Lecture 5

Basic Nuclear Physics – 3

Nuclear Cross Sections and Reaction Rates

The reaction rate for the two reactants, *I* and *j* as in e.g., I (j,k) L is:

$$n_I n_j \sigma_{Ij} \, v$$

which has units "reactions cm^{-3} s^{-1}"

It is often more convenient to write abundances in terms of the mole fractions,

$$Y_I = \frac{X_I}{A_I} \qquad n_I = \rho N_A Y_I \qquad \left(\frac{gm}{cm^3}\right)\left(\frac{atoms}{Mole}\right)\left(\frac{Mole}{gm}\right)$$

so that the rate becomes

$$(\rho N_A)^2 Y_I Y_j \sigma_{Ij} \, v$$

and a term in a rate equation decribing the destruction of I might be

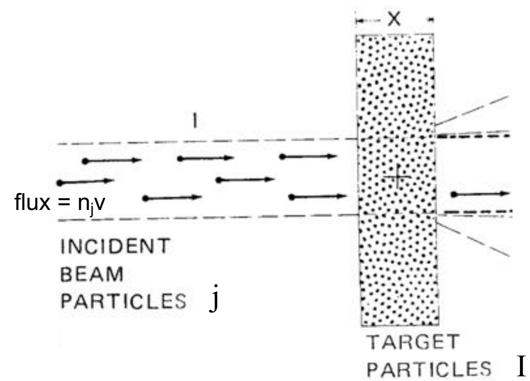
$$\frac{dY_I}{dt} = -\rho Y_I Y_j N_A \langle \sigma_{Ij} v \rangle + ...$$

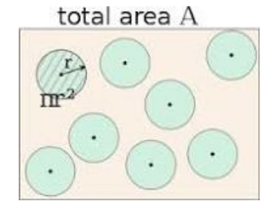
Equivalent to

$$\frac{dn_I}{dt} = -n_I n_j \langle \sigma_{Ij} v \rangle + ...$$

Here $\langle \ \rangle$ denotes a suitable average over energies and angles and the reactants are usually assumed to be in thermal equilibrium. The thermalization time is short compared with the nuclear timescale.

Cross Sections





Total area of target nuclei per cm^3 = $n_I \sigma_I$

Reaction rate per cm^3 per sec = $n_j v \, n_I \, \sigma_I$

For example, a term in the rate equation for ^{12}C(p,γ)^{13}N during the CNO cycle might look like

$$\frac{dY(^{12}C)}{dt} = -\rho Y(^{12}C) Y_p N_A \langle \sigma_{p\gamma}(^{12}C) v \rangle + ...$$

For a Maxwell-Boltzmann distribution of reactant energies

$$f(v) = \sqrt{\left(\frac{m}{2\pi kT}\right)^3} 4\pi v^2 e^{-\frac{mv^2}{2kT}},$$

The average, over angles and speed, of the cross section times velocity is

$$\langle \sigma_{Ij} \mathrm{v} \rangle = 4\pi \sqrt{\left(\frac{m}{2kT}\right)^3} \int_0^\infty \sigma_{Ij}(v) v^3 \, e^{-mv^2/2kT} \, dv$$

$$\langle \sigma_{Ij} \mathrm{v} \rangle = \left(\frac{8}{\pi\mu}\right)^{1/2} \left(\frac{1}{kT}\right)^{3/2} \int_0^\infty \sigma_{Ij}(E) \, E \, e^{-E/kT} \, dE$$

where μ is the "reduced mass"

$$\mu = \frac{\mathrm{M_I m_j}}{\mathrm{M_I + m_j}}$$

for the reaction I (j, k) L

$$v = \left(\frac{2E}{m}\right)^{1/2} \qquad dv = \left(\frac{1}{2}\right)\left(\frac{2}{mE}\right)^{1/2} dE$$

$$\sigma v^3 \, dv = \sigma \left(\frac{2E}{m}\right)^{3/2} \frac{1}{2} \left(\frac{2}{mE}\right)^{1/2} dE$$

$$= \frac{2}{m^2} \sigma E \, dE$$

Center of mass system – that coordinate system in which the total momenta of the reactants is zero.

The energy implied by the motion of the center of mass is not available to cause reactions.

Replace mass by the "reduced mass"

$$\mu = \frac{\mathrm{M_1 M_2}}{\mathrm{M_1 + M_2}}$$

Read Clayton – Chapter 4.1

For T in 10^9 K = 1 GK, s in barns (1 barn = 10^{-24} cm^2), E_6 in MeV, and k = 1/11.6045 MeV/GK, the thermally averaged rate factor in cm^3 s^{-1} is:

$$\langle \sigma_{\mathrm{jk}} \mathrm{v} \rangle = \frac{6.199 \times 10^{-14}}{\hat{\mathrm{A}}^{1/2} \mathrm{T_9}^{3/2}} \int_0^\infty \sigma_{jk}(E_6) \, E_6 \, e^{-11.6045 E_6/T_9} \, dE_6$$

$$\hat{\mathrm{A}} = \frac{\mathrm{A_I A_j}}{\mathrm{A_I + A_j}} \qquad \text{for the reaction I(j,k)L}$$

Ideally one would just measure the cross section as a function of energy, put σ(E) in the integral, integrate numerically, tabullate the result as a function of temperature and proceed. There are several reasons why this doesn't usually work

- The energies of importance in stars, which can wait a long time for a reaction to occur, are generally so low that the cross section is too small to measure directly.
- The targets are of sometimes radioactive and can't be made or handled in the laboratory
- There are too many reactions of interest

Consequently one must use a combination of measurement, extrapolation, and theory to get useful answers

The actual form of σ may be very complicated and depends upon the presence or absence of resonances however, it is of the form …

$$\lambda\!\!\!^{-} = \frac{\hbar}{p} = \frac{1}{k}$$

Area subtended by a de Broglie wavelength in the c/m system. Characteristic quantum mechanical dimension of the system

How much the nucleus I+j looks like the target nucleus I with j sitting at its surface. Liklihood of staying inside R once you get there.

$$\sigma(E) = \pi\lambda\!\!\!^{-2} \quad \rho P_l(E) \quad X(E)$$

geometry term (Cla 4-180) — penetration factor — nuclear structure

probability a flux of particles with energy E at infinity will reach the nuclear surface. Must account for charges and QM reflection.

see Clayton Chapter 4

The barrier penetration term and an overall quantum mechanical dimension don't depend on what happens inside the nucleus

$$\pi\lambda\!\!\!^{-2} \quad \rho P_l(E)$$

all the uncertain physics that goes on inside the nucleus once the reactants have penetrated within the (well-defined) boundary of the nucleus is in

$$X(E)$$

X can be slowly varying with energy – as in "non-resonant" reactions – or rapidly varying – as in resonant reactions.

Here $\lambda\!\!\!^{-}$ is the de Broglie wavelenth in the c/m system

$$\pi\lambda\!\!\!^{-2} = \frac{\pi\hbar^2}{\mu^2 v^2} = \frac{\pi\hbar^2}{2\mu E} = \frac{0.656 \text{ barns}}{\hat{A}\ E(\text{MeV})}$$

and 1 barn = 10^{-24} cm^2 is large for a nuclear cross section.
Note that generally E(MeV) < 1 and $\lambda\!\!\!^{-} > R_{nucleus}$ but much smaller than the interparticle spacing.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad KE = \frac{1}{2}\mu v_{1,2}^2 \quad \vec{v}_{1,2} = \vec{v}_1 - \vec{v}_2$$

$$\hat{A} = \frac{A_1 A_2}{A_1 + A_2} \sim 1 \text{ for neutrons and protons if } A_1 \text{ is large}$$

~ 4 for α-particles if A_1 is large

e.g., ^{12}C(p,γ)^{13}N $\quad \hat{A} = \frac{(12)(1)}{(12+1)} = \frac{12}{13}$

For discussion of center of mass energy see

https://www.youtube.com/watch?v=lhwxK49d28Q

https://www.youtube.com/watch?v=mjrQHIJj1iI

Barrier Penetration
See Clayton Chapter 4.5 and
Appendix 1 to this lecture for derivation

ρP_l gives the probability of barrier penetration to the nuclear radius R with angular momentum l. Sometimes the ρ is absorbed ino the definition of P_ℓ. Here it is not. Under stellar conditions for charged particles it is usually very small,

$$\rho P_l = \frac{\rho}{F_l^2(\eta,\rho) + G_l^2(\eta,\rho)}$$

e.g., Illiadis 2.162

where F_l is the regular Coulomb function and G_l is the irregular Coulomb function

See Abramowitcz and Stegun, *Handbook of Mathematical Functions,* p. 537
These are functions of the dimensionless variables

$$\eta = \frac{Z_I Z_j e^2}{\hbar v} = 0.1575 Z_I Z_j \sqrt{\hat{A}/E}$$ contains all the charge dependence

$$\rho = \sqrt{\frac{2\mu E}{\hbar^2}}\, R = 0.2187\sqrt{\hat{A}E}\, R_{fm}$$ contains all the radius dependence

Physical meaning of $\eta = \frac{Z_I Z_j e^2}{\hbar v}$ *nb.*, both η and ρ are dimensionless.

The classical turning radius, r_0, is given by

$$\frac{1}{2}\mu v^2 = \frac{Z_I Z_j e^2}{r_0}$$

The de Broglie wavelength on the other hand is

$$\lambda\!\!\!^{-} = \frac{\hbar}{p} = \frac{\hbar}{\mu v} \qquad r_o = \frac{2 Z_I Z_j e^2}{\mu v^2} = \eta \frac{2\hbar}{\mu v} = 2\eta \lambda\!\!\!^{-}$$

Hence $\eta = \frac{r_0}{2\lambda\!\!\!^{-}}$ i.e., half the turning radius measured in units of the DeBroglie wavelength

The probability of finding the particle inside of its classical turning radius decreases exponentially with this ratio.

On the other hand,

$$\rho = \sqrt{\frac{2\mu \mathrm{E}}{\hbar^2}}\, R = \frac{R}{\lambda\!\!\!^{-}} \qquad \lambda\!\!\!^{-} = \frac{\hbar}{p} = \frac{\hbar}{\mu v} = \frac{\hbar}{\sqrt{2(\mu)\left(\frac{1}{2}\mu v^2\right)}}$$

is just the size of the nucleus measured in de Broglie wavelengths.

This enters in, even when the angular momentum and charges are zero, because an abrupt change in potential at the nuclear surface leads to reflection of the wave function.

