Lecture 4

Basic Nuclear Physics – 2

Nuclear Stability and the Shell Model

Nuclear Stability

A necessary condition for nuclear stability is that, for a collection of $^A_Z$ nucleons, there exists no more tightly bound aggregate.

- E.g., a single $^8_{\text{Be}}$ nucleus. Though it has finite binding energy (56.4995 MeV), has less binding energy than two $^4_{\text{He}}$ nuclei ($2 \times 28.2957 = 56.591$), hence $^8_{\text{Be}}$ quickly ($6.7 \times 10^{-17}$ s) splits into two heliunds (i.e. two alpha particles).

- An equivalent statement is that the nucleus $^A_Z$ is stable if there is no collection of $A$ nucleons that weighs less.

- However, one must take care in applying this criterion, because while unstable, some nuclei live a very long time. An operational definition of "stable" is that the isotope has a measurable abundance and no decay has ever been observed (ultimately all nuclei heavier than the iron group are unstable, but it takes almost forever for them to decay). One must also include any lepton masses emitted or absorbed in a weak decay.

Most collections of nucleons have positive binding energy, i.e., are temporarily bound, but a nucleus is still considered "unbound" if it can gain binding by ejecting a neutron or proton. or ion (like $^4_{\text{He}}$). If energetically feasible, this ejection occurs on a very short time scale (e.g. $^5_{\text{Li}}$ 3 x 10^{-22} s).

The neutron and proton "drip lines" are defined by

$$BE(^{A+1}_Z) < BE(^A_Z) \quad S_n < 0$$
$$BE(^{A+1}_Z) < BE(^A_{Z-1}) \quad S_p < 0$$

Note that by definition

$$BE(n) = BE(p) = 0$$

Even a nucleus that is bound is usually unstable to weak decay or alpha-decay.

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**Classification of Decays**

- **α-decay:** (no weak interaction)
  - emission of Helium nucleus
  - \( Z \rightarrow Z-2 \)
  - \( N \rightarrow N-2 \)
  - \( A \rightarrow A-4 \)

- **β-decay (or β-decay)**
  - emission of e⁻ and ν
  - \( Z \rightarrow Z+1 \)
  - \( N \rightarrow N-1 \)
  - \( A = \text{const} \)

- **β⁺-decay**
  - emission of e⁺ and ν
  - \( Z \rightarrow Z-1 \)
  - \( N \rightarrow N+1 \)
  - \( A = \text{const} \)

**Electron Capture (EC)**
- absorption of e⁻ and emit ν
- \( Z \rightarrow Z \)
- \( N \rightarrow N+1 \)
- \( A = \text{const} \)

The lightest known alpha-unstable nuclei (except for \(^8\)Be) are \(^{104-110}\)Te

**Energy can often be released by adding nucleons or other nuclei to produce a more tightly bound product:**

\[
\begin{align*}
\text{BE}({}^{56}\text{Fe}) & = 492.247 \text{ MeV} \\
\text{BE}({}^{57}\text{Fe}) & = 499.863 \text{ MeV} \\
Q_{\gamma} ({}^{56}\text{Fe}) & = 7.646 \text{ MeV}
\end{align*}
\]

Both \(^{56}\)Fe and \(^{57}\)Fe are stable

The reaction \(^{56}\)Fe(n,γ\(^{57}\)Fe) provides 7.646 MeV of kinetic energy and radiation. To go the other way, \(^{57}\)Fe(γ,n\(^{56}\)Fe), would require 7.646 MeV. The locus of nuclei with \( Q_{\gamma} = 0 \) is known as the “neutron-drip line”. Similarly \( Q_{\gamma} = 0 \) defines the “proton-drip line”.

The criterion for weak decay is a little more complicated because of the mass difference between the neutron and proton and because electrons or positrons may be created or destroyed.

