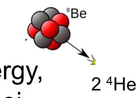
Lecture 4

Basic Nuclear Physics – 2

Nuclear Stability and the Shell Model

Nuclear Stability

A necessary condition for nuclear stability is that, for a collection of "A" nucleons, there exists no more tightly bound aggregate.



- E. g., a single ^{8}Be nucleus. Though it has finite binding energy, (56.4995 MeV), has less binding energy than two ^{4}He nuclei (2 * 28.2957 = 56.591), hence ^{8}Be quickly (6.7 x 10^{-17} s) splits into two heliums (i.e. two alpha particles).

- An equivalent statement is that the nucleus ^{A}Z is stable if there is no collection of A nucleons that weighs less.

- However, one must take care in applying this criterion, because while unstable, some nuclei live a very long time. An operational definition of "stable" is that the isotope has a measurable abundance and no decay has ever been observed (ultimately all nuclei heavier than the iron group are unstable, but it takes almost forever for them to decay). One must also include any lepton masses emitted or absorbed in a weak decay.

Most collections of nucleons have positive binding energy, i.e., are temporarily bound, but a nucleus is still considered "unbound" if it can gain binding by ejecting a neutron or proton. or ion (like ^{4}He). If energetically feasible, this ejection occurs on a very short time scale (e.g. ^{5}Li 3 x 10^{-22} s).

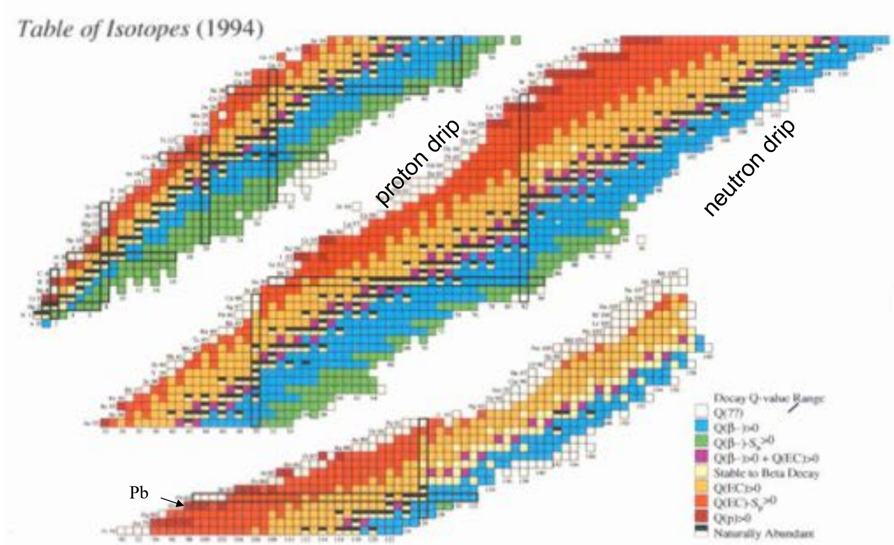
The neutron and proton "drip lines" are defined by

$$\mathrm{BE}(^{A+1}Z) < \mathrm{BE}(^{A}\,Z) \quad S_n < 0$$
$$\mathrm{BE}(^{A+1}Z) < \mathrm{BE}(^{A}\,Z\text{-}1) \quad S_p < 0$$

Note that by definition

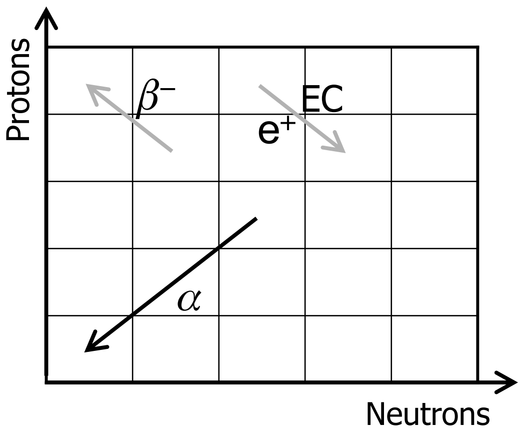
$$\mathrm{BE}\,(\mathrm{n}) = \mathrm{BE}\,(\mathrm{p}) = 0$$

Even a nucleus that is bound is usually *unstable* to weak decay or alpha-decay.



Only the half black squares are stable nuclei, all the squares are bound but most are unstable

Classification of Decays



The lightest known alpha-unstable nuclei (except for ^{8}Be) are $^{104\text{-}110}$Te

α-decay: (no weak interaction)

- emission of Helium nucleus
- Z → Z-2
- N → N-2
- A → A-4

e^--decay (or β-decay)

- emission of e^- and $\bar{\nu}$
- Z → Z+1
- N → N-1
- A = const

e^+-decay

- emission of e^+ and ν
- Z → Z-1
- N → N+1
- A = const

Electron Capture (EC)

- absorbtion of e^- and emiss $\bar{\nu}$
- Z → Z-1
- N → N+1
- A = const

Examples:

^{2}He - diproton - BE < 0 unbound (~700 keV)

^{3}He BE = 7.718 MeV stable

BE(n) = BE(p) = 0

^{4}He 28.296 stable

^{5}He 27.56 unstable n-emission 7.6×10^{-22} s

^{6}He 29.27 bound but decays to ^{6}Li in 807 ms

^{7}He 28.86 unstable n-emission 3×10^{-21} s

^{5}Li 26.33 unstable p-emission $\rightarrow ^4$He + p in 3×10^{-22} s

^{6}Li 31.99 stable

^{7}Li 39.24 stable

^{8}Li 41.27 bound (but decays to ^{8}Be in 840 ms)

^{8}Be 56.50 (barely) unbound - decays to 2 ^{4}He in 6.7×10^{-17} sec

etc

The difference in binding energies for reactions other than weak interactions is also the "Q-value for the reaction"
e.g. ^{3}He(n,γ)^{4}He Q= 20.56 MeV

Energy can often be released by adding nucleons or other nuclei to produce a more tightly bound product:

$$\mathrm{BE}(^{56}\mathrm{Fe}) = 492.247 \text{ MeV}$$
$$\mathrm{BE}(^{57}\mathrm{Fe}) = 499.893 \text{ MeV}$$
$$\mathrm{Q}_{n\gamma}(^{56}\mathrm{Fe}) = 7.646 \text{ MeV}$$

Both ^{56}Fe and ^{57}Fe are stable

The reaction ^{56}Fe(n,γ)^{57}Fe provides 7.646 MeV of kinetic energy and radiation. To go the other way, ^{57}Fe(γ,n)^{56}Fe, would require 7.646 MeV. The locus of nuclei with Q$n\gamma$ = 0 is known as the "neutron-drip line". Similarly Q$p\gamma$ = 0 defines the "proton=drip line".

The criterion for weak decay is a little more complicated because of the mass difference between the neutron and proton and because electrons or positrons may be created or destroyed.

The mass of the *neutral* atom, defined as the "atomic mass' can be written

For Fe the neutron drip line is found at A = 73; the proton drip is at A = 45.

Nuclei from ^{46}Fe to ^{72}Fe are stable against strong decay but only four ^{54,56,57,58}Fe are stable against weak decay.

nuclear part (but m_H contains e^-)

$$M(^AZ) = Z\, m_H + N\, m_n - BE(^AZ)\ /c^2 - + [15.73\, Z^{5/3}\ \text{eV} - Z(13.6\ \text{eV})]\ /c^2$$

electronic binding energy

where m_H is the mass of the neutral hydrogen atom (including m_e), m_n is the mass of the neutron, and the term in the brackets is an approximation to the difference in *electronic* binding energy. The $Z^{5/3}$ term is a Thomas-Fermi approximation to the total binding energy of Z electrons and the Z(13.6) eV term is clearly the electronic binding energy of Z hydrogen atoms. Usually the term in the brackets is negligible and neglected.