For low interaction energy, ($2\eta >> \rho$, i.e., $\mathrm{E} << \frac{Z_I Z_j e^2}{R}$)

and $\mathrm{Z_j} \neq 0$, ρP_l has the interesting limit

$$\eta = \frac{Z_I Z_j e^2}{\hbar v} = 0.1575 Z_I Z_j \sqrt{\hat{A}/E}$$

$$\rho = \sqrt{\frac{2\mu E}{\hbar^2}}\, R = 0.2187\sqrt{\hat{A}E}\, R_{fm}$$

Abramowitz and Stegun, 14.6.7

$$\rho P_l \approx \sqrt{2\eta\rho}\, \exp\left[-2\pi\eta + 4\sqrt{2\eta\rho} - \frac{2l(l+1)}{\sqrt{2\eta\rho}}\right]$$

where

$$\sqrt{2\eta\rho} = 0.2625\left(Z_I Z_j \hat{A} R_{fm}\right)^{1/2}$$

is independent of energy and angular momentum but depends on nuclear size.

Note:

rapid decrease with smaller energy and increasing charge ($\eta \uparrow$)

rapid decrease with increasing angular momentum

The leading order term for any constant ℓ is proportional to

$$\rho P_l \propto \exp(-2\pi\eta)$$

There exist other interesting limits for ρP_l, for example when η is small - as for neutrons where it is 0

$$\rho \propto \mathrm{E}^{1/2} \qquad \rho P_0 = \rho$$

$$\rho P_1 = \frac{\rho^3}{1+\rho^2} \qquad \rho << 1 \text{ for cases of interest for neutron capture}$$

$$\rho P_2 = \frac{\rho^5}{9 + 3\rho^2 + \rho^4}$$

This implies that for $l = 0$ neutrons the cross section will go as 1/v.

$$\eta = \frac{Z_1 Z_j e^2}{\hbar v} = 0$$

$$\rho = \sqrt{\frac{2\mu E}{\hbar^2}}\, R = 0.2187\sqrt{\hat{A}E}\, R_{fm}$$

$$i.e., \;\; \pi\lambda\!\!\!^{-2} \rho P_0 \propto \frac{E^{1/2}}{E} \propto E^{-1/2}$$

For low energy neutron induced reactions, the cross section times velocity, i.e., the reaction rate term, is approximately a constant w/r temperature

For particles with charge, providing X(E) does not vary rapidly. with energy (exception to come), i.e., the nucleus is "structureless"

$$\sigma(E) = \pi \lambdabar^2 \rho P_l X(E) \propto \frac{e^{-2\pi\eta}}{E}$$

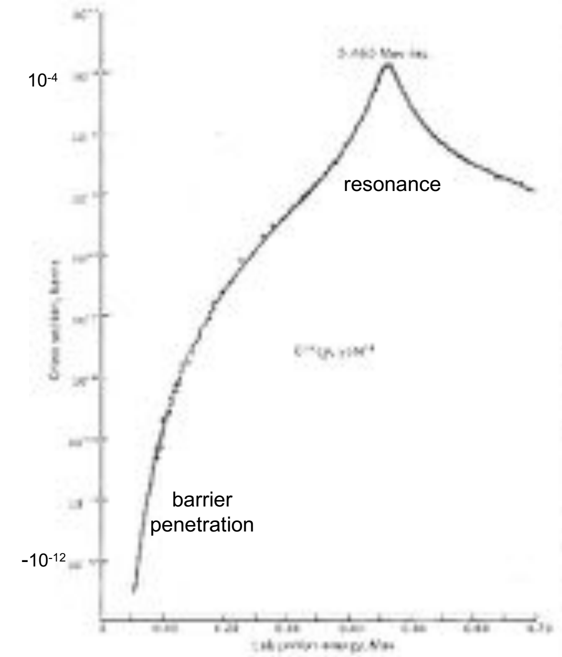
This motivates the definition of an "S-factor"

$$S(E) = \sigma(E)\, E \exp(2\pi\eta)$$

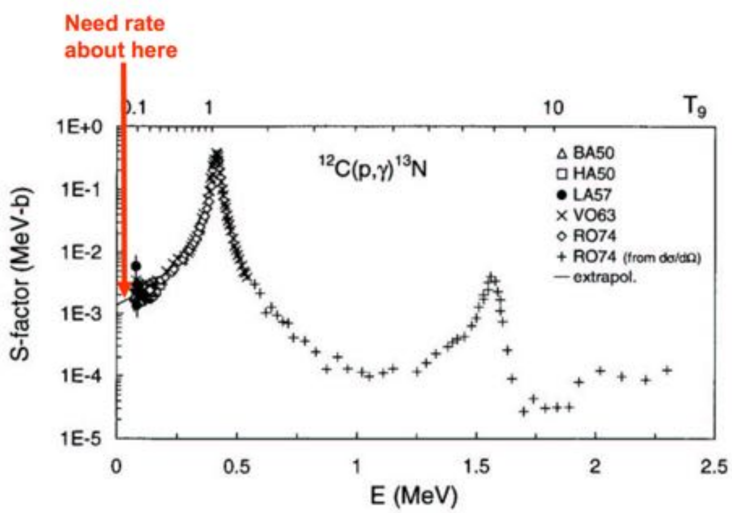
$$\eta = 0.1575 Z_I Z_j \sqrt{\hat{A}/E}$$

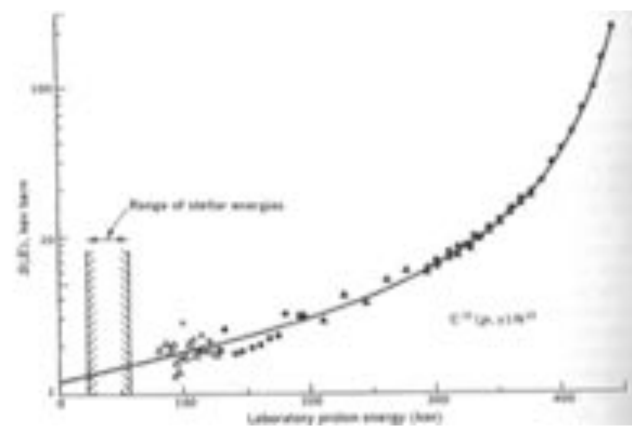
$$\hat{A} = \frac{A_I A_j}{A_I + A_j}$$

This S-factor should vary slowly with energy. The first order effects of the Coulomb barrier and Compton wavelength have been factored out. This is what was plotted in the figure several slides back. Its residual variation reflects nuclear structure and to a lesser extent corrections to the low energy approximation.



Cross section with the DeBroglie and barrier penetration part divided out. Proportional to X(E).





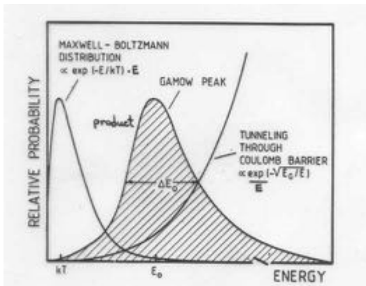
For those reactions in which S(E) is a slowly varying function of energy in the range of interest and can be approximated by its value at the energy where the integrand is a maximum, E_0,

$$\sigma(\mathrm{E}) \simeq \frac{S(E_0)}{E}\exp(-2\pi\eta)$$

$$N_A\langle\sigma v\rangle \approx N_A\left(\frac{8}{\pi\mu}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2} S(E_o)\int_0^\infty \exp(-E/kT-2\pi\eta(E))\,dE$$

where $\eta(E) = 0.1575\sqrt{\hat{A}/E(MeV)}\;Z_I Z_j$

The quantity in the integral looks like



For accurate calculations we would just enter the energy variation of S(E) and do the integral numerically. However, Clayton shows (p. 301 - 306) that

$\exp\left(\frac{-E}{kT}-2\pi\eta\right)$ can be replaced to good accuracy by

$C\exp\left(\frac{-(E-E_0)^2}{(\Delta/2)^2}\right)$, i.e. a Gaussian with the same maximum and second derivative at maximum

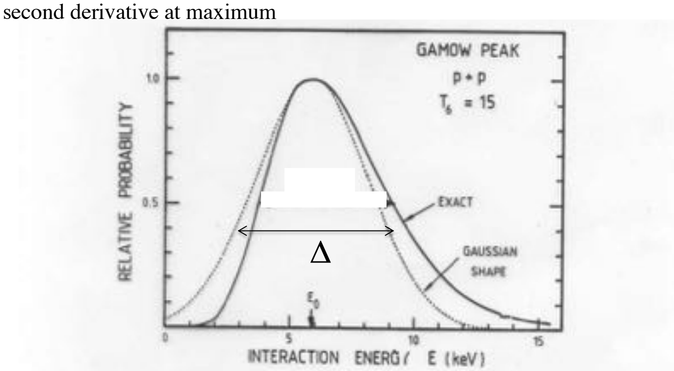


FIGURE 4.7. Curves for the Gamow peak for the p + p reaction at T_6 = 15, as obtained from the exact expression and from the approximation using the Gaussian function.

where E_o is the *Gamow Energy*, where the Gaussian has its peak

$$E_0 = \left(\pi\eta E^{1/2}kT\right)^{2/3};\ \eta E^{1/2} = 0.1575\sqrt{\hat{A}}\ Z_I Z_j;\ kT = \frac{T_9}{11.6045}$$

$$E_o = 0.122\left(Z_I^2 Z_j^2 \hat{A} T_9^2\right)^{1/3}\ \text{MeV}$$

and Δ is its full width at 1/*e* times the maximum

$$\Delta = \frac{4}{\sqrt{3}}\left(E_o kT\right)^{1/2} = 0.237\left(Z_I^2 Z_j^2 \hat{A} T_9^5\right)^{1/6}\ \text{MeV}$$

Δ is approximately the harmonic mean of kT and E_0 and it is always less than E_0

e.g. $^3He(\alpha,\gamma)^7Be$ at 1.5 x 10^7 K

$E_o = 0.122\left(Z_I^2 Z_j^2 \hat{A} T_9^2\right)^{1/3}$ MeV

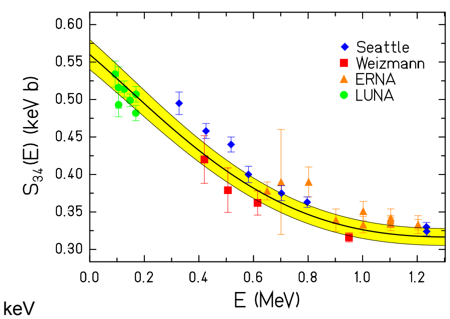
$\hat{A} = \frac{(3)(4)}{3+4} = 1.714;\ T_9 = 0.015;\ Z_I = Z_j = 2$

$E_o = 0.122\left((2)^2(2)^2(1.71)(0.015)^2\right)^{1/3}$ MeV

= 0.02238 MeV = 22.4 keV

Similarly

$\Delta = 0.237\left(Z_I^2 Z_j^2 \hat{A} T_9^5\right)^{1/6}$ =0.0124 MeV = 12.4 keV



In that case, the integral of a Gaussian is analytic*

$$N_A \langle \sigma v \rangle = \frac{4.34\times10^8}{\hat{A} Z_I Z_j} S(E_0)\, \tau^2 e^{-\tau} \text{ cm}^3 / (\text{Mole s})$$

Clayton 4-54 and 56 uses S in keV b and leaves out N_A otherwise the same answer.

where $S(E_0)$ is measured in MeV barns. If we define

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

$$\lambda_{jk} = N_A \langle \sigma_{jk} v \rangle$$

then a term in the rate equation for species I such as $Y_j \rho \lambda_{jk}$ has units

$$\left(\frac{Mole}{gm}\right)\left(\frac{gm}{cm^3}\right)\left(\frac{cm^3}{Mole\ s}\right) = s^{-1}$$

Different people use different conventions for λ which sometimes do or do not include ρ or N_A. This defines mine. Clayton does not innclude N_A.