The mass of the neutral atom, defined as the “atomic mass” can be written

\[
M(A,Z) = Z \, m_H + N \, m_n - \frac{BE(A,Z)}{c^2} + \left[ 15.73 \times Z^{5/3} \text{ eV} - Z(13.6 \text{ eV}) \right]/c^2
\]

electronic binding energy

The difference in binding energies for reactions other than weak interactions is also the “Q-value for the reaction” e.g. \(^{3}\)He(\(n,\gamma\))\(^{4}\)He \( Q = 20.56 \text{ MeV} \)

**Examples:**
- \(^{2}\)He - diproton - BE < 0 unbound (~700 keV)
- \(^{3}\)He \( \text{BE} = 7.718 \text{ MeV} \) stable \( \text{BE}(n) = \text{BE}(p) = 0 \)
- \(^{4}\)He \( \text{BE} = 28.296 \text{ MeV} \) stable
- \(^{5}\)He \( \text{BE} = 27.56 \text{ MeV} \) unstable n-emission \( 7.6 \times 10^{-22} \text{ s} \)
- \(^{6}\)He \( \text{BE} = 29.27 \text{ MeV} \) bound but decays to \(^{6}\)Li in 807 ms
- \(^{7}\)He \( \text{BE} = 28.86 \text{ MeV} \) unstable n-emission \( 3 \times 10^{-21} \text{ s} \)
- \(^{6}\)Li \( \text{BE} = 26.33 \text{ MeV} \) unstable p-emission \( 4 \times 10^{-17} \text{ s} \)
- \(^{6}\)Li \( \text{BE} = 31.99 \text{ MeV} \) stable
- \(^{7}\)Li \( \text{BE} = 39.24 \text{ MeV} \) stable
- \(^{8}\)Li \( \text{BE} = 41.27 \text{ MeV} \) (bound but decays to \(^{8}\)Be in 840 ms)
- \(^{8}\)Be \( \text{BE} = 56.50 \text{ MeV} \) (barely) unbound - decays to \( ^2\)He in \( 6.7 \times 10^{-17} \text{ s} \)

etc.
More commonly used is the Atomic Mass Excess

\begin{align*}
1 \text{ amu} &= 1/12 \text{ the mass of the neutral } ^{12}\text{C atom} \\
&= 931.494 \text{ MeV} / c^2 \\
\Delta^p &= 1.00727647 \text{ amu} \\
\Delta^e &= 0.00054860 \text{ amu} \\
\Delta^H &= 1.007825037 \text{ amu} \\
\Delta^{16}O &= 15.994915 \text{ amu} \\
\Delta^{12}C &= 12.000000 \text{ amu}
\end{align*}

Neutral atoms

The atomic mass excess is then defined:

\[
\Delta = \text{ atomic mass excess} = 931.494 \text{ MeV} [M(^A Z) - A]
\]

The mass excess of \(^{12}\text{C}\) is obviously zero.

The mass excess of \(^{16}\text{O}\) is -4.737 MeV. That is the neutral \(^{16}\text{O}\) atom weighs less than 16 times 1/12 of the neutral \(^{12}\text{C}\) atom.

This automatically includes the electron masses

Wilhelm Ostwald suggested \(\text{O}\) as the standard in 1912 (before isotopes were known) In 1961 the carbon-12 standard was adopted. \(\text{O}\) was not really pure \(^{16}\text{O}\)

The binding energy (MeV) is given in terms of the mass excess by the previous definition of mass excess (neglecting electronic binding energy) and the definition of the binding energy

\[
\begin{align*}
\text{BE} &= Z \Delta^H + N \Delta^N - M(^A Z) \\
&= Z (1.007825 \text{ amu}) + N (1.008649 \text{ amu}) - 931.494 \text{ MeV} \\
&= Z \Delta^H + N \Delta^N - M(^A Z) \\
\end{align*}
\]

where \(\Delta^H = 7.288969 \text{ MeV} = \text{mass excess of H in amu} \times 931.494 \text{ MeV} \\
\Delta^N = 8.071323 \text{ MeV} = \text{mass excess of N in amu} \times 931.494 \text{ MeV}

eg. \(^{4}\text{He}\) \(\Delta = +2.425 \text{ MeV} \)

BE = (8.071323+2)(7.2889) - 2.425 = 29.296 MeV

6p, 6n, 6 e\(^{-}\) in the atom

\[
\begin{align*}
\text{M}(^A Z) &\text{ the amonic mass} \\
\text{or \ } M(^A Z) &= A + \frac{\Delta}{931.494} \text{ amu's} \\
A &\text{ a natural integer}
\end{align*}
\]

http://t2.lanl.gov/nis/data/astro/molnix96/massd.html
Add Z electrons

\[ \text{electron capture: } p + e^- \rightarrow n + \nu_e \]

Also possible at high T

\[ e^- + p \rightarrow n + p \]

positron capture

At high density even “stable”

nuclei capture electrons

\[ \Delta \]

The energy released in the decay

\[ ^{13}\text{N}(e^-\nu)^{13}\text{C} \quad Q_{\beta^+} = 1.20 \text{ MeV} \]

\[ ^{13}\text{B}(e^-\nu)^{13}\text{C} \quad Q_{\beta} = 13.437 \text{ MeV} \]

Frequently nuclei are unstable to both electron-capture

and positron emission.