More commonly used is the *Atomic Mass Excess*

$$\begin{aligned} 1 \text{ amu} &= 1/12 \text{ the mass of the neutral } {}^{12}C \text{ atom} \\ &= 931.494 \text{ MeV}/c^2 \\ m_p &= 1.00727647 \text{ amu} \\ m_n &= 1.008665012 \text{ amu} \\ m_H &= 1.007825037 \text{ amu} \\ {}^{16}O &= \cancel{15.94915 \text{ amu}} \quad 15.994915 \text{ amu} \\ {}^{12}C &= 12.00000 \text{ amu} \end{aligned}$$

6p, 6n, 6 e^- in the atom

i.e., $m_p + 0.511/931.494$

neutral atoms

$M(^AZ)$ the amomic mass

The *atomic mass excess* is then defined:

$$\boxed{\begin{aligned} \Delta &= \text{atomic mass excess} \\ &= 931.494 \text{ MeV} \left[M(^AZ) - A\right] \end{aligned}}$$

or $\boxed{M(^AZ) = A + \frac{\Delta}{931.494} \text{ amu's}}$

A is an integer

$(15.994915 - 16) / 931.494$
$= -4.737$

The mass excess of ^{12}C is obviously zero. The mass excess of ^{16}O is -4.737 MeV. That is the neutral ^{16}O atom weighs less than 16 times 1/12 of the neutral ^{12}C atom.

This automatically includes the electron masses

Wilhelm Ostwald suggested O as the standard in 1912 (before isotopes were known) In 1961 the carbon-12 standard was adopted. O was not really pure ^{16}O

https://www.nndc.bnl.gov/wallet/

The binding energy (MeV) is given in terms of the mass excess by the previous definition of mass excess (neglecting electronic binding energy) and the definition of the binding energy

$$\boxed{\frac{BE}{c^2} = Z\, m_H + N\, m_n - M(^AZ)}$$

include electron mass but neglect electron binding energy

$$M(^AZ) = A + \frac{\Delta}{931.494} \text{ amu's} \qquad (1 \text{ amu})\, c^2 = 931.494 \text{ MeV}$$

$$\frac{BE(\text{MeV})}{931.494} = Z\,(1.007825 \text{ amu}) + N\,(1.008649 \text{ amu}) \overbrace{- Z - N}^{A} - \frac{\Delta(^AZ)}{931.494}$$

$$= Z\,(0.007825 \text{ amu}) + N(0.008649 \text{ amu}) - \frac{\Delta(^AZ)}{931.494}$$

$$\boxed{BE = Z\,\Delta_H + N\,\Delta_n - \Delta(^AZ)}$$

$.007825 * 931.494 = 7.2889$

where $\Delta_H = 7.288969$ MeV = mass excess of H in amu $\times$ 931.494 MeV
$\Delta_n = 8.071323$ MeV = mass excess of n in amu $\times$ 931.494 MeV

eg. ^{4}He $\Delta = +2.425$
BE = 2(8.07132)+2(7.2889) - 2.425
= 29.296 MeV

Audi and Wapstra, Nuc. Phys A., 595, 409 (1995)

http://t2.lanl.gov/nis/data/astro/molnix96/massd.html

WEAK DECAY

2.2 NUCLEAR STABILITY 67

$^{14}C(e^-\bar{\nu})^{14}N$ $\boxed{\beta\text{-decay:} \quad n \to p + e^- + \bar{\nu}_e}$ unstable if

^{14}N $\Delta = 2.863$
^{14}C $\Delta = 3.020$

$$\begin{aligned} {}^A(Z-1) &\to {}^AZ + e^- + \bar{\nu} \\ m_{\rm nuc}(^AZ-1) &> m_{\rm nuc}(^AZ) + m_e \\ M(^AZ-1) &> M(^AZ) \\ \Delta(^AZ-1) &> \Delta(^AZ) \end{aligned}$$

Add Z-1 electron masses $+(Z-1)m_e$

Nuclear masses
Atomic masses
Mass excesses

$\Delta = M - A$

$^{23}Mg(e^+\nu)^{23}Na$ $\boxed{\text{positron-decay:} \quad p \to n + e^+ + \nu_e}$

$$\begin{aligned} {}^A(Z+1) &\to {}^AZ + e^+ + \nu \\ m_{\rm nuc}(^AZ+1) &> m_{\rm nuc}(^AZ) + m_e \\ M(^AZ+1) &> M(^AZ) + 2m_e \\ \Delta(^AZ+1) &> \Delta(^AZ) + 2m_e \end{aligned}$$

Add Z+1 electron masses $+(Z+1)m_e$

This is a little tricky since one electron mass has to be paid to create the positron, but another also must be paid for the electron that disappears when a neutral atom (Z+1) turns into Z. That is, $m_{\rm nuc}(^AZ+1) = M(^AZ+1) - (Z+1)m_e$ but $m_{\rm nuc}(^AZ) = M(^AZ) - Zm_e$

^{23}Na $\Delta = -9.53$
^{23}Mg $\Delta = -5.47$

$^{7}Be\,(e^{-},\nu)^{7}Li$

^{7}Be $\Delta = 15.768$

^{7}Li $\Delta = 14.907$

Add Z electrons

Also possible at high T

$e^{+} + n \rightarrow p + \bar{\nu}_e$

positron capture

At high density even “stable” nuclei capture electrons

electron capture: $p + e^{-} \rightarrow n + \nu_e$

$$^{A}(Z+1) + e^{-} \rightarrow {}^{A}Z + \cancel{xxxx} + \nu$$

$$\left.\begin{aligned} m_e + m_{\rm nuc}(^{A}Z+1) &> m_{\rm nuc}(^{A}Z) \\ M(^{A}Z+1) &> M(^{A}Z) \end{aligned}\right\} +Zm_e$$

$$\Delta(^{A}Z+1) > \Delta(^{A}Z)$$

These decays may proceed to excited states of the daughter nucleus in which case one or more Γ-rays will be emitted. This is the basis for γ-ray line astronomy.

An example of weak instability

	Z	N	Δ(MeV)
^{13}C	6	7	3.125
^{13}N	7	6	5.345
^{13}B	5	8	16.562

	Δ	Z	N
^{13}B	16.562	5	8
^{13}C	3.125	6	7
^{13}N	5.345	7	7
^{13}O	23.114	8	5

The “Q-value”, or energy carried away by the products, is just the difference in the mass excesses, adjusted in the case of positron-

The energy released in the decay

emission by $2m_ec^2$.

$$Q_{\rm decay} = \begin{cases} \Delta(^{A}Z) - \Delta(^{A}Z-1) & e-\text{decay} \\ \Delta(^{A}Z+1) - \Delta(^{A}Z) - 2m_e & e^{+}-\text{decay} \\ \Delta(^{A}Z+1) - \Delta(^{A}Z) & e-\text{capture} \end{cases}$$

For example:

$$^{13}N(e^{+}\nu)^{13}C \qquad Q_{\beta^{+}} = 1.20 \text{ MeV}$$

$-2m_ec^2$

where 1.20 = 5.345 - 3.125. Note in the same example, that for electron capture the Q-value would be $Q_{ec} = 2.22$ MeV, i.e., $2m_ec^2$ larger. Also, 16.562 - 3.125 = 13.437, and

$2m_ec^2 = 1.02$ MeV

$$^{13}B(e^{-}\nu)^{13}C \qquad Q_{\beta} = 13.437 \text{ MeV}$$

Frequently nuclei are unstable to both electron-capture and positron emission.