Note that τ here is

$$\tau = \frac{3E_0}{kT} = 4.248\left(\frac{Z_I^2 Z_j^2\, \hat{A}}{T_9}\right)^{1/3}$$

differs from Clayton which measures T in 10^6 K

***See Appexdix 2 for integral**

Adelberger (2006) gives corrections (from Bahcall 1966) for derivatives of S. His eq 4

$$S_{eff} = S(E_0)\left[1 + \tau^{-1}\left(\frac{5}{12} + \frac{5S' E_0}{2S} + \frac{S'' E_0^2}{S} + \ldots\right)_{E=E_0}\right]$$

If derivatives are known use S_{eff} instead of $S(E_0)$ in the integral.

Adelberger et al, RMP, (2011)
The standard solar values

TABLE I The Solar Fusion II recommended values for S(0), its derivatives, and related quantities, and for the resulting uncertainties on S(E) in the region of the solar Gamow peak – the most probable reaction energy – defined for a temperature of 1.55 × 10^7K characteristic of the Sun's center. See the text for detailed discussions of the range of validity for each S(E). Also see Sec. VIII for recommended values of CNO electron capture rates, Sec. XI.B for other CNO S-factors, and Sec. X for the ^{8}B neutrino spectral shape. Quoted uncertainties are 1σ.

Reaction	Section	S(0) (keV-b)	S′(0) (b)	S″(0) (b/keV)	Gamow peak uncertainty (%)
$p(p,e^+\nu_e)d$	III	$(4.01 \pm 0.04)\times10^{-22}$	$(4.49 \pm 0.05)\times10^{-24}$	–	± 0.7
$d(p,\gamma)^3He$	IV	$(2.14^{+0.17}_{-0.16})\times10^{-4}$	$(5.56^{+0.18}_{-0.20})\times10^{-6}$	$(9.3^{+3.9}_{-3.4})\times10^{-9}$	± 7.1 [a]
$^3He(^3He,2p)^4He$	V	$(5.21 \pm 0.27) \times 10^3$	-4.9 ± 3.2	$(2.2 \pm 1.7) \times 10^{-2}$	± 4.3 [a]
$^3He(^4He,\gamma)^7Be$	VI	0.56 ± 0.03	$(-3.6 \pm 0.2)\times10^{-4}$ [b]	$(0.151 \pm 0.008)\times10^{-6}$ [c]	± 5.1
$^3He(p,e^+\nu_e)^4He$	VII	$(8.6 \pm 2.6)\times10^{-20}$	–	–	± 30
$^7Be(e^-,\nu_e)^7Li$	VIII	See Eq. (40)	–	–	± 2.0
$p(pe^-,\nu_e)d$	VIII	See Eq. (46)	–	–	± 1.0 [d]
$^7Be(p,\gamma)^8B$	IX	$(2.08 \pm 0.16)\times10^{-2}$ [e]	$(-3.1 \pm 0.3)\times10^{-5}$	$(2.3 \pm 0.8)\times10^{-7}$	± 7.5
$^{14}N(p,\gamma)^{15}O$	XI.A	1.66 ± 0.12	$(-3.3 \pm 0.2)\times10^{-3}$ [b]	$(4.4 \pm 0.3)\times10^{-5}$ [c]	± 7.2

Adelberger et al, RMP, (2011)
The standard solar values

TABLE XII Summary of updates to S-values and derivatives for CNO reactions.

Reaction	Cycle	S(0) keV b	S′(0) b	S″(0) keV^{-1} b	References
$^{12}C(p,\gamma)^{13}N$	I	1.34 ± 0.21	2.6×10^{-3}	8.3×10^{-5}	Recommended: Solar Fusion I
$^{13}C(p,\gamma)^{14}N$	I	7.6 ± 1.0	-7.83×10^{-3}	7.29×10^{-4}	Recommended: Solar Fusion I
		7.0 ± 1.5			NACRE: Angulo *et al.* (1999)
$^{14}N(p,\gamma)^{15}O$	I	1.66 ± 0.12	-3.3×10^{-3}	4.4×10^{-5}	Recommended: this paper
$^{15}N(p,\alpha_0)^{12}C$	I	$(7.3 \pm 0.5)\times10^4$	351	11	Recommended: this paper
$^{15}N(p,\gamma)^{16}O$	II	36 ± 6			Mukhamedzhanov *et al.* (2008)
		64 ± 6			Rolfs and Rodney (1974)
		29.8 ± 5.4			Hebbard (1960)
$^{16}O(p,\gamma)^{17}F$	II	10.6 ± 0.8	-0.054		Recommended: this paper
$^{17}O(p,\alpha)^{14}N$	II		Resonances		Chafa *et al.* (2007)
$^{17}O(p,\gamma)^{18}F$	III	6.2 ± 3.1	1.6×10^{-3}	-3.4×10^{-7}	Chafa *et al.* (2007)
$^{18}O(p,\alpha)^{15}N$	III		Resonances		See text
$^{18}O(p,\gamma)^{19}F$	IV	15.7 ± 2.1	3.4×10^{-4}	-2.4×10^{-6}	Recommended: Solar Fusion I

Temperature dependence of reaction rates (constant S(E))

$$f = \tau^2 e^{-\tau} \qquad \tau = \frac{A}{T^{1/3}} \qquad \frac{d\tau}{dT} = -\frac{A}{3T^{4/3}} = -\frac{\tau}{3T}$$

$$\frac{df}{dT} = 2\tau e^{-\tau}\frac{d\tau}{dT} - \tau^2 e^{-\tau}\frac{d\tau}{dT}$$

$$\frac{T}{f}\left(\frac{df}{dT}\right) = \frac{T}{\tau^2 e^{-\tau}}(2\tau e^{-\tau})(-\frac{\tau}{3T}) - \frac{T}{\tau^2 e^{-\tau}}(\tau^2 e^{-\tau})(-\frac{\tau}{3T})$$

$$= \left(\frac{d\ln f}{d\ln T}\right) = \frac{\tau - 2}{3}$$

$$\therefore\ f \propto T^n \qquad n = \frac{\tau - 2}{3}$$

For example, $^{12}C + {}^{12}C$ at 8 x 10^8 K

$$\tau = 4.248\left(\frac{6^2 6^2 \frac{12\cdot 12}{12+12}}{0.8}\right)^{1/3}$$

$$= 90.66$$

$$n = \frac{90.66 - 2}{3} = 29.5$$

p + p at 1.5 x 10^7 K

$$\tau = 4.248\left(\frac{1\cdot 1\cdot \frac{1\cdot 1}{1+1}}{0.015}\right)^{1/3}$$

$$= 13.67$$

$$n = \frac{13.67 - 2}{3} = 3.89$$

This is all predicated upon $S(E_o)$ being constant, or at least slowly varying within the “Gamow window”

$$E_o \pm \Delta / 2$$

This is true in many interesting cases, especially for light nuclei (no resonances or a single broad resonance) and very heavy ones (very many resonances in the window so that average properties apply). But it is not always true.

S(E) ~ const	• Truly non-resonant reactions (direct capture and the like)
S(E) ~ const	• Reactions that proceed through the tails of broad distant resonances
S(E) highly variable	• Reactions that proceed through one or a few “narrow” resonances within the “Gamow window”
S(E)~ const	• Reactions that have a very large number of resonances in the “Gamow window”

Reaction Mechanisms

1) Direct Capture - an analogue of atomic radiative capture

The target nucleus and incident nucleon (or nucleus) react without a sharing of energy among all the nucleons. An example be the direct radiative capture of a neutron or proton and the immediate ejection of one or more photons. The ejected photons are strongly peaked along the trajectory of the incident projectile. The reaction time is very short, ~ R/c ~10^{-21} s.

This sort of mechanism dominates when there are no strong resonances in or near the Gamow window. It is especially important at low energies in light nuclei where there are few resonances

The S-factor for direct capture is smooth and featureless.

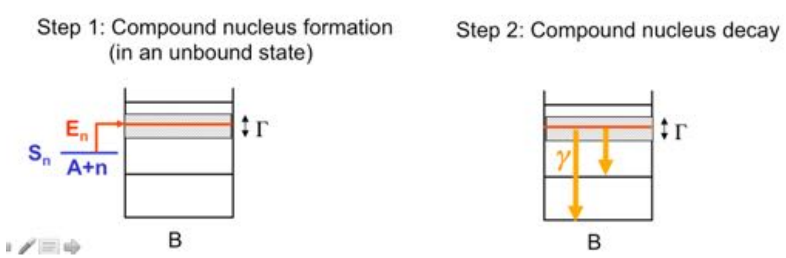
Examples:

$^{3}He(\alpha,\gamma)^{7}Be$, $^{2}H(p,\gamma)^{3}He$, $^{3}He(^{3}He, 2p)^{4}He$

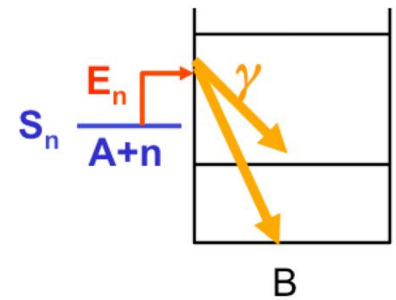
$^{12}C(n,\gamma)^{13}C$, $^{48}Ca(n,\gamma)^{49}Ca$

2) Resonant Reaction:

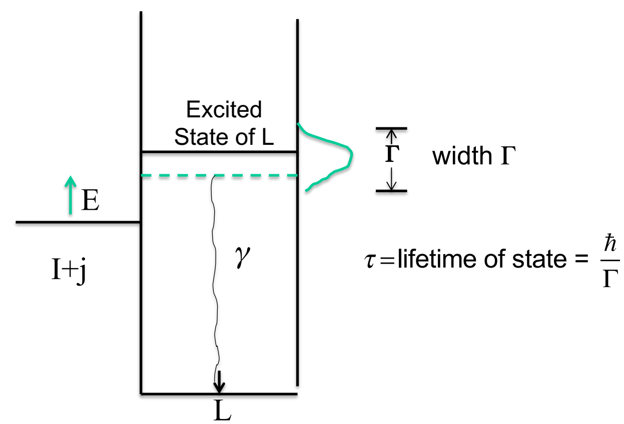
A two step reaction in which a relatively long-lived excited state of the "compound nucleus" is formed – the "resonance". This state decays statistically without any memory (other than energy and quantum numbers) of how it was produced. The outgoing particles are not peaked along the trajectory of the incident particle. (This is called the "Bohr hypothesis" or the "hypothesis of nuclear amnesia"). The presence of a resonance says that the internal structure of the nucleus is important and that a "long-lived" state is being formed.