Example: \( p(p,e^+\nu)^2\text{H} \)

Mass excess \( ^2\text{H} = 2 \times 7.289 \text{ MeV} \)

= 14.578 \text{ MeV} \)

Mass excess \( \Delta\) \( ^2\text{H} = 13.136 \text{ MeV} \). This is a

smaller number so the diproton is unstable to

weak decay. The Q value is given by

\[ 14.578 - 13.136 = 1.442 \text{ MeV} \]

- \( 2m_e c^2 = 0.420 \text{ MeV} \)

but the electron and positron annihilate and

so we get the \( 2m_e c^2 \) back and the reaction

yields 1.442 \text{ MeV} \)

But the neutrino carries away a variable amount

of energy that averages to 0.262 \text{ MeV} so really

only deposit 1.18 \text{ MeV} of energy locally

Decays may proceed though excited states
In terms of binding energy

\[ Q_0 = BE(^4 Z + 1) - BE(^4 Z) + 0.782 \text{ MeV} \]
\[ Q_0 = BE(^4 Z - 1) - BE(^4 Z) - 1.804 \text{ MeV} \]
\[ Q_0 = BE(^4 Z - 1) - BE(^4 Z) - 0.782 \text{ MeV} \]

Another example, pick out the stable isotopes:

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>( \Delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>40Cl</td>
<td>-27.54</td>
</tr>
<tr>
<td>40Ar</td>
<td>-35.04</td>
</tr>
<tr>
<td>40K</td>
<td>-33.54</td>
</tr>
<tr>
<td>40Ca</td>
<td>-34.85</td>
</tr>
<tr>
<td>40Sc</td>
<td>-20.53</td>
</tr>
</tbody>
</table>

The ones with the bigger (less negative) mass excesses are unstable.

40Cl and 40Sc are obviously unstable. 40K can decay either to 40Ar (30.7%) or to 40Ca (89.3%), but both 40Ar and 40Ca are stable,

\[ \Delta BE = -\left( \frac{a_3}{A^{5/3}} \right) (Z^2 - Z_{\text{stab}}^2) - \left( \frac{a_1}{A} \right) \left( [A - 2Z]^2 - [A - 2Z_{\text{stab}}]^2 \right) \]

\[ = -\left( \frac{a_3}{A^{5/3}} \right) (Z^2 - Z_{\text{stab}}^2) - \left( \frac{a_1}{A} \right) \left( [A - 4AZ + 4Z^2 - A^2 + 4AZ_{\text{stab}} - 4Z_{\text{stab}}^2] \right) \]

\[ = -\left( \frac{a_3}{A^{5/3}} \right) (Z^2 - Z_{\text{stab}}^2) - \left( \frac{4a_1}{A} \right) \left( [Z^2 - Z_{\text{stab}}^2 - AZ + AZ_{\text{stab}}] \right) \]

\[ = -\left( \frac{a_3}{A^{5/3}} \right) (Z^2 - 2ZZ_{\text{stab}} + Z_{\text{stab}}^2 + 2ZZ_{\text{stab}} - 2Z_{\text{stab}}^2) \]

\[ - \left( \frac{4a_1}{A} \right) \left( Z^2 - 2ZZ_{\text{stab}} + Z_{\text{stab}}^2 - 2Z_{\text{stab}}^2 - AZ + AZ_{\text{stab}} + 2ZZ_{\text{stab}} \right) \]

\[ = K(Z - Z_{\text{stab}})^2 - \left( \frac{a_3}{A^{5/3}} \right) (2ZZ_{\text{stab}} - 2Z_{\text{stab}}^2) \]

\[ - \left( \frac{4a_1}{A} \right) (-2Z_{\text{stab}}^2 + AZ + AZ_{\text{stab}} + 2ZZ_{\text{stab}}) = K(Z - Z_{\text{stab}})^2 + F \]