Example: $p(p,e^{+}\nu)^{2}H$

Mass excess 2 ^{1}H = 2 x 7.289 MeV
= 14.578 MeV

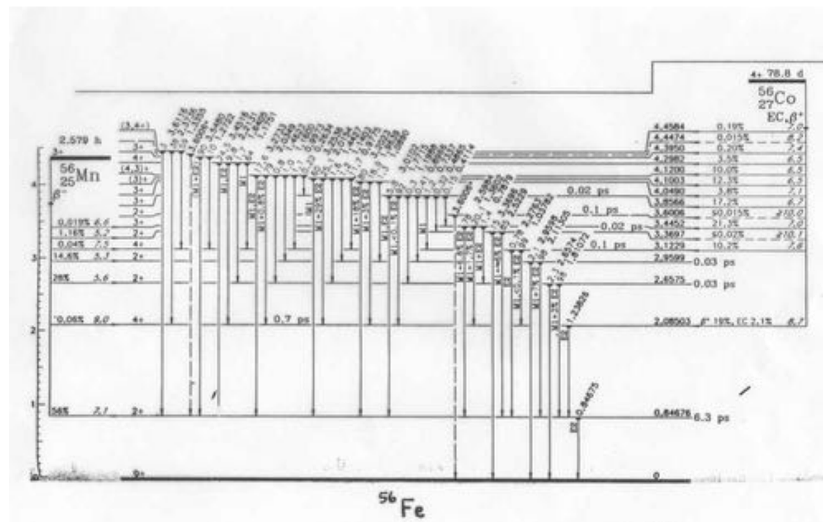
Mass excess ^{2}H = 13.136 MeV. This is a smaller number so the diproton is unstable to weak decay. The Q value is given by

14.578 - 13.136 = 1.442 MeV

$-\,2m_ec^2$ = 0.420 MeV

but the electron and positron annihilate and so we get the $2m_ec^2$ back and the reaction yields 1.442 MeV

But the neutrino carries away a variable amount of energy that averages to 0.262 MeV so really only deposit 1.18 MeV of energy locally



Decays may proceed though excited states

prominent:
847 KeV
1238 KeV

In terms of binding energy

$$Q_\beta = BE(^A Z+1) - BE(^A Z) + 0.782\,\text{MeV}$$

$$Q_{e^+} = BE(^A Z-1) - BE(^A Z) - 1.804\ \text{MeV}$$

$$Q_{ec} = BE(^A Z-1) - BE(^A Z) - 0.782\ \text{MeV}$$

Another example, pick out the stable isotopes:

Nucleus	Δ
^{40}Cl	-27.54
^{40}Ar	-35.04
^{40}K	-33.54
^{40}Ca	-34.85
^{40}Sc	-20.53

The ones with the bigger (less negative) mass excesses are unstable.

^{40}Cl and ^{40}Sc are obviously unstable. ^{40}K can decay either to ^{40}Ar (10.7%) or to ^{40}Ca (89.3%), but both ^{40}Ar and ^{40}Ca are stable,

How many stable isotopes are there for each A? Recall the mass formula

$$BE(^A Z) = a_1 A - a_2 A^{2/3} - a_3 \frac{Z^2}{A^{1/3}} - a_4 \frac{(A-2Z)^2}{A} \pm \delta(A)$$

neglecting shell corrections

We previously solved for Z_{stable} such that the partial of BE with respect to Z at constant A was zero

$$Z_{\text{stable}} = \frac{2a_4 A}{a_3 A^{2/3} + 4a_4}$$

A little algebra (omitted here) shows that if A= constant and $\delta = 0$ (i.e., A is odd), then the differences in binding energy for two nuclei, one having arbitrary Z and the other having Z_{stable} will be parabolic in Z

$$\boxed{\begin{aligned} \Delta BE(\text{odd A}) &= \text{const}\,(Z - Z_{\text{stable}})^2 \\ \text{const} &= -\frac{4a_4}{A} - \frac{a_3}{A^{1/3}} \end{aligned}}$$

for constant A

See the figure on the next page. This means

Proof

$$\Delta BE = -\left(\frac{a_3}{A^{1/3}}\right)\left(Z^2 - Z_{stab}^2\right) - \left(\frac{a_4}{A}\right)\left(\left[A-2Z\right]^2 - \left[A-2Z_{stab}\right]^2\right)$$

$$= -\left(\frac{a_3}{A^{1/3}}\right)\left(Z^2 - Z_{stab}^2\right) - \left(\frac{a_4}{A}\right)\left(A^2 - 4AZ + 4Z^2 - A^2 + 4AZ_{stab} - 4Z_{stab}^2\right)$$

$$= -\left(\frac{a_3}{A^{1/3}}\right)\left(Z^2 - Z_{stab}^2\right) - \left(\frac{4a_4}{A}\right)\left(Z^2 - Z_{stab}^2 - AZ + AZ_{stab}\right)$$

$$= -\left(\frac{a_3}{A^{1/3}}\right)\left(Z^2 - 2ZZ_{stab} + Z_{stab}^2 + 2ZZ_{stab} - 2Z_{stab}^2\right) - \left(\frac{4a_4}{A}\right)\left(Z^2 - 2ZZ_{stab} + Z_{stab}^2 - 2Z_{stab}^2 - AZ + AZ_{stab} + 2ZZ_{stab}\right)$$

$$= K\left(Z - Z_{stab}\right)^2 - \left(\frac{a_3}{A^{1/3}}\right)\left(2ZZ_{stab} - 2Z_{stab}^2\right) - \left(\frac{4a_4}{A}\right)\left(-2Z_{stab}^2 - AZ + AZ_{stab} + 2ZZ_{stab}\right) = K\left(Z - Z_{stab}\right)^2 + F$$

$$F = -\left(\frac{a_3}{A^{1/3}}\right)\left(2ZZ_{stab} - 2Z_{stab}^2\right) - \left(\frac{4a_4}{A}\right)\left(-2Z_{stab}^2 - AZ + AZ_{stab} + 2ZZ_{stab}\right)$$

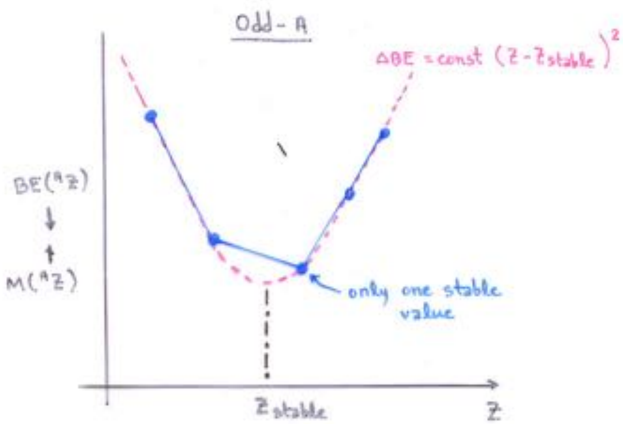
$$= -2Z_{stab}\left(\frac{a_3}{A^{1/3}} + \frac{4a_4}{A}\right)\left(Z - Z_{stab}\right) - \left(4a_4\right)\left(Z_{stab} - Z\right)$$

$$= \left(\frac{2Z_{stab}}{A}\right)\left(a_3 A^{2/3} + 4a_4\right)\left(Z_{stab} - Z\right) - \left(4a_4\right)\left(Z_{stab} - Z\right)$$

$$= \left(\frac{2\left[\dfrac{2a_4 A}{a_3 A^{2/3} + 4a_4}\right]}{A}\right)\left(a_3 A^{2/3} + 4a_4\right)\left(Z_{stab} - Z\right) - \left(4a_4\right)\left(Z_{stab} - Z\right)$$

