AY 12 Homework #3 Solutions
Winter 2016

Longer Problems

1. We begin with Kepler's third law, modified for the case of binary stars:

$$P^2 = \frac{(r_1 + r_2)^3}{M_1 + M_2},$$

where P is measured in years, r in AU, and M in $M_\odot$. The subscripts 1 and 2 refer to the two binary stars. The problem states that $P = 50\,\text{yr}$ and $r_1 + r_2 = 20\,\text{AU}$. Substituting this into Kepler's third law, and solving for the sum of the masses, we find that

$$P^2 = \frac{(r_1 + r_2)^3}{M_1 + M_2} \Rightarrow M_1 + M_2 = \frac{(r_1 + r_2)^3}{P^2} = \frac{(20)^3}{(50)^2} = 3.2\,M_\odot$$

Now we use the fact that the ratio of the two masses is inversely proportional to the ratio of their orbital speeds, that is:

$$\frac{M_1}{M_2} = \frac{v_2}{v_1}.$$

The problem states that Star 1 is moving 1/2 as fast as Star 2, i.e., $v_1 = v_2/2$. Thus the ratio above gives that $M_1 = 2M_2$. We now have two equations in two unknowns,

$$\begin{aligned} M_1 + M_2 &= 3.2\,M_\odot \\ M_1 &= 2M_2 \end{aligned}$$

which can be solved by substitution. In the end we get $\boxed{M_1 = 2.1\,M_\odot, M_2 = 1.1\,M_\odot}$. As a reality check, note that since Star 1 is more massive, it *must* be moving slower, as stated in the problem.

2. a) Coronal gas is made of hot (10^6–10^7 K), ionized gas (including H II). Since it is so hot, it emits UV and X-ray radiation. However, coronal gas has very low density ($n \sim 10^{-3}$ cm^{-3}). The gas is found about 1-3 kpc above and below the plane of the galactic disk.

b) Molecular clouds contain molecular gas (mostly H_2) and are very cold (10-20 K). These clouds emit mostly in the radio (i.e., 21 cm radiation). These are extremely dense regions ($n \sim 10^4$ cm^{-3}) and are the sites of star formation.

c) H II regions surround young O and B stars. These stars emit profusely in the UV, which ionizes the surrounding hydrogen gas, creating H II regions. They have temperatures around 10000 K and densities of about 1000 cm^{-3}. They emit primarily visible light, e.g., Hα. There are about 700 such regions in the Milky Way.

d) T-Tauri stars are very young stars that have not yet settled on the main sequence. They are powered by gravitational contraction (not nuclear fusion). Their brightness can vary dramatically, and they are often observed to have jets and disks of material surrounding them.

3. a) The mass of a cloud that is gravitationally bound (i.e., will just barely collapse) is given by the *Jeans mass*, which can be written conveniently as

$$M_{\text{Jeans}} = 34 \times \frac{T^{3/2}}{n^{1/2}}\, M_\odot,$$

where T is the temperature and n is the number density. Substituting the provided values $T = 40\,\text{K}$, $n = 10000\,\text{cm}^{-3}$ I find that

$$M_{\text{Jeans}} = 34 \times \frac{40^{3/2}}{10000^{1/2}}\, M_\odot = \boxed{86\, M_\odot}.$$

b) A more massive cloud with the same values of T and n will be $\boxed{\text{more tightly bound}}$ by gravity—the Jeans mass is the *minimum* mass required for a cloud to be gravitationally bound.

4. a) From your notes, "F" for a composition of 35% hydrogen and 65% helium is F=1.19. The equation for ideal gas pressure is

$$P_{ideal} = 8.31 \times 10^7 F \rho T \; \frac{dynes}{cm^2}$$

Substituting the values of $\rho = 160 \frac{g}{cm^3}$ and $T = 15 \times 10^6$ K, we get:

$$P_{ideal} = \boxed{2.37 \times 10^{17} \; \frac{dynes}{cm^2}}$$

b) The equation for radiative pressure is

$$P_{rad} = \frac{1}{3} a T^4$$

Using $a = 7.56 \times 10^{-15} \; \frac{dynes}{cm^2 K^4}$, we get:

$$P_{rad} = \boxed{1.28 \times 10^{14} \frac{dynes}{cm^2}}$$

c) The equation for non-relativistic degeneracy pressure is

$$P_{deg,NR} = 1.00 \times 10^{13} (\rho Y_e)^{5/3} \; \frac{dynes}{cm^2}$$

Using $Y_e = 0.68$, this gives:

$$P_{deg,NR} = \boxed{2.48 \times 10^{16} \; \frac{dynes}{cm^2}}$$

d) Ideal gas pressure dominates at the center of the Sun.

5. a) Recall that density is defined as $\rho = M/V$, where M is the mass of the object and V is its volume. For a star $V \propto R^3$, so $\rho \propto M/R^3$. We are given that $R \propto M^{0.65}$ for stars on the main sequence, and thus

$$\rho \propto \frac{M}{(M^{0.65})^3} \propto \frac{M}{M^{1.95}} \propto M^{-0.95},$$

i.e., $\boxed{\rho \propto M^{-0.95}}$. If the mass is halved, the density increases by a factor of $\boxed{2^{0.95} = 1.9}$, i.e., the density is 1.9 times the original density.

b) The central pressure in a star of uniform density and composition is $P_c = GM\rho/2R \propto M\rho/R$. Assuming ideal gas pressure, we know that $P_c \propto \rho T_c$ by the ideal gas law (the c means "central"). Equating the two expressions gives us the needed relation between central temperature, mass, and radius:

$$\rho T_c \propto \frac{M\rho}{R} \Rightarrow T_c \propto \frac{M}{R}.$$

Substituting in the mass-radius relation, $R \propto M^{0.65}$, we find that the central temperature depends on mass to the 0.35 power:

$$T_c \propto \frac{M}{M^{0.65}} \propto M^{0.35} \Rightarrow \boxed{T_c \propto M^{0.35}}.$$