We previously solved for \( Z_{\text{stable}} \), such that the partial of BE with respect to Z at constant A was zero

\[ Z_{\text{stable}} = \frac{2a_4 A}{\epsilon A^{7/3} + 4} \]

A little algebra (omitted here) shows that if \( A = \text{constant and } \epsilon = 0 \) (i.e., A is odd), then the differences in binding energy for two nuclei, one having arbitrary Z and the other having \( Z_{\text{stable}} \) will be parabolic in Z

\[ \Delta BE(\text{odd } A) = \text{const} \cdot (Z - Z_{\text{stable}})^2 \]

\[ \text{const} = -\frac{4a_4}{A - a_3 A^{7/3}} \]

See the figure on the next page. This means

\[ F = -\left( \frac{a_3}{A^{5/3}} \right) (2ZZ_{\text{stab}} - 2Z_{\text{stab}}^2) \]

\[ - \left( \frac{4a_1}{A} \right) (-2Z_{\text{stab}}^2 - AZ + AZ_{\text{stab}} + 2ZZ_{\text{stab}}) \]

\[ = -2Z_{\text{stab}} \left( \frac{a_3}{A^{5/3}} + \frac{4a_1}{A} \right) (Z - Z_{\text{stab}}) \]

\[ - \left( \frac{4a_4}{A} \right) (Z_{\text{stab}} - Z) \]

\[ = 2Z_{\text{stab}} \frac{A}{a_3 A^{2/3} + 4a_1} \left( a_3 A^{2/3} + 4a_1 \right) (Z_{\text{stab}} - Z) \]

\[ - \left( \frac{4a_4}{A} \right) (Z_{\text{stab}} - Z) \]

\[ = 0 \]
At constant A

- Even A:
  - two parabolas
  - one for o-o & one for e-e
  - lowest o-o nucleus often has two decay modes
  - most e-e nuclei have two stable isotopes
  - there are nearly no stable o-o nuclei in nature because these can usually decay to an e-e nucleus
- Exceptions: $^2$H, $^6$Li, $^{10}$B, $^{14}$N

Odd A. A=135

Single parabola
even-odd and odd-even

Only $^{135}$Ba is stable.

that for all A = odd, there is one and only one stable isotope, e.g., $^{13}$C, $^{15}$N, $^{17}$O, $^{19}$F, $^{21}$Ne, $^{23}$Na, $^{27}$Al, etc. There are some near calls - $^{113}$Cd decays to $^{113}$In with a half life of $9 \times 10^6$ y; $^{115}$In decays to $^{115}$Sn with a half life of $4 \times 10^{11}$ y; and $^{123}$Te decays to $^{123}$Sn with a half life of $1 \times 10^{13}$ y. These special cases are because of shell closures e.g., at Z = 50 for In and Sn.

Things are more complicated for even A because of the pairing correction and the two different ways of making even A (even Z; odd Z, N).

$$\Delta BE(\text{even } A) = \text{const} (Z - Z_{\text{stab}})^2 + \delta \text{ odd } Z - \delta \text{ even } Z$$

As a result one gets two curves, one for the odd-Z, even-A isotopes, and one for the even-Z, even-A isotopes. Depending on the placement of points on these curves one can have 1, 2, or even 3 stable isotopes at each
To summarize:

odd A

There exists one and only one stable isotope

odd Z – odd N

Very rarely stable. Exceptions $^2$H, $^6$Li, $^{10}$B, $^{14}$N. Large surface to volume ratio. Our liquid drop model is not really applicable.

even Z – even N

Frequently only one stable isotope (below sulfur). At higher A, frequently 2, and occasionally, 3.