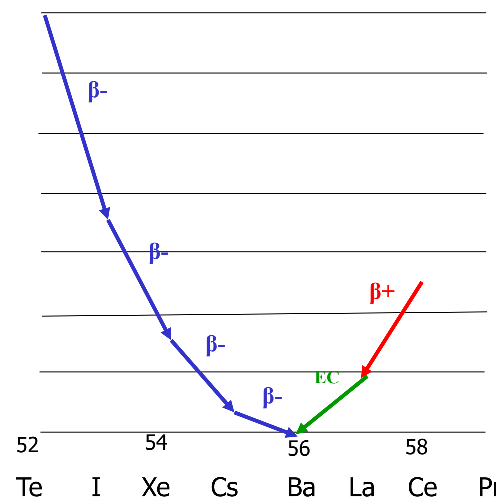
$$= 0$$

At constant A



Odd A. A=135

Single parabola
even-odd and odd-even



Only ^{135}Ba is stable.

that for all A = odd, there is one and only one stable isotope, e.g., ^{13}C, ^{15}N, ^{17}O, ^{19}F, ^{21}Ne, ^{23}Na, ^{27}Al, etc. There are some near calls - ^{113}Cd decays to ^{113}In with a half life of 9×10^{15} y; ^{115}In decays to ^{115}Sn with a half life of 4×10^{14} y; and ^{123}Te decays to ^{123}Sb with a half life of 1×10^{13} y. These special cases are because of shell closures. e.g., at Z = 50 for In and Sn.

Things are more complicated for even A because of the pairing correction and the two different ways of making even A (even Z,N; odd Z,N).

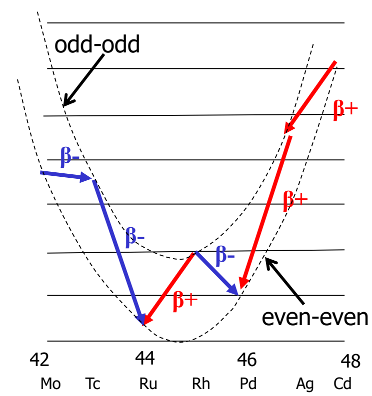
$$\begin{aligned} \Delta BE(\text{even A}) &= \text{const}\,(Z - Z_{\text{stab}})^2 \\ &\quad + \delta \quad \text{odd Z} \\ &\quad - \delta \quad \text{even Z} \end{aligned}$$

As a result one gets *two* curves, one for the odd-Z, even-A isotopes, and one for the even-Z, even-A isotopes. Depending on the placement of points on these curves one can have 1, 2, or even 3 stable isotopes at each

- Even A:
- two parabolas
- one for o-o & one for e-e
- lowest o-o nucleus often has two decay modes
- most e-e nuclei have two stable isotopes
- there are nearly no stable o-o nuclei in nature because these can usually decay to an e-e nucleus
- Exceptions ^{2}H, ^{6}Li, ^{10}B, ^{14}N

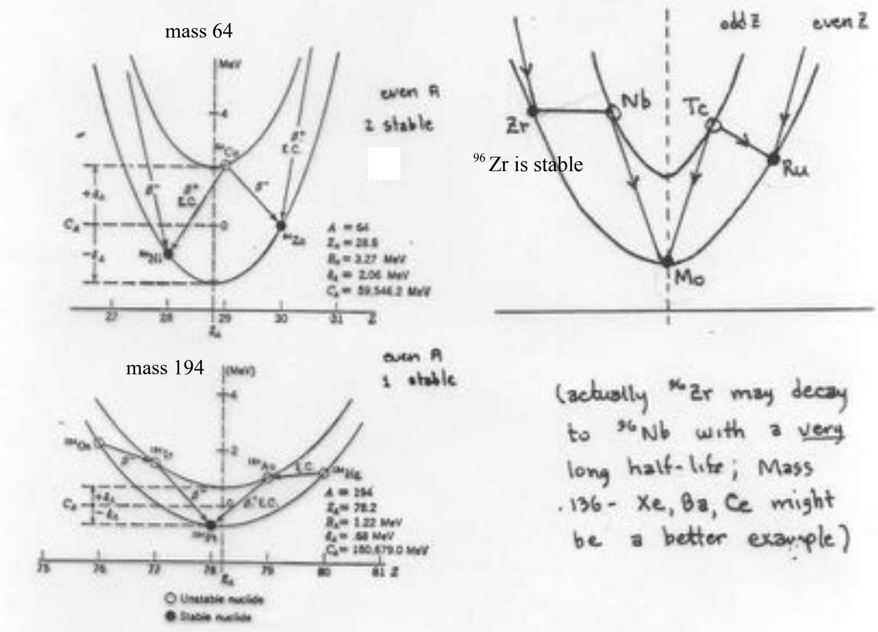
Even A. A=102

Two parabolas separated by 2δ,
odd-odd and even-even



Stable ^{102}Ru and ^{102}Pd

an “even-even” nucleus must decay to
an “odd-odd” nucleus and vice versa.



To summarize:

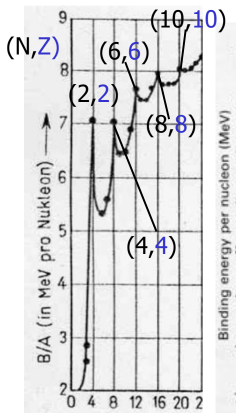
odd A	There exists one and only one stable isotope
odd Z – odd N	Very rarely stable. Exceptions ^{2}H, ^{6}Li, ^{10}B, ^{14}N. Large surface to volume ratio. Our liquid drop model is not really applicable.
even Z – even N	Frequently only one stable isotope (below sulfur). At higher A, frequently 2, and occasionally, 3.

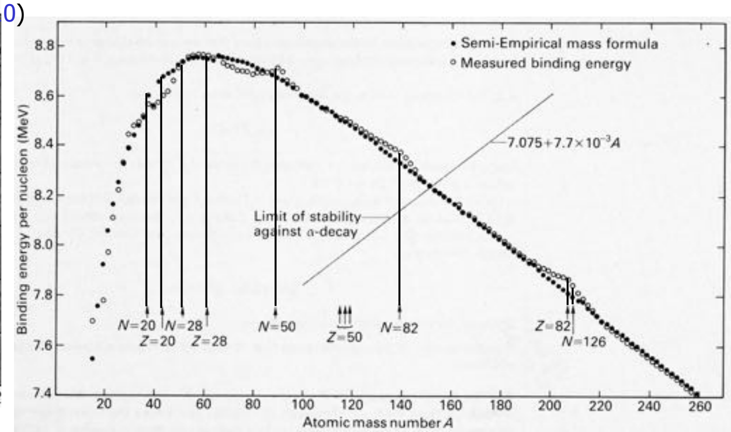
A. For example ^{12}C, ^{14}N, and ^{16}O; but also ^{40}Ar, ^{40}Ca, ^{54}Cr, ^{54}Fe, ^{64}Ni ^{64}Zn; and even ^{136}Xe, ^{136}Ba, ^{136}Ce. Because the pairing energy gets smaller as one goes to large A, the two parabolas lie closer and it is easier to have multiplets. For light elements below sulfur, 1 isotope is typical for even A. Above about calcium, two isotopes are typical, but there are exceptions, especially in the vicinity of closed shells. Nuclei with both odd Z and odd N are very rarely bound, but there are notable exceptions, ^{2}H, ^{6}Li, ^{10}B, ^{14}N, but these are so light that our liquid drop model is quite inadequate.

