the brighter stars having higher mass. Since higher mass stars have shorter lives, the age of a cluster of stars (all of whom were born at the same time) can be determined from the "turn off mass", the brightest main sequence star still left in the cluster. Globular clusters and open clusters tus have distinctively different HR diagrams. If two open clusters have different turn off masses, the one with the heaviest main sequence star still remaining will be the younger.

Lecture 3

Stellar spectra were observed a long time before they were understood and a lot of the nomenclature like calcium H and K lines dates to that era. Even though calcium lines are strong in the sun, calcium is not as abundant as hydrogen, which has weaker lines.

For hydrogen several spectral series are seen that originate or end on the ground state (Lyman series, ultraviolet); first excited state (n = 2, Balmer series, optical); second excited state (n = 3, Paschen series, infrared), etc. An especially ubiquitous optical line is the Balmer alpha line corresponding to the 3<-> 2 transition at 6563 Angstroms (red).

Depending on the temperature, different ionization states are present and lines have different strengths in the spectrum. This gives us another way to determine the photospheric temperature (besides Wiens law and $L = 4 \pi R^2 \sigma T^4$). The spectral sequence of temperatures is OBAFGKM. The sun is a G2 star. O stars (the hottest, most massive and rarest stars) show lines of helium, A stars have the strongest hydrogen (at about 10,000 K). G stars show strong Ca II lines and weakening H lines; M stars show refractory molecules like TiO and VO. Others are in between.

Lines can be seen in emission or absorption. Most stars including all main sequence stars have absorption lines. These are created as the blackbody passes through cool overlying layers that absorb light at preferred wavelengths. Nebulae show emission lines. They are radiated from the side and re-emit the atomic radiation isotropically.

In astronomy H+ is written as H II. Neutral hydrogen is H I. Triply ionized carbon is C IV, etc.

The Saha equation can be used to calculate the ionization state of atoms in local thermodynamic equilibrium. This equation gives the abundaces of e.g., H I and H II in terms of n_e, the temperature, statistical factors, and the ionization potential. The latter enters in as an exponential and so the answer is very dependent on in. Similarly, the population of excited states in a given atom can be calculated using the Boltzmann equation. This gives the population of various levels also in terms of their statistical weights and excitation energies. Using these equations one can show that hydrogen in the stellar photosphere is almost entirely neutral, but Ca is almost entirely singly ionized. The population of the n = 2 level in H is very small and hence the Balmer hydrogen line is quite weak. On the other hand, Ca absortption lines based on the ground state of Ca II are very strong despite the very small abundance ratio of H/Ca in the sun.

The Saha equation also shows that for typical densities, hydrogen will remain neutral until about 10,000 K and then become ionized. The population of the n = 2 level increases with T until the H is ionized. This explains why H lines are strongest in A stars and weak in both M stars and O stars. Similar arguments apply to other spectral features seen on OBAFGKM stars.

As pressure increases, the density increases and atoms tend to be more neutral. Also as the density increases lines are broadened by “pressure broadening” or the Stark effect, as the quantum levels become less precise due to the electric fields of nearby passing atoms. This effect can be used to distinguish stars of the same temperature and spectral class (OBAFGKM) with different surface gravities. High gravity means, in hydrostatic equilibrium, high pressure. So main sequence stars have broader lines than red giants of the same temperature. One can this segregate stars in the HR diagram not only by T but by radius. There are thus stars of class I, II, III, IV and V. V is the main sequence. The sun is a G2 – V star. Supergiant stars are class I. The rest are in between.

Elemental abundances are measured in both the sun and in meteorites. A class of very primitive, organic bearing meteeorites called C I Carbonaceous Chondrites are the sample used. They show little signs of melting and chemical fractionation. They are however deficient in H, He, C, N, O and noble gases like Ar. They are after al, rocks. For the rest of the elements, agreement between the solar photosphere and the meteoritic abundances is quite good, usually better than 10%. This set of concurrent abundances, called the solar abundance set, is typical of Population I stars and the interstellar medium in our own and other galaxies.