Direct capture provides a mechanism for reaction in the absence of resonances. Usually DC cross sections are much smaller than resonant cross sections on similar nuclei - if a resonance is present.



For the reaction I(j,γ)L



E is the energy of I + j in the center of mass frame and the state is characterized by a width Γ (in energy units) given by its lifetime against all the ways it can decay, photon emission being one of them. The excited state has a certain spin and parity and, depending on the values might serve as a resonance for the reaction. Some reactions proceed directly to the ground state.

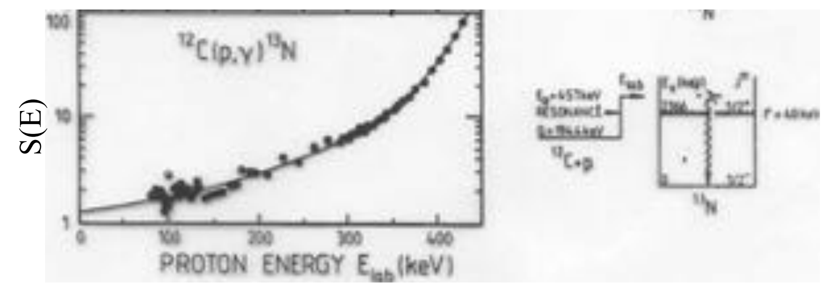
Resonances may be broad or narrow. The width is given by the (inverse of the) lifetime of the state and the uncertainty principle.

$$\Delta E\, \Delta t \sim \hbar$$

Generally states that can decay by emitting a neutron or proton will be broad (if the proton has energy greater than the Coulomb barrier. Resonances will be narrow if they can only decay by emitting a photon or if the charged particle has energy << the Coulomb barrier..

$$\tau = \frac{\hbar}{\Gamma_{tot}} \qquad \Gamma_{tot} = \sum \Gamma_k \qquad \hbar = 6.582 \times 10^{-22}\ \text{MeV sec}$$

E.g., a broad resonance



$$\frac{13}{12}(422) = 457$$

2.366	Excitation energy
- 1.944	Q value for (pγ)
0.422 MeV	Threshold c/m

The energy scale is given in the center of mass fram (422 keV) needs to be converted to the lab frame to compare with lab data. Multiply by $(A_1+A_2)/(A_1A_2)$

For this case the S factor is slowly varying in the Gamow "window".

Say hydrogen burning at 2 x 10^7 K, or T_9 = 0.020

$$^{12}C(p,\gamma)^{13}N$$

$$E_{Gamow} = 0.122\left(6^2\, 1^2\, \frac{12\cdot 1}{12+1} 0.02^2\right)^{1/3} = 0.0289\ \text{MeV} = 28.9\ \text{keV}$$

$$\Delta = 0.237\left(6^2\, 1^2\, \frac{12\cdot 1}{12+1}\ 0.02^5\right)^{1/6} = 0.0163\ \text{MeV} = 16.3\ \text{keV}$$

Note there is no data at energies this low.
As is generally the case, one must extrapolate the experimental date to lower energies than are experimentally accessible. The S-factor is useful for this.

Consider, however, the reaction $^{24}Mg(p,\gamma)^{25}Al$

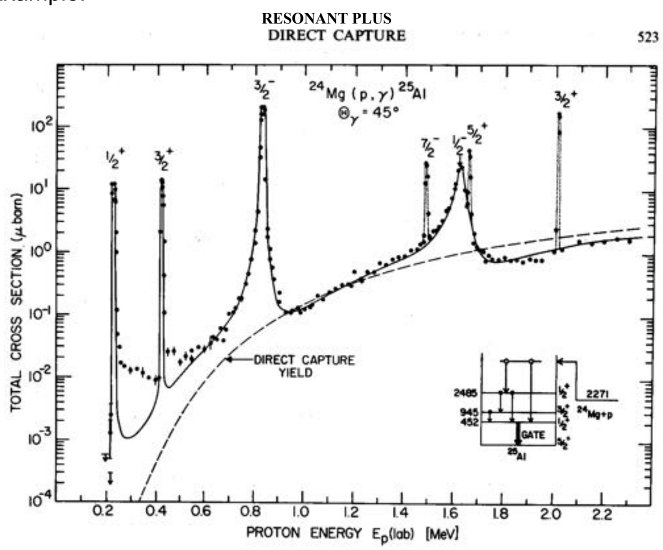
This reaction might be of interest either in hot hydrogen burning at 30 million K or in carbon burning at 800 million K. Consider the latter.

$$E_{Gamow} = 0.122\left(12^2\, 1^2\, \frac{24\cdot 1}{24+1} 0.8^2\right)^{1/3} = 0.543\ \text{MeV}$$

$$\Delta = 0.237\left(12^2\, 1^2\, \frac{24\cdot 1}{25+1}\ 0.8^5\right)^{1/6} = 0.447\ \text{MeV}$$

That is energies up to 1 MeV are important
Now three resonances and direct capture contribute.

Another Example:



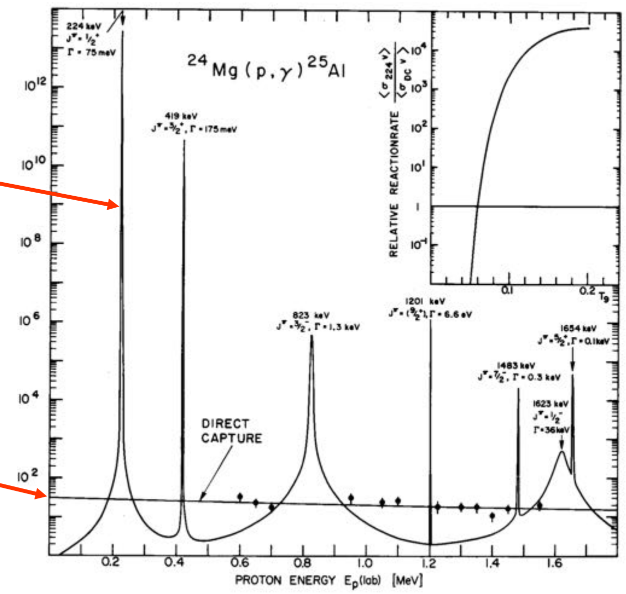
Resonance contributions are on top of direct capture cross sections

… and the corresponding S-factor

Note varying widths and effects for E >> Γ !

Not constant S-factor for resonances (log scale !!!!)

~ constant S-factor for direct capture



The cross section contribution due to a single resonance is given by the Breit-Wigner formula:

$$\sigma(E) = \pi\lambda\!\!\!^{-2} \cdot \omega \cdot \frac{\Gamma_1\Gamma_2}{(E-E_r)^2 + (\Gamma/2)^2}$$

Usual geometric factor

$$= \frac{0.656}{\hat{A}}\frac{1}{E}\,\text{barn}$$

Spin factor:

$$\omega = \frac{2J_r + 1}{(2J_1+1)(2J_2+1)}$$

$\propto \Gamma_1$ Partial width for decay of resonance by emission of particle 1
= Rate for formation of Compund nucleus state

$\propto \Gamma_2$ Partial width for decay of resonance by emission of particle 2
= Rate for decay of Compund nucleus into the right exit channel

Γ = Total width is in the denominator as a large total width reduces the maximum probabilities (on resonance) for decay into specific channels.

See appendix 3 and Clayton for derivation.

One can perform the Maxwell Boltzman integral analytically (Clayton 4-193):

For the contribution of a single narrow resonance to the stellar reaction rate:

$$N_A < \sigma v >= 1.54 \cdot 10^{11} (AT_9)^{-3/2}\, \omega\gamma\,[\text{MeV}]\, e^{\frac{-11.605\, E_r\,[\text{MeV}]}{T_9}} \quad \frac{\text{cm}^3}{\text{s mole}}$$

The rate is entirely determined by the "resonance strength" $\omega\gamma$

$$\omega\gamma = \frac{2J_r+1}{(2J_j+1)(2J_I+1)}\frac{\Gamma_1\Gamma_2}{\Gamma}$$

Which in turn depends mainly on the total and partial widths of the resonance at resonance energies.

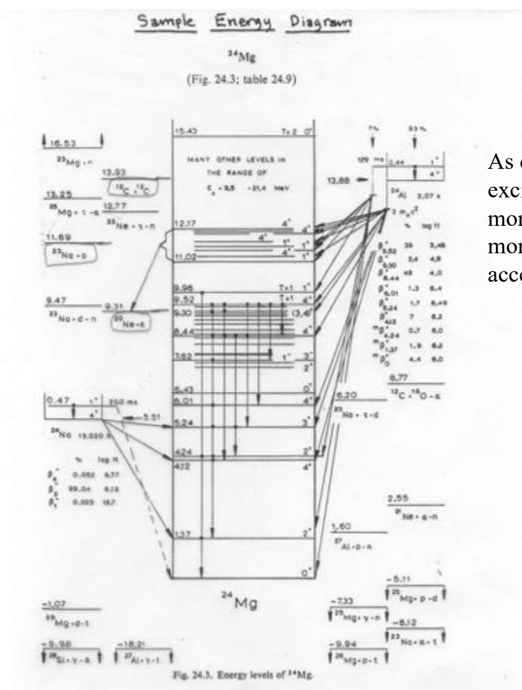
Often $\Gamma = \Gamma_1 + \Gamma_2$ Then for $\Gamma_1 << \Gamma_2 \longrightarrow \Gamma \approx \Gamma_2 \longrightarrow \frac{\Gamma_1\Gamma_2}{\Gamma} \approx \Gamma_1$

$\Gamma_2 << \Gamma_1 \longrightarrow \Gamma \approx \Gamma_1 \longrightarrow \frac{\Gamma_1\Gamma_2}{\Gamma} \approx \Gamma_2$

And reaction rate is determined by the smaller one of the widths !