The Shell Model

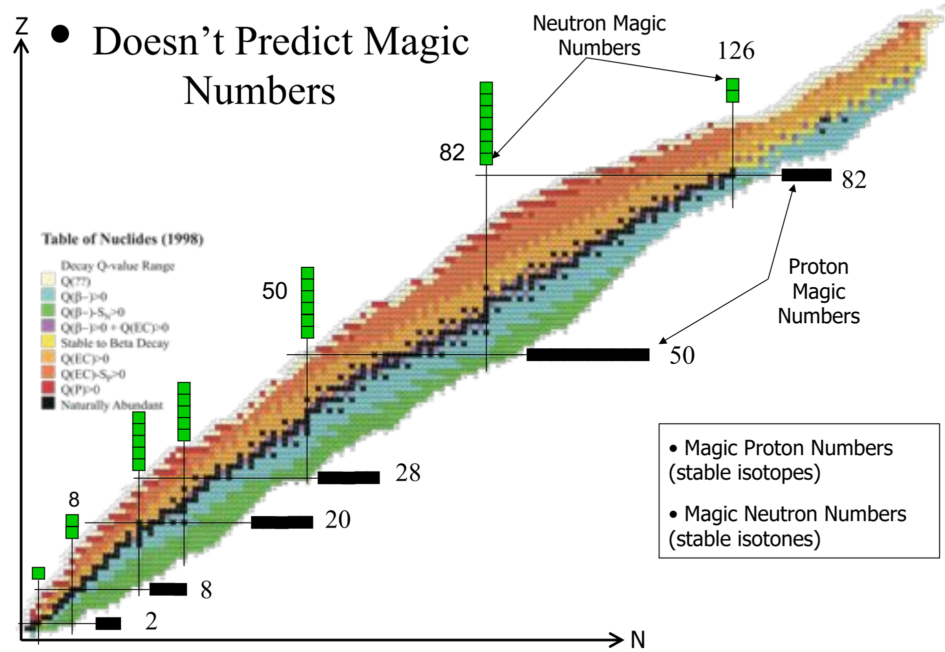
Shortcomings of the Liquid Drop Model

- Simple model does not apply for A < 20



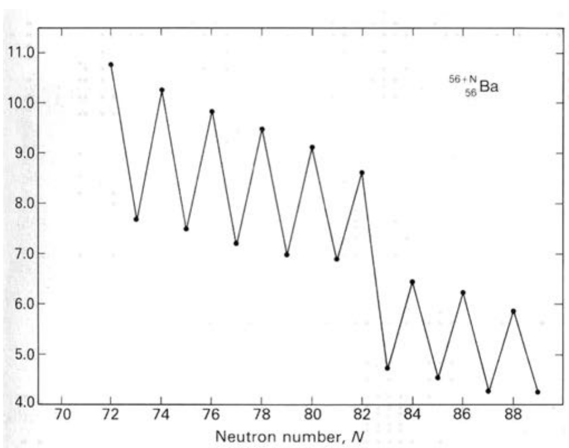


- Doesn't Predict Magic Numbers

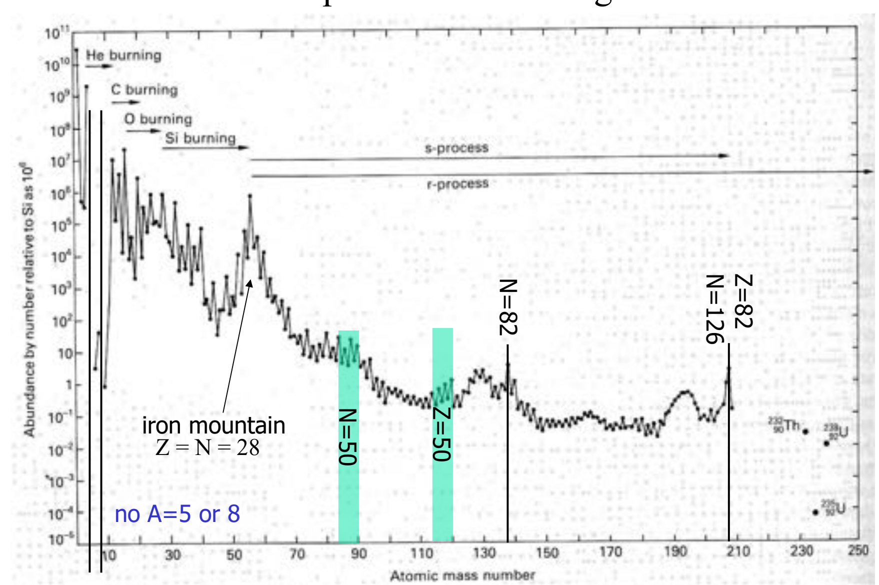


- Neutron separation energies
  - saw tooth from pairing term
  - big step down when N goes across magic number at 82

Ba Neutron separation energy in MeV



Abundance patterns reflect magic numbers



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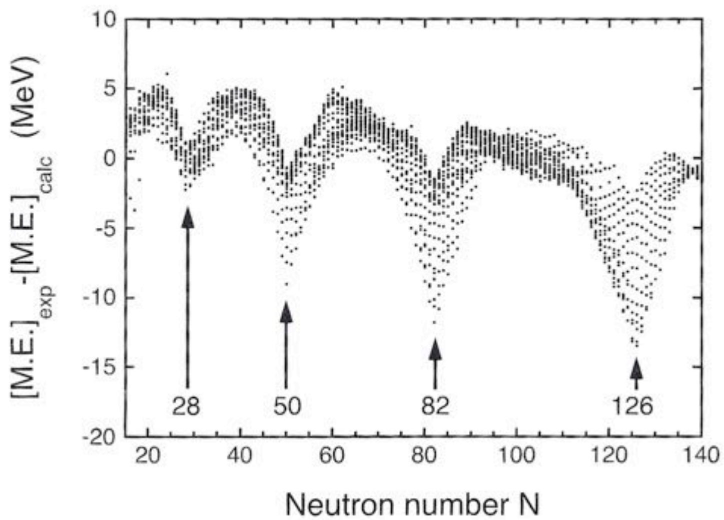
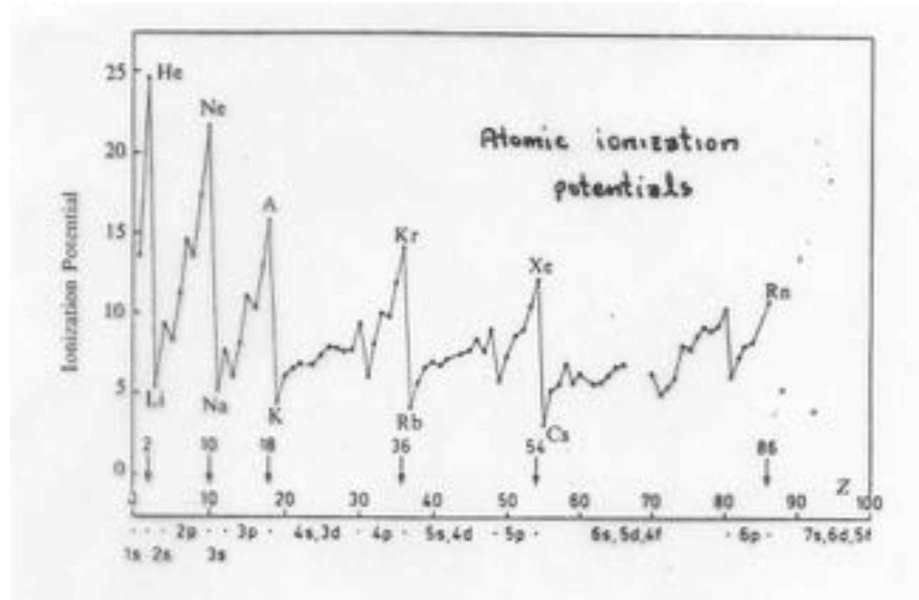


Fig. 1.11 Difference between experimental ground-state atomic mass excess (Audi et al. 2003) and the mass excess predicted by the spherical macroscopic part of the finite-range droplet (FRDM) mass formula (Möller et al. 1995) versus neutron number.



