Each nucleus is a bound collection of N neutrons and Z protons. The mass number is \( A = N + Z \), the atomic number is \( Z \) and the nucleus is written with the elemental symbol for \( Z \):

\[
^A_Z \text{A} \]

E.g. \(^{12}\text{C},^{13}\text{C},^{14}\text{C}\) are isotopes of carbon all with \( Z = 6 \) and neutron numbers \( N = 6, 7, 8 \)

The neutrons and protons are bound together by the residual strong or color force

Mesons are quark-antiquark pairs and thus carry net spin of 0 or 1. They are Bosons while the baryons are Fermions. They can thus serve as coupling particles. Since they are made of quarks they experience both strong and weak interactions. The lightest three mesons consist only of combinations of \( u, d, \bar{u}, \bar{d} \) and are

<table>
<thead>
<tr>
<th>Name</th>
<th>Made of</th>
<th>Charge</th>
<th>Mass ( \times c^2 )</th>
<th>( \tau ) (sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi^0 )</td>
<td>( \frac{u\bar{d} - d\bar{u}}{\sqrt{2}} )</td>
<td>0</td>
<td>135 MeV</td>
<td>8.4(-17)</td>
</tr>
<tr>
<td>( \pi^\pm )</td>
<td>( u\bar{d}, d\bar{u} )</td>
<td>( \pm 1 )</td>
<td>139.6</td>
<td>2.6(-8)</td>
</tr>
</tbody>
</table>

\( \pi^0 \rightarrow \mu^+ + \nu_\mu \) \hspace{1cm} \( \pi^0 \rightarrow 2\gamma \), occasionally \( e^+ + e^- \)

There are many more mesons. Exchange of these lightest mesons give rise to a force that is complicated, but attractive. But at a shorter range, many other mesons come into play, notably the omega meson (782 MeV), and the nuclear force becomes repulsive.
There are many mesons with different masses. The heavier the mass, the shorter the lifetime by the uncertainty principle $\Delta E \Delta t \sim h$ and hence the shorter the range $r = c \Delta t = \frac{ch}{mc^2} = \frac{h}{mc}$.

The nuclear force is only felt among hadrons (2 or 3 bound quarks – e.g., nucleons, mesons).

At typical nucleon separation (1.3 fm) it is a very strong attractive force.

At much smaller separations between nucleons the force is very powerfully repulsive, which keeps the nucleons at a certain average separation.

Beyond about 1.3 fm separation, the force exponentially dies off to zero. It is less than the Coulomb force beyond about 2.5 fm (1 fm = $10^{-13}$ cm).

The $NN$ force is nearly independent of whether the nucleons are neutrons or protons. This property is called charge independence or isospin independence.

The $NN$ force depends on whether the spins of the nucleons are parallel or antiparallel.

The $NN$ force has a noncentral or tensor component.

Since the nucleons are Fermions they obey FD statistics and have a Fermi energy $p_F = \left( \frac{3h^3}{8\pi n} \right)^{1/3}$.

where $n = \frac{0.17}{2}$ fm$^{-3}$ = $8.5 \times 10^{37}$ cm$^{-3}$ is the density of n or p.

Here $h = 6.626 \times 10^{-27}$ erg s. This implies a speed for the nucleons of about $c/4$, and a peak Fermi energy, $\varepsilon_F = \frac{p_F^2}{2M} = 39$ MeV.

The average Fermi energy is 3/5 of this

Coulomb energies are much smaller than this. To zeroth order the nucleus is a degenerate gas of nucleons confined by the (residual) strong force. For heavy nuclei though the electrical energy does become important since it goes as $Z^2$. 

Because of the nature of the force, putting nucleons in the nucleus is like putting (magnetic) marbles into a (spherical) fishbowl. The nucleus is virtually incompressible and its volume is proportional to $A = N + Z$. 

[Image]
Removing each marble from the bowl takes the same energy

\[ R \propto A^{1/3} \]

The binding energy of a nucleus:

- the energy available to hold nucleus together
- For a bunch of well-separated nucleons: the binding energy is zero
- Bring them together: strong force glues them together. However, energy has to come from somewhere: binding energy must come from a reduction in nuclear mass

Formally, it is the difference between mass of component protons and neutrons and that of actual nucleus, related through \( E = mc^2 \):

\[
\text{BE}(A,Z) = Z m_p c^2 + N m_n c^2 - M(A,Z) c^2
\]

Binding energy is a positive quantity
(even though the strong potential in which the nucleons sit is negative)

Binding energy per nucleon

- the average energy state of nucleon is a sum of high energy “surface” nucleons with low energy “bulk” nucleons
- nucleus minimizes energy by minimizing surface area – a sphere

The nuclear force is spin dependent

The nuclear crossing time for a typical nucleon, \( \tau \sim 10 R/c \sim 10^{-22} \text{ s} \) sets the time scale for the shortest nuclear reactions.

\[
E_{\text{ Coul}} = \frac{e^2}{R} = \frac{1.44 \text{ MeV}}{R(\text{fm})} \ll \epsilon_F
\]

for any one nucleon. However, since electrical energy rises as \( Z^2 \) and nuclear binding goes only as \( A \), for large mass nuclei, the electrical force does become non-negligible.