Isotopic ratios come from meteorites, not the sun.

The most abundant elements are H, He, O, C, and N in that order. Interestingly except for He, they are what life is mostly made out of.

The solar abundances show strong correlation with nuclear processes, e.g., the large abundance of iron reflects the strong stability of the iron group elements. The odd-even effect in the abundances of both elements (K less abundant than Ca for example) and isotopes reflects pairing effects in the nucleus (to be discussed). Li is depleted in the sun by a large factor compared with meteorites reflecting the burn up of Li, a fragile element, in the solar convective zone.

Metal deficient stars show different abundance patters from the sun, especially a more pronounced odd-even effect and a deficiency of iron compared with oxygen. This reflects the history of different kinds of supernovae in the galaxy.

Lecture 4

The most basic stellar structure equation is the equation of hydrostatic equilibrium. For matter at rest, $dP/dr = -Gm\rho/r^2$. This is worth remembering. Also easy to remember is the mass conservation equation $dm = 4\pi r^2\rho\, dr$.

Starswill stay in hydrostatic equilibrium on a hydrodynamical times scale $\sim$ 1000 s/sqrt(ρ), less than an hour for the sun.

Gravitational binding energy is defined as Omega = - integral from 0 to M (Gm/r) dm which is some constant of order unity, α , times – GM^2/R. For a sphere of constant density α= 3/5. The total internal energy U is a similar integral over u, the internal energy per gram. For an ideal gas u = 3/2 P/ρ. For radiation and other relativistic gases u = 3 P/ρ.

The Virial theorem states that -3 integral from 0 to M (P/ρ) dm = Ω. This is true regardless of equation of state. Also the integral need not extend all the way to the surface, but can apply to portions of the star.

For an ideal gas, the Virial theorem more specifically says that $\Omega = -2$ U and thus the total energy E = Ω + U = - U = $\Omega/2$. Stars supported by ideal gas pressure have a net binding equal to their internal energy or ½ of their gravitational binding energy. The other half of the gravitational binding energy got radiated away during star formation. One can thus compute a Kelvin Helmholtz time scale for the star, $\tau = \alpha GM^2/(2RL)$ which is how long it can shine with luminosity L using gravity as a source and reaching radius R. For the sun this is about 30 My. Were nuclear reactions to go out in the sun, it would still shine at about the same L and roghly the same radius for 30 My. This is also the time scale for a star to make structural (as opposed to thermal) rearrangements.

Assuming an ideal gas equation of state, one can also derive a Virial temperature for the sun of about 3 million K. Assuming constant density

$$\frac{3MN_AkT}{2\mu} = \frac{\alpha GM^2}{R}.$$

The first law of thermodynamics describes the evolution of energy in the star

$\frac{du}{dt} + P\frac{d}{dt}\left(\frac{1}{\rho}\right) = \varepsilon - \frac{dF}{dm}$ where u is the internal energy, the second term is PdV work, ε is the internal energy creation (e.g., by nuclear reactions) and the last term is the net flux entering or leaving the matter. Integrating this equation gives essentially the global variation of these quantities in the star, e.g, the integral of ε is the total nuclear power developed by the star. An important consequence is the equation $\frac{d}{dt}\left(\frac{\Omega}{2}\right) = L_{nuc} - L$ which describes how the nuclear binding evolves in response to an imbalance in energy creation by nuclear reactions and luminosity leaving the surface of the star. If L_{nuc} is 0 for example, the binding energy becomes more negative in order to power the star. This means it gets hotter (as shown e.g., in the above Virial equation for the temperature). If the two L's are balanced, Ω doesn't change. The star' structure stays roughly the same except for a gradually evolving composition.

The time scale for the composition to evolve is the nuclear time scale, which is longest by far for hydrogen burning. Burning heavier fuels gives less energy and the stars tend to be more luminous too.

The time scale for L and L_{nuc} to come into global balance is the thermal time scale (power deposited at the center of a main sequence star equals luminosity leaving the surface). In some situations, these two may not be balanced, e..g., when the outer layers of the star are expanding and it is becoming a red giant.