<u>Shell Model</u> – Mayer and Jensen 1963 Nobel Prize

Our earlier discussions treated the nucleus as sets of identical nucleons and protons comprising two degenerate Fermi gases. That is OK so far as it goes, but now we shall consider the fact that the nucleons have spin and angular momentum and that, in analogy to electrons in an atom, are in ordered discrete energy levels characterized by conserved quantized variables – energy, angular momentum and spin.

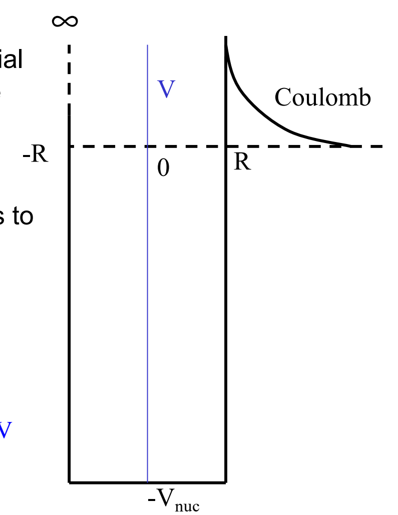
Clayton 311 – 319

A highly idealized nuclear potential looks something like this "infinite square well".

As is common in such problems one applies boundary conditions to Schroedinger's equation.

$$V = -V_{nuc} \qquad r < R$$
$$= \infty \qquad r \geq R$$

$$\Psi(R) = 0 \qquad V_{nuc} \approx 50 - 60 \text{ MeV}$$



(In the case you have probably seen before of electronic energy levels in a hydrogen atom, one would follow the same procedure, but the potential would be the usual [attractive] 1/r potential.)

Schroedinger's Equation:

$$-\frac{\hbar^2}{2M}\nabla^2\Psi + (\mathrm{V} - \mathrm{E})\Psi = 0$$

Spherical symmetry:

$$\Psi_{\mathrm{n,l,m}}(r,\theta,\varphi) = f_{n,l}(r)\,Y_l^m(\theta,\phi)$$

Radial equation:

$$-\frac{\hbar^2}{2\mathrm{M}}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right) f_{n,l}(r) + \left[\underbrace{\frac{l(l+1)\hbar^2}{2\mathrm{M}r^2}}_{\text{Rotational energy}} + \underbrace{\mathrm{V}_{\mathrm{nuc}}(r)}_{\text{Nuclear potential}}\right] f_{n,l}(r) = \underbrace{\mathrm{E}}_{\text{Energy eigenstate}}\, f_{n,l}(r)$$

Clayton 4-102

Solve for E.

Substitute:

$$\rho = \sqrt{\frac{2\mathrm{M}(\mathrm{E} - \mathrm{V}_{\mathrm{nuc}})}{\hbar^2}}\; r \qquad V_{nuc} \text{ is} < 0$$

To obtain:

$$\rho^2\frac{\partial^2 f}{\partial \rho^2} + 2\rho\frac{\partial f}{\partial \rho} + (\rho^2 - l(l+1))\,f = 0$$

Solution is:

$$f = \sqrt{\frac{\pi}{2\rho}}\;\mathrm{J}_{l+1/2}(\rho)$$

Spherical Bessel Functions

Abramowitz and Stegun 10.1.1

http://people.math.sfu.ca/~cbm/aands/

Classically, centrifugal force goes like

$$F_c = \frac{mv^2}{R} = \frac{m^2v^2R^2}{mR^3} = \frac{L^2}{mR^3}$$

One can associate a centrifugal potential with this,

$$\int F_c\, dR = \frac{-L^2}{2mR^2}$$

Taking the usual QM eigenvaluens for the operator L^2 one has

$$\frac{-l(l+1)\,\hbar^2}{2m\mathrm{R}^2}$$

The solutions to the infinite square well potential are then the zeros of spherical Bessel functions (Landau and Lifshitz, Quantum Mechanics, Chapter 33, problem 2)

$$\boxed{E_{n,l} = -\left|V_{nuc}\right| + \frac{\hbar^2}{2MR^2}\left[\pi^2\left(n + \frac{\ell}{2}\right)^2 - \ell(\ell+1)\right]}$$

more negative means more bound

We follow the custom of labeling each state by a principal quantum number, n, and an angular momentum quantum number, ℓ, e.g. 3d (n = 3, ℓ = 2) ℓ = 0, 1, 2, 3, 4, 5, etc = s, p, d, f, g, h etc

- States of higher n are less bound as are states of larger ℓ
  ℓ can be greater than n
- Each state is 2 (2ℓ +1) degenerate. The 2 out front is for the spin and the 2 ℓ + 1 are the various z projections of ℓ
- E.g., a 3d state can contain 2 (2(2) +1) = 10 neutrons or protons

This gives an energy ordering

$$\pi^2\left(n+\frac{\ell}{2}\right)^2-\ell(\ell+1)$$

$1s^2$	$1p^6$	$1d^{10}$	$2s^2$	$1f^{14}$	$etc.$
π^2	$\frac{9\pi^2}{4}-2$	$4\pi^2-6$	$4\pi^2$	$\frac{25}{4}\pi^2-12$	
9.87	20.20	33.48	39.48	49.69	

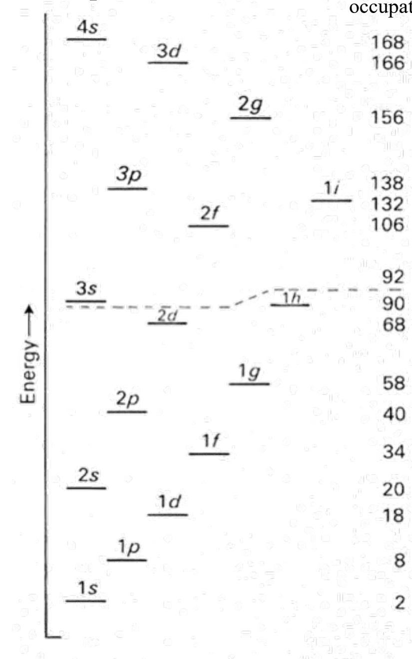
This simple progression would predict shell closures at Z = N = 2, 8, 18, 20, 34 etc, i.e, ^{4}He, ^{16}O, ^{36}Ar, ^{40}Ca, etc

A good beginning but increasingly in error at high Z, N

So far we have considered the angular momentum of the nucleons but have ignored the fact that they are Fermions and have spin

Infinite Square Well Solutions

cumulative occupation



desired magic numbers

126

82

50

28

20

8

2

dotted line is to distinguish 3s, 2d, and 1h.