The Shell Model

A. For example $^{12}$C, $^{14}$N, and $^{16}$O; but also $^{34}$Ar, $^{54}$Cr, $^{58}$Fe, $^{84}$Ni $^{84}$Zn; and even $^{136}$Xe, $^{136}$Ba, $^{136}$Ce. Because the pairing energy gets smaller as one goes to large A, the two parabolas lie closer and it is easier to have multipoles. For light elements below sulfur, 1 isotope is typical for even A. Above about calcium, two isotopes are typical, but there are exceptions, especially in the vicinity of closed shells. Nuclei with both odd Z and odd N are very rarely bound, but there are notable exceptions, $^{2}$H, $^{6}$Li, $^{10}$B, $^{14}$N, but these are so light that our liquid drop model is quite inadequate.
Shortcomings of the Liquid Drop Model

- Simple model does not apply for $A < 20$

- Doesn’t Predict Magic Numbers

- **Proton Magic Numbers**
  - Stable isotopes
  - Magic Proton Numbers

- **Neutron Magic Numbers**
  - Stable isotones

- Neutron separation energies
  - Saw tooth from pairing term
  - Big step down when $N$ goes across magic number at 82

Abundance patterns reflect magic numbers

- Iron mountain
  - $Z = N = 28$
  - No $A = 5$ or 8
Our earlier discussions treated the nucleus as sets of identical nucleons and protons comprising two degenerate Fermi gases. That is OK so far as it goes, but now we shall consider the fact that the nucleons have spin and angular momentum and that, in analogy to electrons in an atom, are in ordered discrete energy levels characterized by conserved quantized variables – energy, angular momentum and spin.

A highly idealized nuclear potential looks something like this “infinite square well”.

As is common in such problems one applies boundary conditions to Schrödinger’s equation.

\[ V = -V_{\text{nuc}} \quad r < R \]
\[ = \infty \quad r \geq R \]

\[ \Psi(R) = 0 \quad V_{\text{nuc}} \approx 50 - 60 \text{ MeV} \]

(In the case you have probably seen before of electronic energy levels in a hydrogen atom, one would follow the same procedure, but the potential would be the usual [attractive] 1/r potential.)
Schroedinger's Equation:

\[-\frac{\hbar^2}{2M} \nabla^2 \Psi + (V - E)\Psi = 0\]

Spherical symmetry:

\[\Psi_{n,lm}(r,\theta,\phi) = f_{nl}(r) Y_{lm}^{m}(\theta,\phi)\]

Radial equation:

\[-\frac{\hbar^2}{2M} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) f_{nl}(r) + \left[ \frac{l(l+1)\hbar^2}{2Mr^2} + V_{\text{nuc}}(r) \right] f_{nl}(r) = \frac{E}{\hbar^2} f_{nl}(r)\]

Solve for \(E\).

Classically, centrifugal force goes like

\[F_c = \frac{mv^2}{R} = \frac{m^2v^2R^2}{mR^3} = \frac{L^2}{mR^3}\]

One can associate a centrifugal potential with this,

\[\int F_c \, dR = \frac{-L^2}{2mR^2}\]

Taking the usual QM eigenvalues for the operator \(L^2\) one has

\[-\frac{l(l+1)}{2mR^2}\]

The solutions to the infinite square well potential are then the zeros of spherical Bessel functions (Landau and Lifshitz, Quantum Mechanics, Chapter 33, problem 2)

\[E_{n,l} = -|V_{\text{nuc}}| + \frac{\hbar^2}{2Mr^2} \left[ \pi^2 \left( n + \frac{l}{2} \right)^2 - l(l+1) \right]\]

more negative means more bound

We follow the custom of labeling each state by a principal quantum number, \(n\), and an angular momentum quantum number, \(l\), e.g.

3d \((n = 3, l = 2)\) \(l = 0, 1, 2, 3, 4, 5\), etc = s, p, d, f, g, h etc

- States of higher \(n\) are less bound as are states of larger \(l\) can be greater than \(n\)

- Each state is 2 \((2l+1)\) degenerate. The 2 out front is for the spin and the 2 \(l+1\) are the various z projections of \(l\)

- E.g., a 3d state can contain 2 \((2(2) + 1) = 10\) neutrons or protons

Substitute:

\[\rho = \sqrt{\frac{2M(E - V_{\text{nuc}})}{\hbar^2}} r \quad V_{\text{nuc}} \text{ is } < 0\]

To obtain:

\[\rho^2 \frac{\partial^2 f}{\partial \rho^2} + 2\rho \frac{\partial f}{\partial \rho} + (\rho^2 - l(l+1)) f = 0\]

Solution is:

\[f = J_{\ell + \frac{1}{2}}(\rho)\]