In general, especially on main sequence the time scales are ordered $\tau_{hydro} << \tau_{thermal} \sim \tau_{KH} << \tau_{nuc}$. The thermal time scale for a star where heat transport is by radiative diffusion is roughly $\tau \approx R^2 / (\ell_{mfp} c) = R^2 \kappa\rho / c$ which is about 150,000 years for the sun.

Lecture 5

Stars are composed of ionized perfect gases. That the gas is completely ionized, at least the H and He is not so straightforward because the simplest Saha equation predicts partial neutrality at the solar center due to the high density. Pressure ionization makes the gas fully ionized. Again (as in the Stark effect), the electric fields from the nearby ions reduce the netionization potential making the gas easier to ionize. Except at their surfaces stars are to good approximation fully ionized gases.

The pressure integral, which you need not remember verbatim, $\frac{1}{3}\int_0^\infty \frac{dn(p)}{dp} p\, v\, dp$ tells how to evaluate the pressure in general for an isotropic "perfect" gas (one where interparticle interactions are negligible except during collisions). Different kinds of pressure arise from different assumptions from statistical mechanics about dn/dp. Ideal gas is treated using Maxwell Boltzmann (MB) statistics; radiation by Bose-Einstein (BE), Fermions by Fermi-Dirac (FD). Each distribution function contains a $4\pi p^2 dp$ for phase space, a statistical weight (usually 2) and an exponential function describing the occupation of each state. The distribution function is proportional to $\left(e^{energy/kT} \pm 1,0\right)^{-1}$ for MB (0), BE (-1), and FD (+1) statistics. The different statistics result from counting the particles differently (distibguishable or indistinguishable) and other rules like the Pauli exclusion principle for FD and the lack of particle number conservation for BE.

For an ideal gas, MB statistics is used and the particle number is conserved. The result is P = nkT. The difficulty in practice is how to evaluate n for a gas with multiple components including electrons and ions from different elements. It is convenient to define an abundance variable $Y_i = X_i/A_i$ where X is mass fraction and A the atomic mass number of the nucleus – 4 for ^{4}He, 12 for ^{12}C, etc. It is the number of neutrons plus protons in the nucleus. For hydrogen A = 1. The number density of a given ion is then $n_i = \rho N_A Y_i$ with N_A Avogadro's number, the number of hydrogen atoms in a gram of hydrogen. A corresponding value of Y_e is defined for the electrons such that $n_e = \rho N_A Y_e$. In both cases n is the number density per cm^3. With these definitions the ideal gas pressure for a fully ionized plasma can be written $P_{ideal} = \frac{\rho N_A kT}{\mu}$ with $\frac{1}{\mu} = \sum Y_i + Y_e$. Y_e is also defined by charge neutrality (each nucleus with charge Z contributes Z electrons to the gas) as $Y_e = \sum Z_i Y_i$. You need not remember these expressions for μ and Y_e for the test.

The internal energy per gram of an ideal gas is $u = \frac{3}{2}\frac{P}{\rho}$. For a relativistic gas $u = 3\frac{P}{\rho}$.

Using the BE distribution one can similarly derive radiation pressure, $P_{rad}=1/3\ a\ T^4$ (you don't need to remember the values of a, G, e, c, h, or any other fundamental constants for the test). Since radiation is relativistic $u_{rad} = 3\frac{P_{rad}}{\rho} = \frac{aT^4}{\rho}$. The number density of photons is 20 T^3 and is not conserved as the temperature is changed. The average photon has energy 2.70 kT.

Electrons (and other Fermions) are the hard part because they can be degenerate or ideal, relativistic or non-relativistic. In general they are always described by FD statistics, but the "1" in the distribution function becomes negligible at high T. Then, if their number is conserved, they are an ideal gas.