Illiadis Table 4.12

Reaction	E_r^{lab} (keV)	J^π	$\omega\gamma_{cm}$ (eV)	Error (%)	Reference
$^{14}N(p,\gamma)^{15}O$	278	$1/2^+$	$1.37(7) \times 10^{-2}$	5.1	h
$^{18}O(p,\gamma)^{19}F$	151	$1/2^+$	$9.7(5) \times 10^{-4}$	5.2	g
$^{23}Na(p,\alpha)^{20}Ne$	338	1^-	$7.16(29) \times 10^{-2}$	4.0	a
$^{23}Na(p,\gamma)^{24}Mg$	512	$(1,2^+)$	$9.13(125) \times 10^{-2}$	13.7	b
$^{24}Mg(p,\gamma)^{25}Al$	223	$1/2^+$	$1.27(9) \times 10^{-2}$	7.1	c
	419	$3/2^+$	$4.16(26) \times 10^{-2}$	6.2	d
$^{25}Mg(p,\gamma)^{26}Al$	435	4^-	$9.42(65) \times 10^{-2}$	6.9	d
	591	1^+	$2.28(17) \times 10^{-1}$	7.4	e
$^{26}Mg(p,\gamma)^{27}Al$	338	$3/2^-$	$2.73(16) \times 10^{-1}$	5.9	d
	454	$1/2^+$	$7.15(41) \times 10^{-1}$	5.7	d
	1966	$5/2^+$	$5.15(45)$	8.7	b
$^{27}Al(p,\gamma)^{28}Si$	406	4^+	$8.63(52) \times 10^{-3}$	6.0	d
	632	3^-	$2.64(16) \times 10^{-1}$	6.1	b
	992	3^+	$1.91(11)$	5.7	b
$^{30}Si(p,\gamma)^{31}P$	620	$1/2^-$	$1.95(10)$	5.1	b
$^{31}P(p,\gamma)^{32}S$	642	1^-	$5.75(50) \times 10^{-2}$	8.7	b
	811	2^+	$2.50(20) \times 10^{-1}$	8.0	b
$^{34}S(p,\gamma)^{35}Cl$	1211	$7/2^-$	$4.50(50)$	11.1	b
$^{35}Cl(p,\gamma)^{36}Ar$	860	3^-	$7.00(100) \times 10^{-1}$	14.3	b
$^{36}Ar(p,\gamma)^{37}K$	918	$5/2^+$	$2.38(19) \times 10^{-1}$	8.0	f
$^{37}Cl(p,\gamma)^{38}Ar$	846	1^-	$1.25(16) \times 10^{-1}$	12.8	b
$^{39}K(p,\gamma)^{40}Ca$	2042	1^+	$1.79(19)$	10.6	b
$^{40}Ca(p,\gamma)^{41}Sc$	1842	$7/2^+$	$1.40(15) \times 10^{-1}$	10.7	b



As one goes up in excitation energy many more states and many more reactions become accessible.

As one goes to heavier nuclei and/or to higher excitation energy in the nucleus, the number of excited states, and hence the number of potential resonances increases exponentially.

Why? The thermal energy of a non-relativistic, nearly degenerate gas (i.e., the nucleus) has a leading term that goes as T^2 where T is the "nuclear temperature. The energy, E, of a degenerate gas from an expansion of Fermi integrals is:

$$E = f(\rho) + a(kT)^2 + b\,(kT)^4 + ...$$

here ρ is the density and Ω is the partition function

One definition of temperature is

$$\frac{1}{kT} = \frac{\partial \ln \Omega}{\partial E}$$

$$\frac{1}{T} = \frac{\partial S}{\partial E} \quad S = k \ln \Omega \ \text{ defines T}$$

where Ω is the number of states (i.e., the partition function)

$$\frac{\partial \ln \Omega}{\partial T} = \frac{\partial \ln \Omega}{\partial E}\frac{\partial E}{\partial T}$$

$$d \ln \Omega \sim \frac{1}{kT}\left(\frac{\partial E}{\partial T}\right) dT \ \sim \ \frac{1}{kT}\left(2ak^2T\right)\ dT$$

$$\ln \Omega \sim 2ak \int dT = 2akT + \text{const}$$

$$\Omega \sim C \exp(2akT)$$

Note that T here is not the stellar temperature but a ficticous temperature for the nucleons in the nucleus. The ground state has T = 0

and if we identify the excitation energy, $E_x \approx a(kT)^2$, i.e., the first order thermal correction to the internal energy, then

$$(kT)^2 \sim \frac{E_x}{a}$$

$$\Omega = C \exp\left(2\sqrt{aE_x}\right)$$

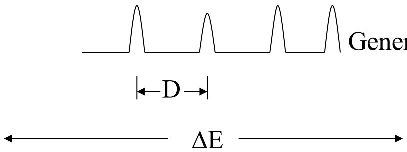
The number of excited states (resonances) per unit excitation energy increases exponentially with the square root of the excitation energy.

Empirically a $\approx$ A/9. There are corrections to a for shell and pairing effects. In one model (back-shifted Fermi gas)

$$C = \frac{0.482}{A^{5/6} E_x^{3/2}}$$

What is the cross section when the density of resonances is large?

Take N (>>1) equally spaced identical resonances in an energy interval ΔE. For example, assume they all have the same partial widths.



Generate an energy averaged cross section

D << ΔE

$$\langle\sigma\rangle = \frac{\int_E^{E+\Delta E}\sigma(E)\,dE}{\Delta E} \propto \frac{1}{\Delta E}\int_E^{E+\Delta E}\sum_1^N \frac{\omega\Gamma_j\Gamma_k\,dE}{(E-\varepsilon_r)^2+\Gamma_r^2/4}$$

$$\approx \frac{\omega\Gamma_j\Gamma_k}{\Delta E}\,N\int_0^\infty \frac{dE}{(E-\varepsilon_r)^2+\Gamma_r^2/4}$$

$$\int_0^\infty \frac{dE}{(E-\varepsilon_r)^2+\Gamma_r^2/4} = \frac{2\pi}{\Gamma_r} \qquad \frac{N}{\Delta E} = \frac{1}{D}$$

$$\langle\sigma\rangle = 2\pi^2\bar{\lambda}^2\omega\frac{\Gamma_j\Gamma_k}{\Gamma_r D} = \pi\bar{\lambda}^2\omega\frac{T_jT_k}{T_{tot}}$$

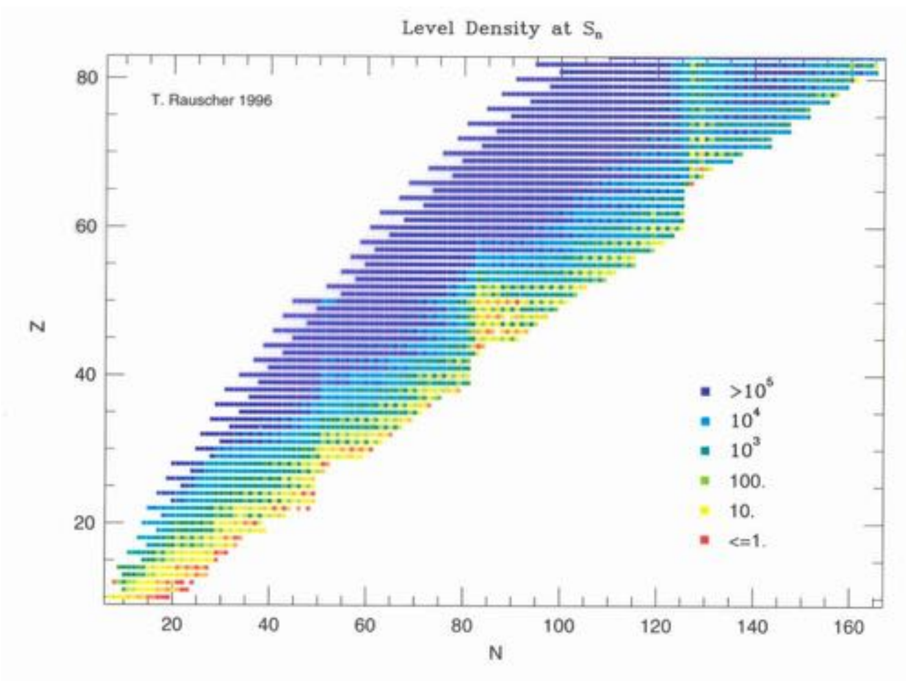
where $T_j = 2\pi\left\langle\frac{\Gamma_j}{D}\right\rangle$

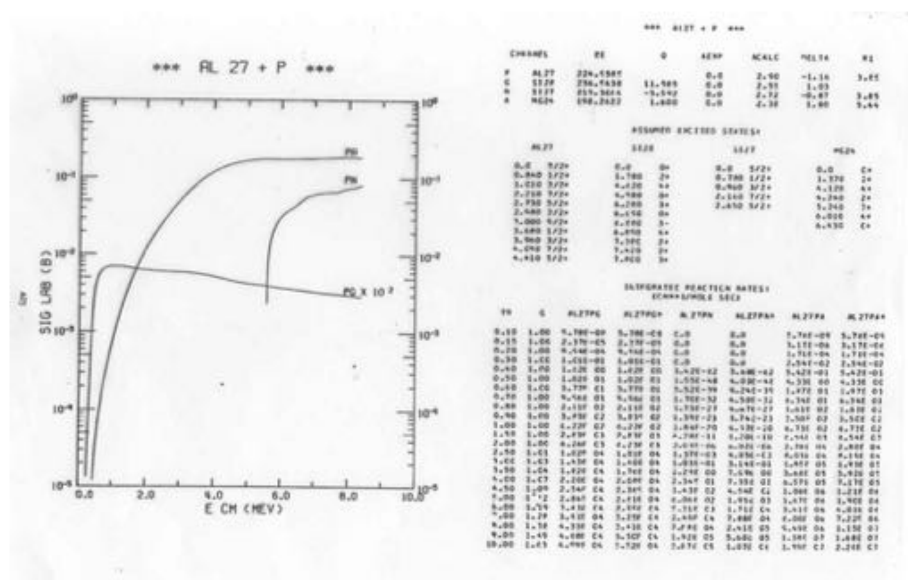
This gives the Hauser-Feshbach formula for estimating cross sections where the density of resonances is high.