Improving the Nuclear Potential Well

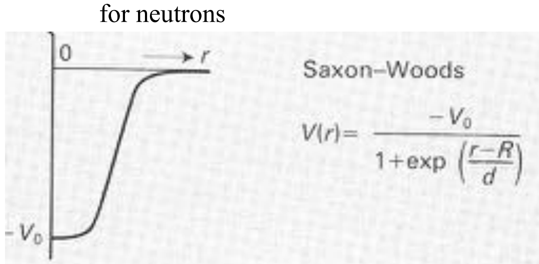
The real potential should be of finite depth and should probably resemble the nuclear density - flat in the middle with rounded edges that fall off sharply due to the short range of the nuclear force.

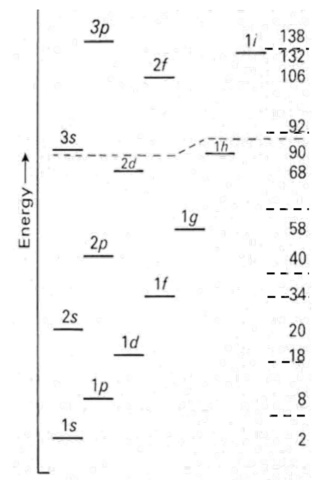
for neutrons

R ≈ Nuclear Radius

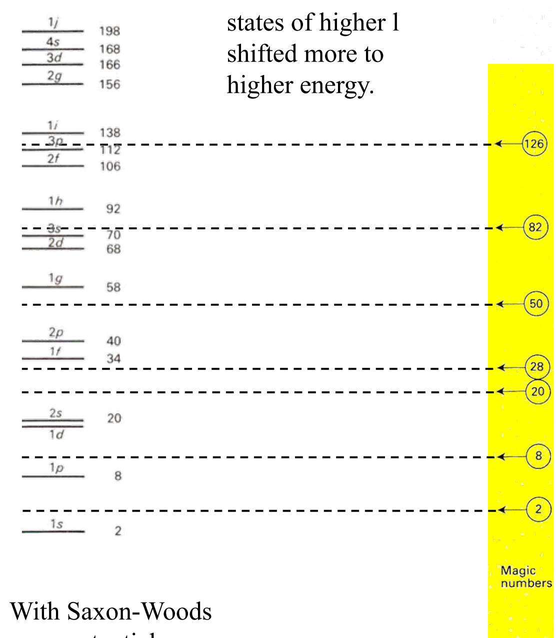
d ≈ width of the edge

R >> d





Infinite square well



states of higher l shifted more to higher energy.

With Saxon-Woods potential

Better, the gap at 20 is now closer to correct.

But this still is not very accurate above Z = 20 because:

- Spin is very important to the nuclear force
- The Coulomb force becomes important for protons but not for neutrons.

Introduce spin-orbit and spin-spin interactions

$$\vec{l}\cdot\vec{s} \text{ and } \vec{s}\cdot\vec{s}$$

Define a new quantum number

$$\vec{j} = \vec{l} + \vec{s}$$

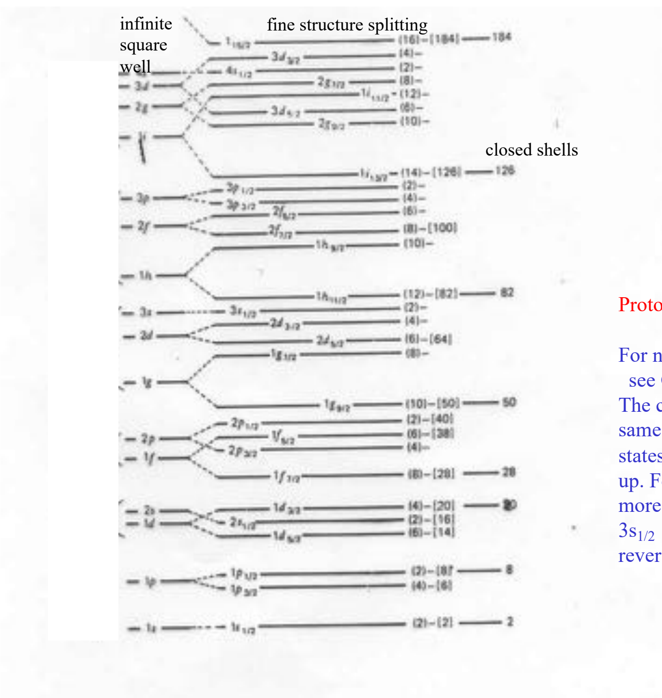
Get spliting of levels into pairs

$1p \rightarrow 1p_{1/2} \quad 1p_{3/2}$

$2f \rightarrow 1f_{5/2} \quad 2f_{7/2}$

etc

Label states by nl_j



Protons:

For neutrons see Clayton p. 315
The closed shells are the same but the ordering of states differs from $1g_{7/2}$ on up. For neutrons $2d_{5/2}$ is more tightly bound. The $3s_{1/2}$ and $2d_{3/2}$ are also reversed.

This interaction is quite different from the fine structure splitting in atoms. It is much larger and *lowers* the state of larger **j** (parallel **l** and **s**) compared to one with smaller **j**. See Clayton p. 311ff). The interaction has to do with the spin dependence of the nuclear force, not electromagnetism.

Empirically $V = -\alpha\, \mathbf{l}\cdot\mathbf{s}$

$\alpha = 13\ A^{-2/3}$ MeV

$$\Delta E = -\frac{\alpha}{2} l \qquad j = (l + \frac{1}{2})$$

$$+\frac{\alpha}{2}(l+1) \qquad j = (l - \frac{1}{2})$$

These can be large compared even to the spacing between the principal levels.

The state with higher j is more tightly bound; the splitting is bigger as l gets larger.

For neutrons the level scheme is the same as for protons up to N = 50. Above that the Coulomb repulsion of the protons has an effect and favors orbits (for protons) with higher angular momentum. Thus for example the 51st neutron is in the d level of j = 5/2 while for protons it is in the g level of j = 7/2. The effect is never enough to change the overall shell closures and magic numbers.

Maria Goeppert Mayer – Nobel - 1963

The correct energy ordering then becomes:

For neutrons:

$$1s_{1/2}^2 \;|\; 1p_{3/2}^4 \; 1p_{1/2}^2 \;|\; 1d_{5/2}^6 \; 2s_{1/2}^2 \; 1d_{3/2}^4 \;|$$
$$1f_{7/2}^8 \;|\; 2p_{3/2}^4 \, 1f_{5/2}^6 \; 2p_{1/2}^2 \, 1g_{9/2}^{10} \;|\; etc.$$

where | denotes a large energy gap – hence “magic number

For protons the ordering is the same up to $1g_{9/2}$ but differs at the next level, $2d_{5/2}$ for neutrons, $1g_{7/2}$ for protons

Each state can hold (2j+1) nucleons.

The numbers where each of these shells close are

2, (6), 8, (14, 16), 20, 28, (32 38, 40), 50

where the calculated shell gaps are relatively small for the numbers in parenthesis

Remember $\boxed{2,\ 8,\ 20,\ 28,\ 50,\ 82,\ 126}$

Examples: ^4He, ^{16}O, ^{40}Ca, ^{56}Ni, ^{90}Zr (Z=40), ^{48}Ca

^{120}Sn (Z=50), ^{208}Pb (Z=82, N=126), ^{209}Bi (N=126) (end of the s-process)

Each state is now (2j+1) degenerate (less than before)

The total number of states of given n + l is still the same 2(2*l*+1)

before 1p (2)(2+1)=6 now $1p_{3/2}$ (4), $1p_{1/2}$ (2)

The states with higher j are more tightly bound

(remember ^2H ↑↑ $j=1^+$ is bound; ^2He ↑↓ $j=0^+$ is not)

Some implications:

A. Ground states of nuclei

Each quantum mechanical state of a nucleus can be specified by an energy, a total spin, and a parity.