For degeneracy pressure, a simplifying assumption is complete degeneracy, all states are filled to a maximum given by the Fermi momentum and no states above that are occupied. The Fermi momentum for electrons is $p_0 = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$ (not needed for the test) and when the upper bound of the pressure integral is so restricted one gets $P_{non-rel} = K_{non-rel}\left(\rho Y_e\right)^{5/3}$ and $P_{rel} = K_{rel}\left(\rho Y_e\right)^{4/3}$ for non-relativistic and relativistic electron degeneracy pressure respectively. The values of K are in the notes and need not be remembered though the dependences on ρY_e do need to be remembered. Typically for everything but hydrogen Y_e = ½. For pure hydrogen Y_e = 1. The electrons become relativistic somewhere between a few x 10^6 g cm^{-3} and 10^7 g cm^{-3}.

There exist other approaches for solving the general case of partial degeneracy and partial relativity. In general the solutions involve Fermi integrals and have the right high T (ideal gas) and low T (degenerate) limits.

An adiabatic expansion (or compression) is one in which no energy flows across the boundary of the gas. Work can be done or absorbed by expansion and compression and internal energy can change. $du + P\, d\left(1/\rho\right) = 0$. Since $u = \phi\left(P/\rho\right)$ it follows that the pressure is a power law of the density $P \propto \rho^{\frac{\phi+1}{\phi}} = \rho^{\gamma_{ad}}$. The adiabatic exponent can be reduced in regions of partial ionization because energy used to ionize atoms does not go directly into raising the temperature. This reduction in adiabatic exponent can make these layers hydrodynamically unstable as will be discussed later.

Lecture 6

Radiation diffuses in stars because of the small temperature gradient that is present. The mean free path is the distance over which a beam of radiation would be attenuated by a factor of "e". It is $\ell_{mfp} = (n\sigma)^{-1}$ or $(\rho\kappa)^{-1}$ where κ is the opacity measured in cm^2 g^{-1}. A rough approximation to the heat transport is given by condidering two surfaces of temperature T and T + dT broadcasting blackbody radiation into each other to a depth of ℓ_{mfp} . The temperature difference, dT, is the temperature gradient dT/dr times this length scale. A more accurate derivation must integrate over angles or use Fick's Law for diffusion. Ficks Law states that the flux of diffusing radiation is $F = -D\frac{du_{rad}}{dr} = -K\frac{dT}{dr}$ where D is the diffusion coefficient, $D = \frac{1}{3}v\ell_{mfp}$ and K is the conductivity equal to the heat capacity at constant volume, C_V. The speed here is c, and the heat capacity for radiation is $C_P = \frac{du_{rad}}{dT} = 4aT^3$. This gives a flux $-\frac{4}{3}\frac{acT^3}{\kappa\rho}\frac{dT}{dr}$ and an equation for the temperature gradient required to carry a luminosity L of $\frac{dT}{dr} = \frac{-3\kappa\rho}{16\pi acT^3}\frac{L}{4\pi r^2}$. This, and its Lagrangian equivalent, is one more basic stellar structure equation. It can be inverted to give L transported by radiative diffusion for a given temperature gradient, opacity, density, etc. You do not have to remember this equation.

This derivation assumed that the opacity κ was independent of frequency. The derivation can be repeated using the Planck function to describe the distribution of blackbody radiation with frequency and the same result is obtained except that opacities with frequency dependence must be averaged over the temperature derivative of the Planck function. This "Rosseland mean opacity" is defined in the notes.

Opacity in nature can have several origins. In general it involves photons interacting with electrons, either free electrons or electrons bound in atoms. The simplest form is electron scattering opacity is frequency independent (and therefore temperature and density independent) so long as the temperature is a small fraction of the electron rest mass and the electrons are neither degenerate or relativistic. $\kappa_{es} = 0.2(1 + X_H)\ \mathrm{cm^2\ g^{-1}}$. Modifications, e.g., Klein Nishina corrections, exist to correct for high temperature effects. In a white dwarf where matter is generate the electrons lack phase space to scatter into so the electron scattering opacity is reduced there. But under the same conditions conduction generally dominates

Other opacities involve interactions between the electrons and ions. Free-free opacity involves electrons near an ion that shares the energy and momentum. Inverse process to bremsstrahlung. Depends on ν^{-3}. The Rosseland mean depends on $\rho T^{-7/2}$. Opacities with this dependence on temperature and density are called "Kramer's opacity". Free free opacity also depends on the ion abundance ad is reduced if the metallicity is low.