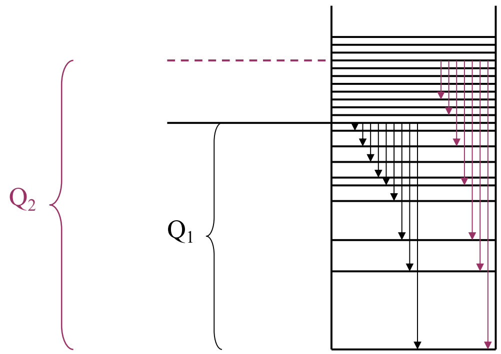
$$\bar{\sigma}_{jk}(E) = \frac{\pi\bar{\lambda}^2}{(2J_I+1)(2J_j+1)}\sum_{\substack{all\\ J_r^\pi}}(2J_r+1)\frac{T_j^l(J^\pi,E)\,T_k^l(J^\pi,E)}{T_{tot}(J^\pi,E)}$$

Expressions for the transmission functions for n, p, α, and γ are given in Woosley et al, ADNDT, 22, 378, (1978). See also the appendix here. A transmission function is like an average strength function for the reaction over the energy range of interest. It includes the penetration function. It is dimensionless and less than 1. **See appendix 4 for derivation and details.**

This formula has been used to generate thousands of cross sections for nuclei with A greater than about 24. The general requirement is many (> 10) resonances in the Gamow window.







More levels to make transitions to at higher Q and also, more phase space for the outgoing photon.

E_γ^3 for electric dipole

$$T_\gamma(Q_2) > T_\gamma(Q_1)$$

and as a result

$$\sigma_{n\gamma} \propto \frac{T_n \, T_\gamma}{T_n + T_\gamma} \approx T_\gamma$$

is larger if Q is larger

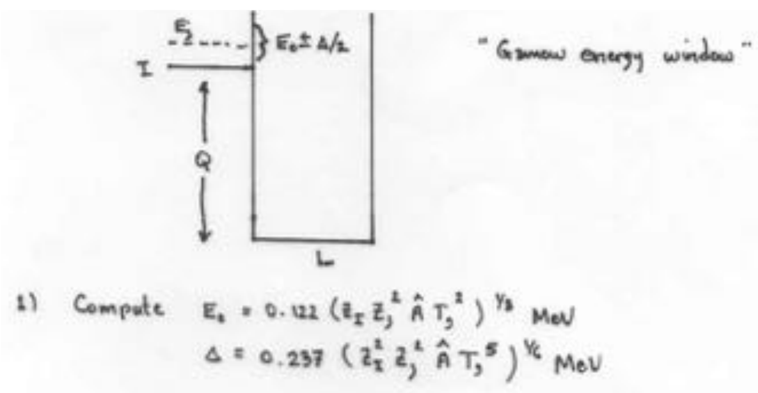
The Q-value for capture on nuclei that are tightly bound (e.g., even-even nuclei, closed shell nuclei) is smaller than for nuclei that are less tightly bound (e.g., odd A nuclei, odd-odd nuclei).

As a result, nuclear stability translates into smaller cross sections for destruction - most obviously for nuclei made by neutron capture, but also to some extent for charged particle capture as well.

This is perhaps the chief reason that tightly bound even mass nuclei above the iron group are more abundant in nature than their less tightly bound odd mass neighbors.

Summary of reaction mechanisms

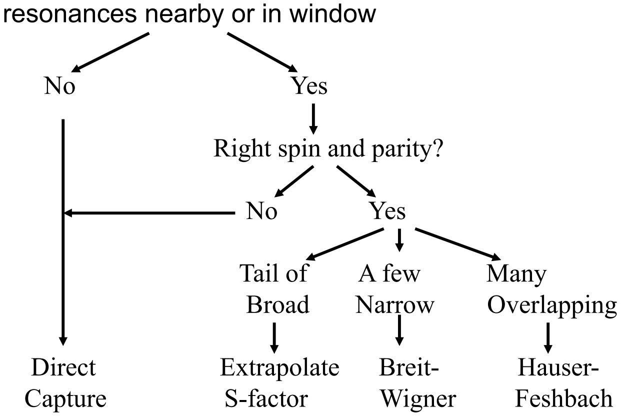
I(j,k)L



Summary of reaction mechanisms

I(j,k)L

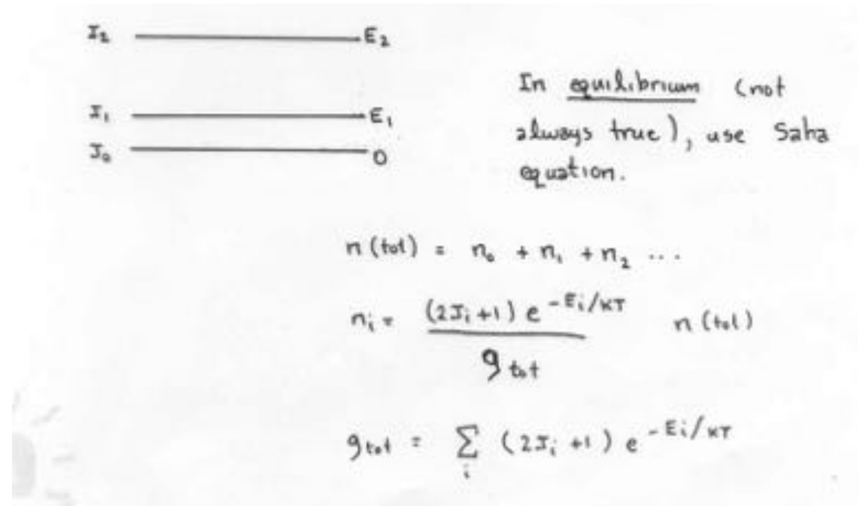
- Add the Gamow energy E_0 to Q-value and look inside nucleus I+j
- Any resonances nearby or in window

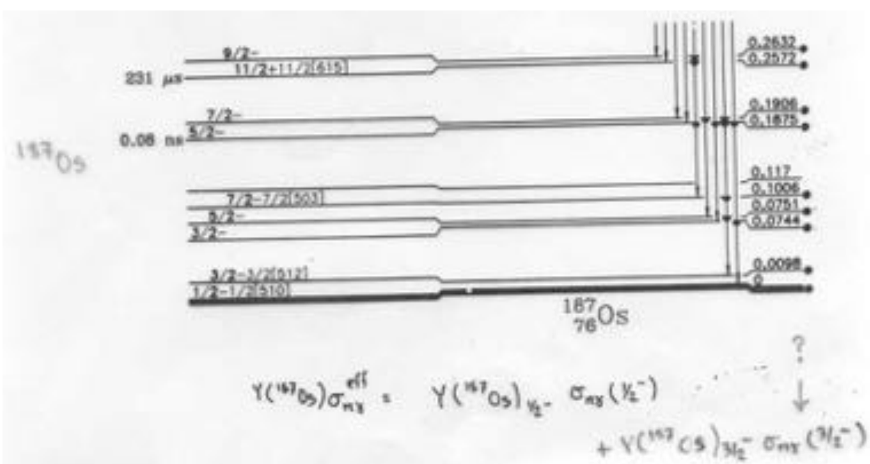


Special Complications in Astrophysics

- Low energy = small cross section – experiments are hard.
- Very many nuclei to deal with (our networks often include 1600 nuclei; more if one includes the r-process)
- The targets are often radioactive and short lived so that the cross sections cannot be measured in the laboratory (^{56}Ni, ^{44}Ti, ^{26}Al, etc)
- Sometimes even the basic nuclear properties are not know - binding energy, lifetime. E.g., the r-process and the rp-process which transpire near the neutron and proton-drip lines respectively.
- Unknown resonances in many situations

- Target in excited state effects – in the laboratory the target is always in its ground state. In a star, it may not be





- Electron screening

  Nuclei are always completely ionized – or almost completely ionized at temperature in stars where nuclear fusion occurs. But the density may be sufficiently high that two fusing nuclei do not experience each others full Coulomb repulsion.

  This is particularly significant in Type Ia supernova ignition.

Electron screening is generally treated in two limiting cases.

Weak screening: (Salpeter 1954)

The electrical potential of the ion is adjusted to reflect the presence of induced polarization in the background electrons. The characteristic length scale for this screening is the Debye length

$$R_D = \left(\frac{kT}{4\pi e^2 \rho N_A \varsigma} \right)^{1/2} \qquad \zeta = \sum (Z_i^2 + Z_i) Y_i$$

Clayton 2-238 and discussion before

This is the typical length scale for the clustering of charge in the plasma. Weak screening holds if $R_D >> n_Z^{-1/3}$

Roughly the ion sphere is the volume over which a given ion can "polarize" the surrounding electron cloud when that cloud has a thermal energy ~kT. Its size is given by equating thermal kinetic energy to electrical potential energy. The charge within such a cloud is (Volume)(n_e e) and the charge on each ion is Ze. The volume is 4/3 πR_D^3 and $n_e = Zn_Z$. So

$$PE = \frac{\left(\frac{4}{3}\pi R_D^3\right)(Zn_Z e)(Ze)}{R_D} \sim kT$$

$$R_D \sim \left(\frac{kT}{\frac{4}{3}\pi e^2 Z^2 n_Z} \right)^{1/2}$$

Compare with Clayton 2-235

Differs by $\sqrt{3}$

$$\rho N_A Y_Z = n_Z$$

In general must include more than one kind of ions and the interaction among electrons and among ions, not only between ions and electrons,

These "Coulomb correction" affect the pressure and energy of a gas, not just reaction rates

The modified Coulomb potential is then

$$V = \frac{e^2 Z}{r} \exp(-r / R_D)$$

Clayton eq. 4-215 and discussion leading up to it shows that, in the limit that $R_D >>$ the inter-ion separation, then the effect of screening is an overall reduction of the Coulomb potential by an energy

$$U_o = \frac{Z_I Z_j e^2}{R_D}$$

This potential does not vary greatly over the region where the rate integrand is large (Gamow energy)

The leading order term in the screening correction (after considering Mawell Boltzmann average) is then (Clayton 4-221; see also Illiadis 3.143)

e.g., the screening for p+p at the solar center is about 5% - Illiadis P 210

$U_0 << kT$

$$f \approx 1 - \frac{U_o}{kT} = 1 + 0.188 Z_I Z_j \rho^{1/2} \varsigma^{1/2} T_6^{-3/2}$$

Strong screening: Salpeter (1954); Salpeter and van Horn (1969)

If R_D becomes less than the inter-ion spacing, then the screening is no longer weak. Each ion of charge Z is individually screened by Z electrons. The radius of the "ion sphere" is

$$R_Z = \left(\frac{3Z}{4\pi n_e} \right)^{1/3} \qquad i.e. \ \frac{4\pi R_Z^3}{3} n_e = Z$$

Clayton 2-262, following Salpeter (1954) shows that the total potential energy of the ion sphere, including both the repulsive interaction of the electrons among themselves and the attractive interaction with the ions, is

$$U = -\frac{9}{10}\left(\frac{(Ze)^2}{R_Z}\right) = -17.6\, Z^{5/3}\left(\rho Y_e\right)^{1/3} \text{ eV} \quad << \text{ Gamow energy } E_0$$

and the correction factor to the rate is $\exp(-U_o / kT) >> 1$ with

$$-U_0 = 17.6\, \left(\rho Y_e\right)^{1/3}\left[\left(Z_I + Z_j\right)^{5/3} - Z_I^{5/3} - Z_j^{5/3}\right] \text{ eV} \quad \text{(Cla 4-225)}$$

More accurate treatments are available, but this can clearly become very large at high density. See Itoh et al. *ApJ*, **586**, 1436, 2003

Suppose X(E) is slowly varying
Consider just the barrier penetration part (R < r < infinity)
where R is the nuclear radius (where the strong interaction dominates).