The nuclear force is independent of charge and is the same between neutrons and protons. However it does depend on spin and orientation. The triplet state (↑↑) of two
Unless weak interactions are involved (to be discussed later), the total energy released or absorbed in a given reaction

\[ Q = \sum BE(\text{species out}) - \sum BE(\text{species in}) \]

\( Q \) is measured in MeV where

1 MeV = 1.6022 \times 10^{-6} \text{ erg.}

\textit{E.g.} the binding energy of a proton is 0

The binding energy of a \(^4\)He nucleus is 28.296 MeV.

4 \( p \rightarrow ^4\) He thus releases 28.296 MeV per helium formed or \( 6.8 \times 10^{18} \text{ erg g}^{-1} \) (modulo some losses to neutrinos and p,n mass changes)

### Semi-Empirical Mass Formulae

- A phenomenological understanding of nuclear binding energies as function of A, Z and N.
- Assumptions:
  - Nuclear density is constant.
  - We can model effect of short range attraction due to strong interaction by a liquid drop model.
  - Coulomb corrections can be computed using electromagnetism (even at these small scales)
  - Nucleons are fermions in separate wells (Fermi gas model \( \Rightarrow \) asymmetry term)
  - Corrections for spin and shell closures.

### Liquid Drop Model

- Phenomenological model to understand binding energies.
- Consider a liquid drop
  - Intermolecular force repulsive at short distances, attractive at intermediate distances and negligible at large distances \( \Rightarrow \) constant density. Molecules on the “inside” are in a lower energy state than those at the surface, so surface area is minimized
  - \( n=\)number of molecules, \( T=\)surface tension, \( BE=\)binding energy
  - \( E=\)total energy of the drop, \( a,b=\)free constants
    \[ E=an + \frac{4\pi R^2 T}{n} \Rightarrow BE=an-bn^{2/3} \]
  - Surface area \( \sim n^{2/3} \)
- Analogy with nucleus
  - Nucleus has constant density
  - From nucleon-nucleon scattering experiments we know:
    - Nuclear force has short range repulsion and is attractive at intermediate distances.

### Volume and Surface Term

- If we can apply the liquid drop model to a nucleus
  - constant density
  - same binding energy for all constituents
- Volume term: \( B_{\text{volume}}(A) = +aA \) \( a \sim 15 \text{ MeV} \)
- Surface term: \( B_{\text{surface}}(A) = -bA^{2/3} \) \( b \sim 18 \text{ MeV} \)
- Since we are building a phenomenological model in which the coefficients \( a \) and \( b \) will be determined by a fit to measured nuclear binding energies we must include any further terms we may find with the same \( A \) dependence together with the above
Coulomb Energy

• The nucleus is electrically charged with total charge $Ze$
• Assume that the charge distribution is spherical and homogeneous and compute the reduction in binding energy due to the Coulomb interaction

$$E_{\text{Coulomb}} = \int_0^R \frac{Q(r)}{r^2} \, dQ = Ze(r/R) dQ = 3Zer^2 / R^3 \, dr$$

to change the integral to $dr$; $R=$outer radius of nucleus

$$E_{\text{Coulomb}} = \int_0^R \frac{3(Ze)^2}{r^5} \, dr = (3/5) (Ze)^2 / R$$

... and remember $R=R_0A^{1/3}$

$$B_{\text{Coulomb}}(Z,A) = -c \frac{Z^2}{A^{1/3}}$$

Asymmetry Term

• Neutrons and protons are spin ½ fermions $\Rightarrow$ obey Pauli exclusion principle.
• If all other factors were equal nuclear ground state would have equal numbers of n & p.

Illustration

• n and p states with same spacing $\Delta$.
• Crosses represent initially occupied states in ground state.
• If three protons were turned into neutrons
  • the extra energy required would be $3 \times 3 \Delta$ but there would now be 6 more neutrons than protons.
• In general if there are $Z-N$ excess neutrons over protons the extra energy is $\sim ((Z-N)/2) \Delta$ relative to $Z=N$.