Bound-bound and bound free opacity involve electrons in the bound states of stoms. They are complicated to calculate, but so long as the photon energy is comparable to or greater than the ionization energy, bound free opacity also has a Kramer's like dependence on temperature and density. Bound-bound is complicated. Both depend on metallicity.

Conduction is heat transport by diffusing electrons instead of photons. It is only important in degenerate matter where the electron mean free path is long. Otherwise the electron mean free path is small compared with radiation and conduction is unimportant. Conduction dominates in white dwarf interiors.

In general, we get opacities from tables derived from calculations with millions of transitions involved. At high temperature electron scattering dominates. At medium temperature, free free is important. Usually bound free I bigger unless the metallicity is low. At very low T, H^- opacity dominates. We have yet to discuss that.

The radiation pressure gradient cannot exceed that required by hydrostatic equilibrium or the star will blow apart due. This gives rise to the concept of an Eddington luminosity, the maximum luminosity a star can have.
$L_{Edd} = \frac{4\pi GMc}{\kappa} = 1.5 \times 10^{38}\ \mathrm{erg\ s^{-1}} \left(\frac{M}{M_\odot}\right)\left(\frac{0.34}{\kappa}\right)$. For electron scattering opacity, which often dominates at these high luminosities, the value 0.34 is appropriate at for a hydrogen helium mixture.

Very massive stars approach the Eddington limit and are also fully convective so they burn their entire mass. Their lifetime is thus a constant, about 3 M y, the shortest lifetime any star can have. Accretion can also be Eddington limited. Any higher accretion rate results in radiation that keeps the matter from accumulating. The Eddington limit for accretion is $\frac{dM}{dt} < \frac{4\pi Rc}{\kappa}$ ~10^{-5} $M_\odot$ y^{-1} for a white dwarf but you do not need to remember this equation.

Lecture 7

Polytropes are analytic models for stars that are characterized by a power law reaction between pressure and density. $P = K\rho^\gamma$ where $\gamma = (n+1)/n$ and n is the "polytropic index". Given this equation and the equation of hydrostatic equilibrium one can derive the Lane Emden equation. $\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = \theta^n$ where ξ and θ are dimensionless variables and $r = \alpha\xi$ and $\rho(r) = \rho_c\theta^n(r)$. You need not remember these equations. α is a radius like variable that depends on P_c, ρ_c and n.

The Lane Emden equation can be solved either analytically (for n = 0, 1, 5) or numerically to give the value of ξ where θ first goes to zero. This is called ξ_1 .

$\alpha\xi_1$ is then the radius of the star and $A_n = \left(\xi^2\frac{d\theta}{d\xi}\right)_{\xi_1}$ can be evaluated there. There are boundary conditions on ξ and $\frac{d\theta}{d\xi}$ namely that θ=1 at $\xi = 0$ and $\frac{d\theta}{d\xi} = 0$ at ξ=0.

You don't need to remember these values. For any value of n, the mass of the star is a given function of α, ρ_c, and A_n. Given the definition of a, the mass can also be expressed as a function of just ρ_c and the polytropic constant K. Interestingly for n = 3, ρ_c drops out of the equation and the mass is just a constant given by the polytropic constant. One can also write down from solutions to the Lane Emden equation the ratio $D_n = \rho_c / \bar{\rho}$ which is a measure of how centrally concentrated the star is. It turns out that $D_n = 1$ for n = 0 polytropes (which are stars of constant density). For n = 5, $D_n = \infty$, and for n = 3, it is 54. As remarked n = 0, 1, 5 have analytic solutions. The n = 0 polytrope corresponds to a star of constant density. Just as in the ocean the star is incompressible. The pressure varies with radius but the density is the same everywhere. The n = 1 polytrope is peculiar in that it has a constant radius no matter what M is. As remarked earlier, n= 3 is interesting because the solution for mass does not contain the radius. The equation just gives a unique mass for each value of K.