The spin and parity of the ground state is given by the spin and parity $(-1)^l$ of the “valence” nucleons, that is the last unpaired nucleons in the least bound shell.

$$1s_{1/2}^2 \, 1p_{3/2}^4 \, 1p_{1/2}^2 \; 1d_{5/2}^6 \, 2s_{1/2}^2 \, 1d_{3/2}^4 \; \ldots$$

i) All ground states of even-even nuclei have spin and parity 0^+ – the nucleons are all paired

eg. ^{12}C $1s_{1/2}^2 \; 1p_{3/2}^4$ n — 6n,6p

$1s_{1/2}^2 \; 1p_{3/2}^4$ p

^{18}O ( ) $1p_{1/2}^2 \; 1d_{5/2}^2$ n — 10n,8p

( ) $1p_{1/2}^2$ p

ii) Odd-mass nuclei – spin and parity usually given by extra (“valence”) nucleon

eg. ^{17}O ( ) $1d_{5/2}$ n — 8 protons, 9 neutrons

$J^\pi = (5/2)^+$ (parity is $(-1)^l$)

^{15}O ( ) $1p_{1/2}$ — 8 protons, 7 neutrons

$J^\pi = (1/2)^-$

iii) The odd-odd nuclei pose special problems

^{14}N $1s_{1/2}^2 \; 1p_{3/2}^4 \; 1p_{1/2}$ n

" " " p

The total J^π is the vector sum of the two extra nucleons which could be 0^+ or 1^+

It turns out to be 1^+ (but the first excited state (2.313 MeV) is 0^+.

(the parity is the product of the parity of the two states)

Obviously, nuclei can have excited states just as atoms can. Key differences -

i) 2 kinds of particles to excite

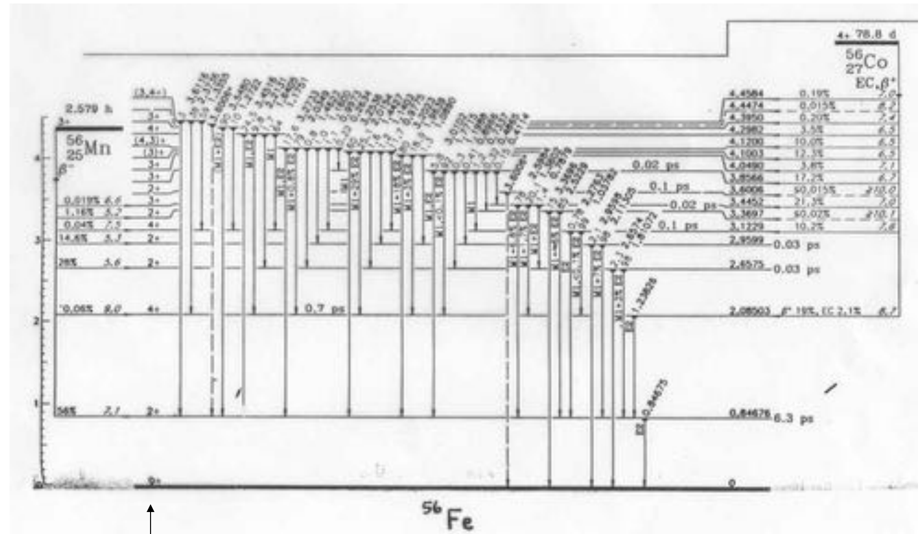
ii) multiple excitations are not uncommon

iii) spin-orbit interaction relatively larger

iv) l can be greater than n

($l < n$ is true for 1/r potentials but not others)

These excited states (and in some cases ground states) can serve as resonances for nuclear reactions.



spin and parity

excited states have either all integer or half-integer spins according to the ground state.

$$1s_{1/2}^2\,1p_{3/2}^4\,1p_{1/2}^2\;1d_{5/2}^6\,2s_{1/2}^2\,1d_{3/2}^4\;\ldots$$

eg, ^{12}C first excited state

$$1s_{1/2}^2 1p_{3/2}^4 \;\rightarrow\; 1s_{1/2}^2 1p_{3/2}^3 1p_{1/2}^1$$

Adding $3/2^-$ and $1/2^-$ gives 1^+ or 2^+

The first excited state of ^{12}C at 4.439 MeV is 2^+

but it is not always, or even often that simple.

Multiple excitations, two kinds of particles, adding holes and valence particles, etc. The whole shell model is just an approximation.

Nuclear reactions:

As will be discussed more next time, the excited states or ground state of a nucleus can serve as a "resonance" for a reaction. The more the product state "looks like" the sum of the reactants, the more likely it is to occur.

Reactions must conserve energy of course, but they must also conserve spin and parity.

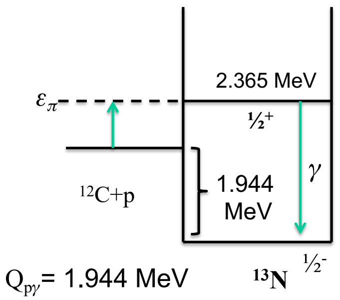
J is the vector sum of the spins of the reactants.

π is the parity of the state or particle

For example, the spin and parity of the ground state of ^{12}C is 0^+. The spin and parity of the α-particle is also 0^+. The reaction ^{12}C(α,γ)^{16}O can thus only make 0^+ states in ^{16}O – if the reactants have no angular momentum. However, there is a quantized angular momentum for the reactants characterized by a quantum number l. The parity of the interaction is $(-1)^l$. So by "l-waves 0, 1, 2, 3 etc states of 0^+, 1^-, 2^+, 3^-, etc in ^{16}O could serve as resonances. 1^+ would be invisible though.

$1s_{1/2}^2\, 1p_{3/2}^4\, 1p_{1/2}^2\; 1d_{5/2}^6\, 2s_{1/2}^2\, 1d_{3/2}^4\; \ldots$

$^{12}C(p,\gamma)^{13}N$



$J^\pi(^{12}C) = 0^+$
$J^\pi(p) = 1/2^+$

So by l = 0 waves can make the ½+ resonance in ^{13}N.

But what if the exited state had some other spin and parity or l was not equal to 0?

Suppose the 2.365 MeV state in ^{13}N had $J^\pi = \frac{1}{2}^-$ instead.

Could the resonant reaction still proceed? Yes but for a different value of ℓ.

$\bar{J}$(target) + $\bar{J}$(projectile) + $\bar{l}$ (projectile)=
$\bar{J}$(product) + $\bar{J}$(outgoing particle) + $\bar{l}$ (outgoing particle)

J(photon)= 0

J(n or p) = 1/2

and we want to couple $1/2^+$ (target) to $1/2^-$ (product). So ℓ=1 works since

$$\frac{\bar{1}}{2} + \vec{1} = \frac{3}{2},\ \frac{1}{2}$$

and the partity is + for the target state and - for $\ell = 1$, so ℓ=1 would make states in ^{13}N with spin and parity, $1/2^-$, and $3/2^-$.

One could make a $3/2^+$ state with an ℓ=2 interaction and so on.

But an ℓ =0 interaction is much more likely (if possible). Cross sections decline rapidly with increasing ℓ