Spin pairing in the liquid drop model:

Spin pairing favours pairs of fermionic nucleons (similar to electrons in atoms)
i.e. a pair with opposite spin have lower energy than pair with same spin

Best case: even numbers of both protons and neutrons
Worst case: odd numbers of both protons and neutrons
Intermediate cases: odd number of protons, even number of neutrons or vice versa

→ Subtract small energy $\delta$ required to decouple nucleons from binding energy:

$$\delta = +a_pA^{-1/2} \quad \text{for both } N \text{ & } Z \text{ odd}$$
$$= 0 \quad \text{for } N \text{ even, } Z \text{ odd / } / Z \text{ even, } N \text{ odd}$$
$$= -a_pA^{-1/2} \quad \text{for both } N \text{ & } Z \text{ even}$$

$\Rightarrow a_p$ collects constants, $A^{-1/2}$ dependence provides best empirical fit to data

→ subtracting $\delta$ reduces BE for N and Z both odd
→ subtracting $\delta$ adds small amount to BE for N and Z both even
Pairing Term

- Nuclei with even number of $n$ or even number of $p$ more tightly bound than with odd numbers.

- Only 4 stable o-o nuclei ($^2$H, $^6$Li, $^{10}$B, $^{14}$N) but 168 stable e-e nuclei.

- These differences in binding energy end up being reflected in cross sections and eventually, abundances, e.g., in the s-process. Elements with odd $Z$ are less abundant.

Similar to atoms, we shall find that nuclei have closed shells that are particularly sable, the nuclear equivalent of the noble gases. One such stable nucleus is $^4$He; another is $^{18}$O. Removing a nucleon from one of these “magic nuclei” with a closed shell takes more energy than a “valence” nucleon. E.g., it is easier to remove a neutron from $^{12}$O than from $^{16}$O (even accounting for the odd-even effect).

The pairing and shell corrections are purely empirical quantum mechanical corrections (for now) to the liquid drop model.

Putting it all together:

$$BE \approx a_1 A - a_2 A^{2/3} - a_3 Z^2 A^{1/3} - a_4 \left(\frac{Z - N}{A}\right)^2 - \frac{\delta(A)}{A} - S(A)$$

where terms 1 through 4 are the volume, surface, Coulomb, and symmetry energies respectively. $\delta(A)$ is the pairing correction, and $S(A)$ is the shell correction. Without the last two terms which are strictly quantum mechanical, this is known as the “liquid drop model” or the Bohr-Weizsäcker semi-empirical mass formula. Empirically, and crudely from fitting to known binding energies (deShalit and Feshbach, *Theoretical Nuclear Physics*, p. 126), we have $a_1 = 15.68$ MeV, $a_2 = 18.56$ MeV, $a_3 = 0.717$ MeV, and $a_4 = 28.1$ MeV. Also $\delta(A) = +34/A^{2/3}$, $-34/A^{2/3}$ MeV for odd $Z$, odd-$N$, odd-$A$; and even-$Z$, even-$N$ nuclei respectively. $S(A)$ is complicated.

Below iron energy can be released by fusion, above iron by fission.
**Semi Empirical Mass Formula**

**Binding Energy vs. A for beta-stable odd-A nuclei**

**Utility**

- Only makes sense for A greater than about 20
- Good fit for large A (<1% in most instances)
- Deviations are interesting - shell effects
- Explains the gross properties of the observed nuclei

<table>
<thead>
<tr>
<th>Fit parameters</th>
<th>in MeV</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>15.56</td>
</tr>
<tr>
<td>b</td>
<td>17.23</td>
</tr>
<tr>
<td>c</td>
<td>23.285</td>
</tr>
<tr>
<td>d</td>
<td>0.697</td>
</tr>
<tr>
<td>d12 (o-o)</td>
<td></td>
</tr>
<tr>
<td>d0 (o-e)</td>
<td></td>
</tr>
<tr>
<td>d12 (e-e)</td>
<td></td>
</tr>
</tbody>
</table>

Abundant light nuclei (up to Ca (Z = 20) have Z = N (e.g., 4He, 12C, 16O, 28Si, etc.)

Above Ca N > Z

Most tightly bound nuclei are in the iron group

Very heavy nuclei are unstable to fission

What is the most tightly bound nucleus?

because of shell effects (Z = 28 is magic).

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>BE/A</th>
<th>Ye</th>
</tr>
</thead>
<tbody>
<tr>
<td>56Ni</td>
<td>8.643</td>
<td>0.500</td>
</tr>
<tr>
<td>56Fe</td>
<td>8.790</td>
<td>0.464</td>
</tr>
<tr>
<td>62Ni</td>
<td>8.794</td>
<td>0.452</td>
</tr>
</tbody>
</table>

**Nuclear Stability, The Shell Model**
What are the equivalent closed shells for nuclei and how many neutrons or protons fit in each shell?