In the general case one solves for a given M, K. and n to get α. From α and K one can also get ρ_c and from that $\bar{\rho}$ and R (since n gives D_n)

There exists a mass radius relation for polytropes, $R \propto M^{\frac{n-1}{n-3}}$. For n = 1 the radius is independent of the mass, as noted before. n = 3 is a singularity but evaluating earlier in the derivation one finds a constant mass independent of R. This is a useful formula in many contexts, but an important example is the mass-radius relation for non-relativistically degenerate (γ = 5/3) and relativistically degenerate (γ = 4/3) white dwarfs. These are n = 3/2 and n = 3 polytropes respectively. For n = 3/2, the radius is proportional to $M^{-1/3}$, or more specifically $R = 8800 \text{ km} \left(\frac{M_\odot}{M}\right)^{1/3}$ if Y_e = ½.

This is the mass radius relation for common white dwarfs. Note that they get smaller as the mass increases. The density thus increases rapidly with the mass. There thus comes a mass where the electrons are mostly relativistic and one needs to use the n = 3 solution. This gives a mass, which again for Y_e = ½ is 1.456 solar masses, the Chandrasekhar mass. No white dwarf can be heavier and an accreting white dwarf approaching this mass would shrink to a point (actualy other things happen along the way).

There is also an important relation between central pressure and central density for polytropes, $P_c = C_n G M^{2/3} \rho_c^{4/3}$ where C_n is a slowly varying function of polytropic index n. For ideal gas, which mostly supports all main sequence stars, this gives, a constant mass $T_c \propto \rho_c^{1/3}$. A contracting star will increase its central temperature in proportion to the cube root of the density increase. This tendency explains a lot about stellar evolution. A star attempting to ignite a given fuel will contract and increase its temperature until it becomes hot enough to ignite a given fuel. If during the contraction it becomes degenerate first, it will cease contracting and never ignite the fuel. Since stars of a given mass are hotter at a given density and thus stay non-degenerate until a higher temperature is reached, this implies the existence of critical masses. A contracting protostar of less than 0.08 solar masses will never ignite hydrogen burning at ~10^7K, but will become a brown dwarf or planet instead. A contracting helium core of less than 0.45 solar masses will not ignite helium burning at ~10^8 K, but will become a helium white dwarf instead. There also exist limits for carbon (8 x 10^8 K) and oxygen burning (2 x 10^9 K), of 1.06 and about 1.4 solar masses, bt more relevant are the main sequence masses that produce such cores. Stars below about 8 solar masses will not ignite carbon burning.

This last conclusion has far reaching implications. Stars that don't ignite carbon burning develop degenerate CO cores surrounded by thin unstable helium burning shells. These are "asymptotic giant branch" stars that ultimately eject the low density envelopes and become first planetary nebulae and then white dwarfs. Stars heavier than 8 to 10 solar masses (the limit is uncertain) go on to develop cores of iron that collapse to neutron stars producing supernovae. Broadly speaking stars over 8 solar masses become supernovae.

The pressure density relation also implies that, for a given mass, more massive stars will burn their fuels somewhat higher temperatures and lower densities. This has important implications for energy generation on the main sequence (CNO cycle vs pp), opacity (electron scattering is more important in massive stars), and convection.

The gravitational binding energy of a polytrope of mass M and index n is $\frac{3}{(5-n)}\frac{GM^2}{R}$. This has a singularity at n = 5 because the polytrope is infinitely centrally condensed (finite central density, infinite radius). Otherwise it gives a better was of estimating a which was an undefined constant of order unity in our previous discussion of the binding energy and Kelvin-Helmholz time). Using n = 3 for the sun gives values much closer to fact, 23 M y for the Kelvin-Helmholtz time for the sun. The best polytropic index for the sun is near 3.5 though n = 3 is frequently used.