Spherical Bessel Functions

Abramowitz and Stegun 10.1.1

http://people.math.sfu.ca/~cbm/aands/
This gives an energy ordering
\[ \pi^2 \left( n + \frac{\ell}{2} \right)^2 - \ell (\ell + 1) \]

\[ \begin{array}{cccc}
1s^2 & 1p^6 & 1d^{10} & 2s^2 & 1f^{14} \text{ etc.}
\end{array} \]
\[ \begin{array}{cccc}
\pi^2 & \frac{9 \pi^2}{4} & 4 \pi^2 & 4 \pi^2 & \frac{25 \pi^2}{4} - 12
\end{array} \]
\[ \begin{array}{cccc}
9.87 & 20.20 & 33.48 & 39.48 & 49.69
\end{array} \]

This simple progression would predict shell closures at Z = N = 2, 8, 18, 20, 34 etc., i.e., \(^4\)He, \(^{16}\)O, \(^{36}\)Ar, \(^{40}\)Ca, etc. A good beginning but increasingly in error at high Z, N.

So far we have considered the angular momentum of the nucleons but have ignored the fact that they are Fermions and have spin.

**Improving the Nuclear Potential Well**

The real potential should be of finite depth and should probably resemble the nuclear density - flat in the middle with rounded edges that fall off sharply due to the short range of the nuclear force.

R \approx \text{Nuclear Radius}
d \approx \text{width of the edge}
R >> d

for neutrons
Better, the gap at 20 is now closer to correct.

But this still is not very accurate above \( Z = 20 \) because:

- Spin is very important to the nuclear force

- The Coulomb force becomes important for protons but not for neutrons.

Introduce spin-orbit and spin-spin interactions

\[ \vec{l} \cdot \vec{s} \text{ and } \vec{s} \cdot \vec{s} \]

Define a new quantum number

\[ j = \vec{l} + \vec{s} \]

Get splitting of levels into pairs

- \( 1p \rightarrow 1p_{1/2} \) \( 1p_{3/2} \)
- \( 2f \rightarrow 1f_{5/2} \) \( 2f_{7/2} \)
- etc

Label states by \( nl_j \)

This interaction is quite different from the fine structure splitting in atoms. It is much larger and lowers the state of larger \( j \) (parallel \( l \) and \( s \)) compared to one with smaller \( j \). See Clayton p. 311ff). The interaction has to do with the spin dependence of the nuclear force, not electromagnetism.

Empirically \( V = -\alpha \cdot l \cdot s \)

\[ \alpha = 13 \text{ A}^{-2/3} \text{ MeV} \]

\[ \Delta E = -\frac{\alpha}{2} l \quad j = (l + \frac{1}{2}) \]

\[ + \frac{\alpha}{2} (l+1) \quad j = (l - \frac{1}{2}) \]

These can be large compared even to the spacing between the principal levels.

The state with higher \( j \) is more tightly bound; the splitting is bigger as \( l \) gets larger.

For neutrons the level scheme is the same as for protons up to \( N = 50 \). Above that the Coulomb repulsion of the protons has an effect and favors orbits (for protons) with higher angular momentum. Thus for example the 51st neutron is in the \( d \) level of \( j = 5/2 \) while for protons it is in the \( g \) level of \( j = 7/2 \). The effect is never enough to change the overall shell closures and magic numbers.

Maria Goeppert Mayer – Nobel - 1963
The correct energy ordering then becomes:

For neutrons:

\[ 1s^{2}_{1/2} \quad 1p^{2}_{3/2} \quad 1d^{2}_{5/2} \quad 2s^{2}_{1/2} \quad 1d^{4}_{3/2} | \]

\[ 1f^{8}_{7/2} \quad 2p^{4}_{3/2} \quad 1f^{6}_{5/2} \quad 2p^{2}_{1/2} \quad 1g^{10}_{9/2} | \text{ etc.} \]

where \(|\) denotes a large energy gap – hence "magic number"

For protons the ordering is the same up to \(1g_{9/2}\) but differs at the next level, \(2d_{5/2}\) for neutrons, \(1g_{7/2}\) for protons

Each state can hold \((2j+1)\) nucleons.

Some implications:

A. Ground states of nuclei

Each quantum mechanical state of a nucleus can be specified by an energy, a total spin, and a parity.

