Midterm Solutions

Part B

1) Sphere of constent density $\Rightarrow n=0$

a) $m(r) = 4/3 \pi r^3 \rho$

b) total gravitational binding energy

in notes $-3/5 \frac{GM^2}{R}$ or intergrate

$$\Omega = -\int_0^M \frac{Gm(r)}{r}\, dm$$

$$= -G\int_0^R (4/3 \pi r^3 \rho)\left(\frac{1}{r}\right) 4\pi r^2 \rho \, dr$$

$$= -\frac{16\pi^2}{3} G\rho^2 \int_0^R r^4\, dr = -\frac{16\pi^2}{3} G\rho^2 \frac{R^5}{5}$$

$$= -(4/3\pi R^3 \rho)(4/3\pi R^3 \rho)\left(\frac{3}{5R}\right)G = -3/5 \frac{GM^2}{R}$$

c) $P(r)$ for $n=0$ polytrope $= \frac{GM\rho}{2R}\left(1-(r/R)^2\right)$

lecture 7 or derive

$$\int_r^R dP = -\int_r^R \frac{Gm(r)}{r^2} \rho\, dr \qquad \text{hydrostatic equil.}$$

$$-P = -G \cancel{4\pi} \rho^2 \int_r^R (4/3 \pi r^3)\left(\frac{1}{r^2}\right) dr = -4\pi G\rho^2 \left(\frac{4\pi}{3}\right)\left(\frac{R^2}{2} - \frac{r^2}{2}\right)$$

$$P = \left(\frac{1}{R}\right) \frac{4\pi}{3} R^3 \rho \; G\rho \left(1/2 - \frac{r^2}{2R^2}\right)$$

$$= \frac{GM\rho}{2R}\left(1 - r^2/R^2\right)$$

d)

$$U = 3/2 \int_0^M P/\rho \, dm = \frac{3}{2}\left(\frac{GM}{2R}\right)\int_0^R \left(1 - \frac{r^2}{R^2}\right) 4\pi r^2 \rho\, dr$$

$$= 3/2 \left(\frac{GM}{2R}\right) 4\pi\rho \int_0^R \left(r^2 - \frac{r^4}{R^2}\right) dr$$

$$= 3/2 \left(\frac{GM}{2R}\right) 4\pi \left(R^3/3 - R^3/5\right)$$

$$= 3/2 \left(\frac{GM}{2R}\right)(4/3\pi R^3 \rho)(1 - 3/5) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\left(2/5\right) \frac{GM^2}{R}$$

$$U = 3/10 \, \frac{GM^2}{R} = -\frac{\Omega}{2}$$

e) $E = U + \Omega = \Omega/2$

2) Given $P_n = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_n} (\rho N_A)^{5/3} = 5.4 \times 10^9 \, \rho^{5/3}$

a) This is of the form $P = K\rho^\gamma$ $\quad \gamma = 5/3 = \frac{n+1}{n}$

$\Rightarrow \boxed{n = 3/2}$

b) As noted in the homework this is just like electron degeneracy pressure but $\frac{K_n}{K_e} = \frac{m_e}{m_n Y_e^{5/3}}$

$(Y_n = 1)$

So neutron stars have the same mass radius relation as white dwarfs, but

$$R_{n*} = \left(\frac{K_n}{K_e}\right) R_{WD}$$

notes lecture 7

$$= 8800 \text{ km} \left(\frac{M_\odot}{M}\right)^{1/3} \left(\frac{m_e}{m_n Y_e^{5/3}}\right)$$

$$= \left(\frac{1}{1838}\right) 8800 \text{ km} \left(\frac{1}{0.5^{5/3}}\right) = \boxed{15 \text{ km} \left(\frac{M_\odot}{M}\right)^{1/3}}$$

c) $M_{CH}^{n*} = 1.456 \, (Y_n/0.5)^2 = \boxed{5.82 \, M_\odot}$

The neutron mass does not enter because $v = c$ instead of P/m.

3) $\nabla = \frac{d \log T}{d \log P} = \nabla_{ad} = 0.4$ for ideal gas

$T \propto P^{0.4} \propto (\rho T)^{0.4}$

$T^{0.6} \propto \rho^{0.4}$ $\qquad T \propto \rho^{2/3}$

$P \propto \rho T \propto \rho^{5/3} \Rightarrow$ an $n = 3/2$ polytrope