Assuming that radiation pressure is a constant fraction of total pressure throughout the star and neglecting convection and degeneracy leads to "Eddington's standard model", an n = 3 polytrope with a unique correspondence between β, the fraction of pressure due to ideal gas and the mass of the star. This relation, "Eddington's quartic equation", shows that stars of common mass (less than 100 solar masses) will be supported mostly by ideal gas pressure $(\beta \approx 1)$ but extremely massive stars will be supported by radiation pressure. The Virial theorem can be shown to imply that stars supported by 100% radiation pressure have zero net energy and are thus unbound. Unlike other polytropes that did not include the radiative diffusion equation, Eddington's standard model also yields a mass luminosity relation that agrees with observations. For most stars, $\beta \approx 1$, and $L \propto M^3$. The proportionality constant is even approximately right for the sun. For very heavy stars, Eddington's model also predicts correctly that the luminosity approaches the Eddington limit (see above) and is this proportional to M, not M^3.

Eddington's model also gives relationships between the central temperature and central density and the central temperature and radius. Without specifying independently a temperature (e.g., from the need to generate L by nuclear reactions) or an empirical relation between radius and, say mass $(R \propto M^{2/3})$ Eddingtons model does not give R or T_c separately. This is consistent with the radius being undefined for an n = 3 polytrope.

Lecture 8

Convection is a much more efficient way of transporting energy than radiative diffusion, but I only occurs when the radial temperature gradient in the star is so steep ("superadiabatic") that an adiabatically expanding, rising plume, after moving some distance outwards, has a lower density than its surroundings. The criterion is more physically expressed in terms of a dimensionless measure of how temperature varies with *pressure* rather than radius (because the adiabatic condition is a relation between density and pressure; at a given pressure a hotter gas will have lower density and be more buoyant). For buoyancy and instability one requires that $\frac{d\log\rho}{d\log P} < \frac{1}{\gamma_{ad}}$ which translates into $\nabla = \frac{d\log T}{d\log P} > \nabla_{ad}$ where ∇_{ad} can be evaluated from the equation of state and is 0.4 for ideal gas and 0.25 for radiation. ∇ can also be written as $\nabla = -\frac{H_P}{T}\frac{dT}{dr}$ where H_P is the pressure scale height (radial distance over which P would decline one e-fold). This is the Schwarzschild criterion for convection. If composition gradients are present this has to be modified by including a composition gradient term and that gives the LeDoux criterion for convection. A zone that is unstable by the Schwarzschild criterion and stable by the LeDoux criterion is said to be "semiconvective".

For an ideal gas the Schwarzschild convection criterion can be written $L(r) > 1.22\times10^{-18}\frac{\mu T^3}{\kappa\rho}m(r)\ \mathrm{erg\ s^{-1}}$. You don't need to remember this equation, but not that convection is favored by a) high luminosity within a given mass (L(r)/m(r), b) high opacity c) high density and d) low temperature.

On the main sequence stars above 1.5 solar masses have convective cores (because of the temperature sensitivity of the CNO cycle makes the energy generation centrally concentrated) and radiative envelopes. Between 0.5 and 1.5 solar masses the opposite is true = radiative core and convective outer layers. Below 0.5 solar masses the whole star is convective on the main sequence.

Heat transport by convection is calculated using "mixing length theory". The heat transport is $F = v_c \rho c_P \Delta T$ where v_c is the convective speed, ρ , the density, C_P, the heat capacity at constant pressure and ΔT, the temperature difference between an adiabatically expanding plume and its surroundings after moving a mixing length. The mixing length is taken to be a multiplier of order unity times the pressure scale height. The temperature difference, which is very small, is given by the difference between the actual temperature gradient and the adiabatic one multiplied by the mixing length. The convective speed is estimated using the acceleration caused by the density contrast with the surroundings following adiabatic expansion. The speed is very subsonic. Putting it all together the convective heat flux is $\rho c_P T \alpha^2 \sqrt{\frac{H_P g}{2}}\left(\nabla - \nabla_{ad}\right)^{3/2}$ where α is a multiplier of order unity times the mixing length. The convective luminosity is 4 π r^2 times that. You need not remember this equation. In the solar convective zone $\left(\nabla - \nabla_{ad}\right) = \Delta T / T \sim 10^{-8}$. Only a very slightly superadiabatic gradient in the temperature is required to drive very efficient heat transport.