The spin and parity of the ground state is given by the spin and parity \((-1)^j\) of the "valence" nucleons, that is the last unpaired nucleons in the least bound shell.

\[ 1s^{2}_{1/2} \quad 1p^{2}_{3/2} \quad 1p^{2}_{1/2} \quad 1d^{6}_{5/2} \quad 2s^{2}_{1/2} \quad 1d^{4}_{3/2} \quad \ldots \]
Obviously, nuclei can have excited states just as atoms can. Key differences:

i) 2 kinds of particles to excite
ii) multiple excitations are not uncommon
iii) spin-orbit interaction relatively larger
iv) $l$ can be greater than $n$

These excited states (and in some cases ground states) can serve as resonances for nuclear reactions.

(i < n is true for 1/r potentials but not others)

Spin and parity excited states have either all integer or half-integer spins according to the ground state.

\[
1s_{1/2}^2 1p_{3/2}^1 1p_{1/2}^2 1d_{5/2}^6 2s_{1/2}^2 1d_{3/2}^4 ...
\]

e.g. $^{12}\text{C}$ first excited state

\[
1s_{1/2}^2 1p_{3/2}^4 \rightarrow 1s_{1/2}^1 1p_{3/2}^3 1p_{1/2}^1
\]

Adding $3/2^-$ and $1/2^-$ gives $1^+$ or $2^+$

The first excited state of $^{12}\text{C}$ at 4.439 MeV is $2^+$

but it is not always, or even often that simple.

Multiple excitations, two kinds of particles, adding holes and valence particles, etc. The whole shell model is just an approximation.

Nuclear reactions:

As will be discussed more next time, the excited states or ground state of a nucleus can serve as a "resonance" for a reaction. The more the product state "looks like" the sum of the reactants, the more likely it is to occur.

Reactions must conserve energy of course, but they must also conserve spin and parity.

$\mathbf{J}$ is the vector sum of the spins of the reactants.

$\mathbf{\pi}$ is the parity of the state or particle

For example, the spin and parity of the ground state of $^{12}\text{C}$ is 0$^+$. The spin and parity of the $\alpha$-particle is also 0$. The reaction $^{12}\text{C}(\alpha,\gamma)^{16}\text{O}$ can thus only make 0$^+$ states in $^{16}\text{O}$ – if the reactants have no angular momentum. However, there is a quantized angular momentum for the reactants characterized by a quantum number $l$.

The parity of the interaction is (-1)$^l$. So by "l-waves 0, 1, 2, 3 etc states of 0$, 1$, 2$, 3$ etc in $^{16}\text{O}$ could serve as resonances. $1^+$ would be invisible though.
\( ^{12}\text{C}(p,\gamma)^{13}\text{N} \)

\[
\begin{align*}
\epsilon_n = 1.944 \text{ MeV} & \quad \begin{array}{c}
\text{12C+p} \\
\text{1.944 MeV} \\
\text{13N} \end{array} \\
\text{J}^\pi(^{12}\text{C}) = 0^+ & \quad \text{J}^\pi(p) = 1/2^+ \\
\text{So by } l = 0 \text{ waves can make the } 1/2^+ \text{ resonance in } ^{13}\text{N}. \\
\end{align*}
\]

But what if the exited state had some other spin and parity or \( l \) was not equal to 0?

Suppose the 2.365 MeV state in \(^{13}\text{N}\) had \( J^\pi = 1^- \) instead.

Could the resonant reaction still proceed? Yes but for a different value of \( \ell \).

\[
\begin{align*}
\text{J(target)} + \text{J(projectile)} + \text{J'(projectile)} &= \\
\text{J(product)} + \text{J'(outgoing particle)} + \text{J''(outgoing particle)} \\
\text{J(photons)} &= 0 \\
\text{J(n or p)} &= 1/2 \\
\end{align*}
\]

and we want to couple \( 1/2^- \) (target) to \( 1/2^- \) (product). So \( \ell = 1 \) works since

\[
\frac{1}{2} - \frac{1}{2} = \frac{1}{2}
\]

and the parity is + for the target state and - for \( \ell = 1 \), so \( \ell = 1 \)

would make states in \(^{13}\text{N}\) with spin and parity, \( 1/2^- \), and \( 3/2^- \).

One could make a \( 3/2^- \) state with an \( \ell = 2 \) interaction and so on.

But an \( \ell = 0 \) interaction is much more likely (if possible). Cross sections
decline rapidly with increasing \( \ell \).