Clayton p. 319ff shows that Schroedinger's equation for two interacting particles in a radial potential is given by (Cla 4-122) [see also our Lec 4]

$$\Psi(\text{r}, \theta, \phi) = \frac{\chi_1(r)}{r} Y_l^m(\theta,\phi) \quad *$$

where χ(r) satisfies

$$\left[\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2} + \frac{l(l+1)\hbar^2}{2\mu r^2} + V(r) - E\right]\chi_l(r) = 0$$

(Clayton 4-103)

potential

$$\begin{cases} V(r) = \dfrac{Z_I Z_j e^2}{r} & r > R \\ V(r) = V_{nuc} & r < R \end{cases}$$

Like the one-electron atom except for r < R

for interacting particles with both charge and angular momentum. The angular momentum term represents the known eigenvalues of the operator L^2 in a spherical potential

*The 1/r cancels the r^2 when integrating $\Psi^*\Psi$ over solid angles (e.g. Clayton 4-114). It is not part of the potential dependent barrier penetration calculation.

Appendix 1:

Solution of Schrodingers Equation for Two Charged Particles with Angular Momentum

Classically, centrifugal force goes like

$$F_c = \frac{mv^2}{R} = \frac{m^2v^2R^2}{mR^3} = \frac{L^2}{mR^3}$$

One can associate a centrifugal potential with this,

$$\int F_c\, dR = \frac{-L^2}{2mR^2}$$

Expressing things in the center of mass system and taking the usual QM eigenvaluens for the operator L^2 one has

$$\frac{-l(l+1)\,\hbar^2}{2\mu R^2}$$

To solve, do some variable substitutions

$$\left[\frac{-\hbar^2}{2\mu}\frac{d^2}{dr^2}+\frac{l(l+1)\hbar^2}{2\mu r^2}+V(r)-E\right]\chi_l(r)=0$$

divide by E and substitute for V(r) for $r > R$

$$\left[\frac{-\hbar^2}{2\mu E}\frac{d^2}{dr^2}+\frac{l(l+1)\hbar^2}{2\mu r^2 E}+\frac{Z_I Z_j e^2}{rE}-1\right]\chi(r)=0$$

Change of radius variable. Substitute for r

$$\rho=\sqrt{\frac{2\mu E}{\hbar^2}}\,r \qquad d\rho\rightarrow\sqrt{\frac{2\mu E}{\hbar^2}}\,dr \qquad d^2\rho\rightarrow\frac{2\mu E}{\hbar^2}\,d^2r$$

chain rule

https://en.wikipedia.org/wiki/Change_of_variables#Differentiation

and for Coulomb interaction

$$\eta=\frac{Z_I Z_j e^2}{\hbar v} \qquad v=\sqrt{\frac{2E}{\mu}}$$

ρ and η are dimensionless numbers

to obtain

$$\left[\frac{-d^2}{d\rho^2}+\frac{l(l+1)}{\rho^2}+\frac{2\eta}{\rho}-1\right]\chi_l(\rho)=0$$

multiply by -1

$$\frac{d^2\chi}{d\rho^2}+\left(1-\frac{2\eta}{\rho}-\frac{l(l+1)}{\rho^2}\right)\chi=0$$

This is the solution for $R < r < \infty$

has solutions (Abromowitz and Stegun 14.1.1)
http://people.math.sfu.ca/~cbm/aands/

$$\chi=C_1 F_l(\eta,\rho)+C_2\,G_l(\eta,\rho) \qquad C_1=1 \qquad C_2=i$$

where F and G, the regular and irregular Coulomb functions are the solutions of the differential equation and the constants come from applying the boundary conditions

The barrier penetration function P_l is then given by

$$P_l=\frac{|\chi_l(\infty)|^2}{|\chi_l(R)|^2}=\frac{F_l^2(\rho=\infty)+G_l^2(\rho=\infty)}{F_l^2(\eta,\rho)+G_l^2(\eta,\rho)}=\frac{1}{F_l^2(\eta,\rho)+G_l^2(\eta,\rho)}$$

Cla 4-115

The "1" in the numerator corresponds to a purely outgoing wave at infinity from a decaying state.

For the one electron atom with a potential $\frac{Ze^2}{r}$, one obtains the same solution but the radial component is Laguerre polynomials.

Appendix 2

$$\exp\left(\frac{-E}{kT}-2\pi\eta\right)\approx e^{-\tau}\exp\left(\frac{E-E_0}{\Delta/2}\right)^2$$

$$\lambda\approx N_A\left(\frac{8}{\pi\mu}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2}e^{-\tau}\int_0^\infty S(E)\exp\left[-\left(\frac{E-E_0}{\Delta/2}\right)^2\right]dE$$

$$=N_A\left(\frac{8}{\pi\mu}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2}e^{-\tau}S(E_0)\int_0^\infty\exp\left[-\left(\frac{E-E_0}{\Delta/2}\right)^2\right]dE$$

Let x = $\left(\frac{E-E_0}{\Delta/2}\right)$ dx = $\frac{2dE}{\Delta}$ $so\ dE=\frac{\Delta\,dx}{2}$

Can replace lower bound to intergral E = $\frac{-2E_0}{\Delta}$ by E = - ∞ with little loss of accuracy (footnote Clayton p 305) so that

$$\lambda=N_A\left(\frac{8}{\pi\mu}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2}e^{-\tau}\frac{\Delta}{2}S(E_0)\int_{-\infty}^{\infty}\exp\left[-x^2\right]dx$$

$$=N_A\left(\frac{8}{\pi\mu}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2}e^{-\tau}\frac{\Delta}{2}S(E_0)\sqrt{\pi}$$

$$=N_A\left(\frac{2}{\mu}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2}e^{-\tau}\Delta\,S(E_0)$$

$$\left(\frac{\Delta}{(kT)^{3/2}}\right)=\frac{4}{9\sqrt{3}\pi\,\eta E^{1/2}}\tau^2$$

$$\frac{\lambda}{N_A}=\left(\frac{2}{\mu}\right)^{1/2}\frac{4}{9\sqrt{3}\,\pi(0.1575\,Z_I Z_j\sqrt{\hat{A}})}\tau^2 e^{-\tau}S(E_0)\ \text{MeV}^{1/2}\,amu^{-1/2}\ \text{barn}$$

$$=\frac{7.2\times10^{-16}}{\hat{A}Z_I Z_j}\tau^2 e^{-\tau}\ S(E_0)\ cm^3\ s^{-1} \qquad (Clay\ \ 4-56)$$

$$\lambda=N_A\langle\sigma v\rangle=\frac{4.34\times10^8}{\hat{A}Z_I Z_j}S(E_0)\,\tau^2 e^{-\tau}\ \text{cm}^3/(\text{Mole s})$$

nb. The unit conversion factor $10^{-24}*(6.02\times10^{23}\bullet1.602\times10^{-6})^{1/2}$ $=9.82\times10^{-16}$ converts $\text{MeV}^{1/2}\,\text{amu}^{-1/2}$ barn to cm^3/s.

Also change μ to $\hat{A}$ amu

Appendix 3: How to calculate resonant cross section?

Decaying states in general have an an energy distribution given by the Breit-Wigner or Cauchy distribution (Clayton 3-103)*. The normalized probability that the state has energy E is

$$P(E)dE = \frac{\Gamma / 2\pi \, dE}{(E-\varepsilon_r)^2 + (\Gamma/2)^2}$$

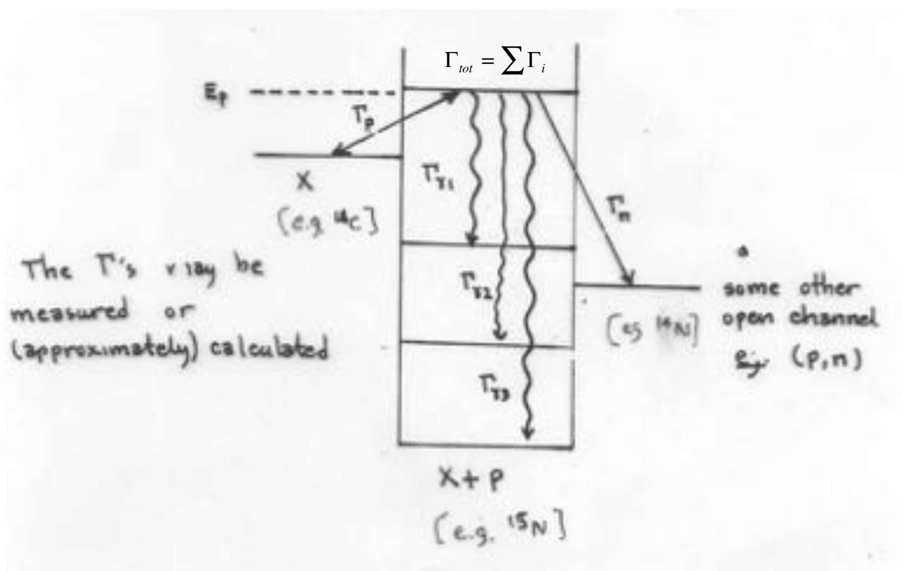
where

$$\Gamma = \frac{\hbar}{\tau}$$

nb. units of energy but rather like a rate

and τ is the lifetime

* Solve wave function for a quasistationary state subject to the constraint that $\int |\psi_k|^2 = \exp(-t/\tau)$. Take Fourier transform of ψ(t) to get φ(E) and normalize.