Iliadis

Empirically the closed shells are known.

Fig. 1.11 Difference between experimental ground-state atomic mass excess (Audi et al. 2003) and the mass excess predicted by the spherical macroscopic part of the finite-range droplet (FRDM) mass formula (Möller et al. 1996) versus neutron number.

Abundance patterns reflect magic numbers

Ingredients in the Shell Model

– Mayer and Jensen 1963 Nobel Prize

Clayton 311 – 319

- Nuclear states are characterized by a wave function of given spin, angular momentum and parity
- Unlike the atom we do not solve for a central potential but instead for a square well (or variations thereon)
- spin-spin and spin-orbit coupling are more important for nuclei than for atoms
- There are two kinds of particles, n and p, not just e
- More than one particle can be excited at the same time. In atoms such states would be autoionizing.
A highly idealized nuclear potential looks something like this “infinite square well”.

As is common in such problems one applies boundary conditions to Schrödinger’s equation.

\[ V = -V_0 \quad r < R \]
\[ = \infty \quad r \geq R \]

\[ \Psi(R) = 0 \]
\[ \Psi(R) = 0 \quad V_0 = 50 - 60 \text{ MeV} \]

(In the case you have probably seen before of electronic energy levels in an atom, one would follow the same procedure, but the potential would be the usual [attractive] \( 1/r \) potential.)

For neutrons the level scheme is the same as for protons up to \( N = 50 \). Above that the Coulomb repulsion of the protons has an effect and favors orbits (for protons) with higher angular momentum. Thus for example the 51st neutron is in the d level of \( j = 5/2 \) while for protons it is in the g level of \( j = 7/2 \). The effect is never enough to change the overall shell closures and magic numbers.

Maria Goeppert Mayer – Nobel talk - 1963
Each state can hold \((2j+1)\) nucleons.

**Nuclear Stability**

A necessary condition for nuclear stability is that, for a collection of \(\textit{A}\) nucleons, there exists no more tightly bound aggregate.

- E. g., a single \(^8\text{Be}\) nucleus has less binding energy than two \(^4\text{He}\) nuclei, hence \(^8\text{Be}\) quickly splits into two heliums.

- An equivalent statement is that the nucleus \(^{\textit{A}}\text{Z}\) is stable if there is no collection of \(\textit{A}\) nucleons that weighs less.

- However, one must take care in applying this criterion, because while unstable, some nuclei live a very long time. An operational definition of “unstable” is that the isotope has a measurable abundance and no decay has ever been observed (ultimately all nuclei heavier than the iron group are unstable, but it takes almost forever for most of them to decay). Exception – nuclei heavier than lead, e.g., U, Pu, etc.

In addition to just splitting up into lighter nuclei, which involves only the strong and electromagnetic forces, a nucleus may be unstable by the weak interaction.

**Weak decays:**

\[
egin{align*}
\text{beta - decay} & : \quad n \rightarrow p + e^- + \nu_e \\
\text{e^+ - decay} & : \quad p \rightarrow n + e^+ + \nu_e \\
\text{e^- - capture} & : \quad p + e^- \rightarrow n \quad \text{or} \quad \nu_e
\end{align*}
\]

While the free neutron can decay, the other two reactions happen only inside of nuclei where the binding energy change between the parent nucleus and products can more than make up for the n-p mass difference.

or, in the last case where there exists a supply of energetic electrons.
Classification of Decays

- **α-decay:**
  - emission of Helium nucleus
  - $Z \to Z-2$
  - $N \to N-2$
  - $A \to A-4$

- **β−-decay (or β-decay):**
  - emission of $e^-$ and $\bar{\nu}$
  - $Z \to Z+1$
  - $N \to N-1$
  - $A = \text{const}$

- **β+ -decay**
  - emission of $e^+$ and $\nu$
  - $Z \to Z-1$
  - $N \to N+1$
  - $A = \text{const}$

- **Electron Capture (EC):**
  - absorption of $e^-$ and emiss $\bar{\nu}$
  - $Z \to Z-1$
  - $N \to N+1$
  - $A = \text{const}$

In terms of binding energy

$$Q_\beta = BE(^A Z + 1) - BE(^A Z) + 0.782 \text{ MeV}$$

$$Q_{\beta^-} = BE(^A Z - 1) - BE(^A Z) - 1.804 \text{ MeV}$$

$$Q_{\beta^+} = BE(^A Z - 1) - BE(^A Z) - 0.782 \text{ MeV}$$

Tables of binding energies (given as BE/A) and reaction Q-values are given at

http://www.nndc.bnl.gov/masses/mass.mas03

Decay properties and lists of isotopes can be found at