Other forms of stellar instability were discussed. You need not remember the equations but the ideas.

Ordinary stars are stable so long as $\gamma > 4/3$ and a substantial fraction of the pressure comes from ideal gas pressure (i.e., they are not degenerate). Analysis shows that the total energy is the negative of the internal energy in the ideal gas component and under such conditions, expansion leads to cooling and the quenching of any potential runaway.

The ignition nuclear burning in a gas that is degenerate on the other hand is violently unstable. There is very little expansion as the temperature rises and what little expansion there is leads to heating not cooling. The star runs away on a nuclear time scale which can become very short due to the escalating temperature. Important examples are the helium core flash in low mass stars and Type Ia supernovae.

Burning in thin shells, with geometrical thickness and mass much less than that of the inner radius of the shell is unstable. The instability ends when the fuel is gone or expansion becomes a significant fraction of the shell radius. An important example is helium shell flashes in asymptotic giant branch stars. This instability also develops on a nuclear time scale.

Hydrodynamical instabilities develop on a hydrodynamical time scale, roughly 1000 s over the square root of the mean density inside the region. Large regions with adiabatic index below 4/3 are unstable. Such instabilities come about because of ionization, photodisintegration, or the pair instability. The latter two afflict only massive stars in their final stages of life. Stars close to $\gamma = 4/3$ can also be unstable to pulsation because with almost no net energy, small energy generation can give a large excursion in radius. If the nuclear energy generation is pulsationally unstable it is said to the unstable by the epsilon instability.

Stars can also be pulsationally unstable because of the behavior of their opacity. Thi sis instability by the kappa mechanism. A necessary condition for the kappa mechanism is that $\dfrac{d\log\kappa}{d\log P}>0$. That is when the stellar layer is compressed, its opacity goes up, trapping the radiation that is flowing through the matter building up an overpressure. In the ensuing expansion the opacity goes down releasing the trapped radiation and the material falls back growing opaque again. In the case of Kramers opacity $(\kappa \propto \rho T^{-7/2})$ the opacity does not increase with pressure. For adiabatic expansion this condition can be expressed $\kappa_P + \kappa_T \nabla_{ad} > 0$ where κ_P and κ_T are the powers of P and T upon which the opacity depends.

For an ideal gas $\kappa \propto PT^{-9/2}$ and $\kappa_P = 1, \kappa_T = -4.5, \nabla_{ad} = 0.4$ and the star is stable against pulsation. However there are other forms of opacity where κ_T is positive (H^- opacity still to be discussed) and if ∇_{ad} is reduced below 0.23, even Kramers opacity is unstable.

∇_{ad} is substantially reduced in an ionization zone. In the hydrogen and helium ionization zones it can drop below 0.23 or even 0.1. This is because energy that might have gone into increasing the pressure goes into an internal heat sink (the ionization). A particularly interesting case is the Cepheid variables where the He II to He III (He^+ to He^{++}) ionization zone at about 40,000 K lies a short distance below the photosphere. It must lie deep enough that enough mass participates in the instability to effectively change the photospheric radius and luminosity of the star but not so deep that the density is so high that ∇_{ad} does not go below 0.23 (see plot in notes). This turns out to require, for a giant-like structure a photospheric temperature of between 5500 and 7500 K. This gives the Cepheid band in the HR diagram.

The pulsations occur because for a time the matter falls in and becomes denser without the pressure rising much due to ionization. Radiation is trapped at the higher density. Eventually so much radiation is trapped that the stars outer layers expand again. Now recombination gives back energy and contributes to the expansion. The cycle continues. The time scale is the hydrodynamical time scale for the low density envelope which is roughly days. More massive Cepheids are less dense and have longer time scales. More massive Cepheids also have high luminosity, hence there is a period-luminosity relation.