If a reaction is dominated by narrow resonances, its cross section will be given by the Breit-Wigner equation (see page 347 Clayton, also probs. 3-7 and eq, 3-103).

$$\sigma_{jk}(E) = \pi \lambda\!\!\!^{-2} \omega \frac{\Gamma_j \Gamma_k}{(E-\varepsilon_r)^2 + \Gamma_{tot}^2/4} \qquad \omega = \frac{2J_r + 1}{(2J_I+1)(2J_j+1)}$$

The Γ' s are the partial widths (like a probability but with dimensions of energy) for the resonance to break up into various channels. These now contain the penetration factors. The lifetime of a resonance is

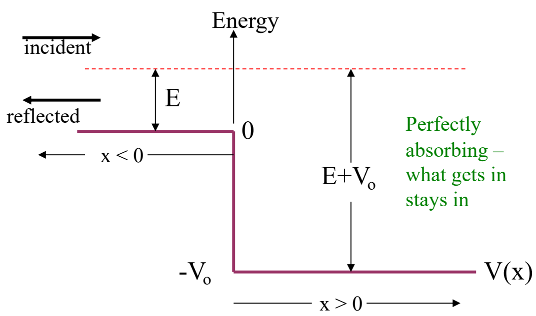
$$\tau = \frac{\hbar}{\Gamma_{tot}} \qquad \Gamma_{tot} = \sum \Gamma_k \qquad \hbar = 6.582 \times 10^{-22} \text{ MeV sec}$$

This cross section will be sharply peaked around ε_r, with a width Γ_{tot}

Appendix 4:
Barrier Penetration
and Transmission Functions

Reflection at a Potential Change

For simplicity consider the case where the incident particle has no charge, i.e., a neutron, and take angular momentum, l = 0.



In QM there exists reflection whether V increases or decreases

$$E = \frac{p^2}{2\mu}$$

$$\frac{\sqrt{2\mu E}}{\hbar} = \frac{p}{\hbar} = \frac{2\pi}{\lambda} = \frac{1}{\bar{\lambda}} \equiv k$$

Wave number for incident particles $\quad k = \frac{\sqrt{2\mu E}}{\hbar} \quad x < 0$

inside well $\quad K = \frac{\sqrt{2\mu(E+V_o)}}{\hbar} \approx \frac{\sqrt{2\mu V_o}}{\hbar}$

$\Psi(x) = Ae^{ikx} + Be^{-ikx} \quad x<0$ Incident wave plus reflected wave

$= Ce^{iKx} \quad x>0$ Wave traveling to the right

$\Psi(x), \Psi'(x)$ continuous implies at x=0, A+B=C

$$ikA - ikB = iKC$$

$$\Rightarrow \quad \frac{B}{A} = \frac{1-\frac{K}{k}}{1+\frac{K}{k}}$$

$$T = 1 - \left|\frac{B}{A}\right|^2 = \frac{(1+\frac{K}{k})^2 - (1-\frac{K}{k})^2}{(1+\frac{K}{k})^2} = \frac{4K/k}{(1+\frac{K}{k})^2} = \frac{4Kk}{(k+K)^2}$$

The fraction that "penetrates" to the region with the new potential.

and if $E \ll V_o$

$$T = \frac{4k}{K} = \frac{4\pi kR}{\pi KR} = \frac{4\pi\rho}{\pi KR} = 4\pi S\, f \rho P_0$$

recall $\rho P_0 = \rho = kR$

where $S = \frac{1}{\pi KR}$ is the "black nucleus strength function"

f corrects empiricaly for the fact that the nucleus is not purely absorptive at radius R

Though for simplicity we took the case l = 0 and Z = 0 here, the result can be generalized to reactants with charge and angular momentum

For Z= 0

$$\rho P_0 = \rho \qquad l = 0$$

$$\rho P_1 = \frac{\rho^3}{1+\rho^2} \qquad l = 1$$

$$\rho P_2 = \frac{\rho^5}{9+3\rho^2+\rho^4} \qquad l = 2$$

For Z > 0

$$\eta = \frac{Z_I Z_j e^2}{\hbar v} = 0.1575\ Z_I Z_j \sqrt{\frac{\hat{A}}{E(MeV)}}$$

$$\rho = \sqrt{\frac{2\mu E}{\hbar^2}} R_0 = 0.2187 \sqrt{\hat{A}E}\ R_0(fm)$$

$$\rho P_l = \frac{\rho}{F_l^2(\eta,\rho) + G_l^2(\eta,\rho)}$$

It is customary to define the transmission function for particles (not photons) as

$$T = 4\pi\, S\, f(\rho P_l)$$

where S, the strength function, could be thought of in terms of resonance properies as

$$S = \frac{\Gamma_j}{D} = \frac{3\hbar^2}{\mu R^2}\frac{\theta_j^2}{D} \quad (see\ 3\ pages\ ahead)$$

which is a constant provided that $\theta_j^2 \propto D$, the level spacing.

This is consistent with the definition

$$T = 2\pi \left\langle \frac{\Gamma}{D} \right\rangle$$

Here "f" is the "reflection factor", empirically 2.7 for n and p and 4.8 for alpha-particles, which accounts for the fact that the reflection is less when the potential does not have infinitely sharp edges at R. Hence the transmission is increased.

But actually the strength function is parameterized in terms of the black nucleus approximation used in the transmission function calculation. Unknown parameters are fit to data.

For nuclei A < 65

$R = 1.25\,A^{1/3} + 0.1$ fm for n,p
$\quad\;\; 1.09\,A^{1/3} + 2.3$ fm for alpha particles

$$S = \frac{1}{\pi K R} \qquad K = \sqrt{\frac{2\mu V_o}{\hbar^2}} \qquad V_o \approx 60\,\text{MeV}$$

This is what is used in the Hauser Feshbach formalism

Analogously the *photon* transmission function is defined as:

$$T_\gamma = 2\pi \left\langle \frac{\Gamma_\gamma}{D} \right\rangle = \text{Strength function * phase space factor}$$

Phase space $\sim E_\gamma^{\;3}$ for dipole radiation
$E_\gamma^{\;5}$ for quadrupole radiation

The strength function is usually taken to be a constant or else given a ``Giant Dipole" (Lorentzian) form.

The transmission functions to the ground state and each excited state are calculated separately and added together to get a total photon transmission function.

Semi-empirical Γ's

Typically $\Gamma_\gamma \sim$ eV – larger for large ΔE in the transition; smaller if a large ΔJ is required or ΔE is small.

For nucleons and alpha particles it can be shown (Clayton 330 – 333) that

$$\Gamma_j^l = \left(\frac{3\hbar^2}{\mu R^2}\right)\theta_j^2\,\rho P_l = \frac{125.41\text{ MeV}}{\hat{A}R^2(fm)}\,\theta_j^2\,\rho P_l$$

where θ_j^2 is the "dimensionless reduced width" which must be evaluated experimentally, but is between 0 and 1 (typically 0.1).

The resulting widths are obviously very energy sensitive (via ρP_l) but for neutrons and protons not too much less than the Coulomb energy, they are typically keV to MeV.

The decay rate of the state is qualitatively given by (Clayton p 331) aside:

$\lambda \equiv$ probability/sec for particle from decaying system to cross large spherical shell

$\lambda = \frac{1}{\tau} =$ velocity at infinity * penetration factor * probability per unit *dr* that the particle is at the nuclear radius ± *dr*

$$= \frac{\Gamma}{\hbar} = \text{v}\;P_l\;\frac{3}{R}\theta^2 = \frac{\hbar\rho}{\mu R}\frac{3}{R}P_l\theta^2 = \frac{3\hbar}{\mu R^2}\rho P_l \theta^2$$

where $\frac{3}{R} = \frac{4\pi R^2\,dr}{4/3\,\pi R^3}$ is the probability per unit radius for finding the nucleon if the density is constant $\frac{d\,(volume)}{volume}$

$\theta^2 =$ dimensionless constant < 1

$$\rho = kR = \frac{\mu \text{v}}{\hbar}R = \sqrt{\frac{2\mu E}{\hbar^2}}\,R$$

Very approximate estimates for Γ

Typically Γ_γ ~ eV – larger for large ΔE in the transition; smaller if a large ΔJ is required or ΔE is small.

For nucleons and alpha particles it can be shown (Clayton 330 – 333 and appendix to this lecture) that

$$\Gamma_j^l = \left(\frac{3\hbar^2}{\mu R^2}\right)\theta_j^2\ \rho P_l = \frac{125.41\ \text{MeV}}{\hat{A}R^2(fm)}\theta_j^2\ \rho P_l$$

use this only in the absence of any experimental data

where θ_j^2 is the "dimensionless reduced width" which must be evaluated experimentally, but is between 0 and 1 (typically 0.1). See appendix to this lecture (last page)

The resulting widths are obviously very energy sensitive (via ρP_l) but for neutrons and protons not too much less than the Coulomb energy, they are typically keV to MeV.

$$\sigma = \pi \lambda\!\!\!^{-2}\omega\,\frac{\Gamma_1\Gamma_2}{(E-E_r)^2 + (\Gamma/2)^2}$$

$$\int_0^\infty \sigma(E)dE \approx \pi\lambda\!\!\!^{-2}_r\omega\ \Gamma_1(E_r)\Gamma_2(E_r)\underbrace{\int_0^\infty \frac{dE}{(E-E_r)^2+(\Gamma_r/2)^2}}_{\frac{2\pi}{\Gamma_r}}$$

Rate of reaction through a narrow resonance

Narrow means: $\Gamma << \Delta E$

In this case, the resonance energy must be "near" the relevant energy range ΔE to contribute to the stellar reaction rate.

Recall:

$$<\sigma v>=\sqrt{\frac{8}{\pi\mu}}\frac{1}{(kT)^{3/2}}\int_0^\infty \sigma(E)E\,e^{-\frac{E}{kT}}dE$$

pull out front

and

$$\sigma(E) = \pi\lambda\!\!\!^{-2}\ \omega\ \frac{\Gamma_1(E)\Gamma_2(E)}{(E-E_r)^2+(\Gamma(E)/2)^2}$$

For a **narrow** resonance assume:

M.B. distribution	$\Phi(E) \propto E\,e^{-\frac{E}{kT}}$	constant over resonance	$\Phi(E) \approx \Phi(E_r)$
All widths Γ(E)		constant over resonance	$\Gamma_i(E) \approx \Gamma_i(E_r)$
$\lambda\!\!\!^{-2}$		constant over resonance	