If the mass of the star is half as large, the central temperature decreases to $(1/2)^{0.35} =$ $\boxed{0.78 \text{ times its original value.}}$

6. In general, the lifetime of the Sun is approximately the total energy content of the Sun divided by the Sun's luminosity:

$$\tau = \frac{E}{L_\odot}.$$

(This equation comes from the definition of luminosity.) The energy content E is equal to the mass that is "burned," M, times the amount of energy released per gram, Q, i.e., $E = MQ$. We set up a ratio to determine the lifetime of a gasoline-powered Sun compared to a nuclear fusion-powered Sun:

$$\frac{\tau_{\text{gas}}}{\tau_{\text{nuc}}} = \frac{E_{\text{gas}}/L_\odot}{E_{\text{nuc}}/L_\odot} = \frac{MQ_{\text{gas}}/L_\odot}{MQ_{\text{nuc}}/L_\odot} = \frac{Q_{\text{gas}}}{Q_{\text{nuc}}} = \frac{4.4 \times 10^{11} \text{ erg/g}}{6.4 \times 10^{18} \text{ erg/g}} = 7 \times 10^{-8}.$$

Since $\tau_{\text{nuc}} = 10^{10}$ yr, the lifetime of a gasoline-powered Sun is $\tau_{\text{gas}} = 7 \times 10^{-8} \tau_{\text{nuc}} = (7 \times 10^{-8})(10^{10} \text{ yr}) = \boxed{700 \text{ yr}}$!

7. a) The distance of closest approach r_{min} is found by using conservation of energy. Initially, the two protons are infinitely far apart and only have kinetic energy, $E_i = (3/2)kT$. At closest approach, the protons are not moving, and all the energy is electric potential energy, $E_f = e^2/r_{\text{min}}$:

$$\begin{aligned} E_i &= E_f \\ \frac{3}{2}kT &= \frac{e^2}{r_{\text{min}}}. \end{aligned}$$

In this problem we are interested in finding the temperature required for $r_{\text{min}} = 10^{-13}$ cm. We solve the above equation for T and substitute numbers:

$$T = \frac{2e^2}{3kr_{\text{min}}} = \frac{2(4.8 \times 10^{-10} \text{ esu})^2}{3(1.38 \times 10^{-16} \text{ erg K}^{-1})(10^{-13} \text{ cm})} = \boxed{1.1 \times 10^{10} \text{ K}}.$$

b) This answer is about 1000 times larger than the actual temperature of 10^7 K. This discrepancy can be resolved if: (1) the colliding protons are actually moving *faster* than the average speed at temperature T (i.e., the protons are those from the high-velocity tail of the velocity distribution), and (2) *quantum mechanical barrier penetration* occurs, which gives a small, but finite, probability of the two protons to approach zero separation.

8. a) Given the Sun's luminosity ($L_\odot = 3.83 \times 10^{33}$ erg s^{-1}) and the amount of energy released from the production of one helium nucleus (4.2×10^{-5} erg), the number of helium nuclei produced *per second* is found by dividing the luminosity by the energy per nuclei:

$$N_{\text{He}}\,(\text{per second}) = \frac{3.83 \times 10^{33} \text{ erg s}^{-1}}{4.2 \times 10^{-5} \frac{\text{erg}}{\text{He nucleus}}} = \boxed{9 \times 10^{37} \text{ He nuclei s}^{-1}}.$$

b) Since the formation of one helium nucleus is accompanied by the release of two neutrinos, the number of neutrinos produced per second is just twice the answer to part a), i.e., $\boxed{1.8 \times 10^{38} \text{ neutrinos s}^{-1}}$.

c) The flux of neutrinos at Earth can be found from the standard definition of flux:

$$\phi = \frac{L}{4\pi d^2},$$

except in this case L is not the luminosity of the Sun due to light, but rather neutrinos. Substituting $L = L_{\text{neutrinos}} = 1.8 \times 10^{38}$ neutrinos s^{-1} into the equation for flux:

$$\phi = \frac{1.8 \times 10^{38} \text{ neutrinos s}^{-1}}{4\pi(1 \text{ AU})^2} = \boxed{6 \times 10^{10} \text{ neutrinos s}^{-1} \text{ cm}^{-2}}.$$

Note that you must convert AU to cm to get the correct answer! Because neutrinos can pass through matter unimpeded (including the Earth), the $\boxed{\text{answer does not depend}}$ on the time of day (or night).

Shorter Questions

1. a) H_2 is found in $\boxed{\text{molecular clouds}}$.

   b) H II is found in the $\boxed{\text{warm ionized medium, hot ionized medium, and H II regions}}$.

   c) H I is found in the $\boxed{\text{cold neutral medium and warm neutral medium}}$.

   Star formation occurs in $\boxed{\text{molecular clouds}}$.

2. The nuclear reactions that power the Sun are described by the *ppI chain*:

$$\begin{aligned} p + p &\rightarrow {}^2\mathrm{H} + e^+ + \nu \\ {}^2\mathrm{H} + p &\rightarrow {}^3\mathrm{He} + \gamma \\ {}^3\mathrm{He} + {}^3\mathrm{He} &\rightarrow {}^4\mathrm{He} + p + p \end{aligned}$$

The net result is the production of a helium-4 nucleus, along with 26.2 MeV of energy. The CNO cycle dominates energy production in stars $\boxed{\text{greater than about } 2M_\odot}$.

3. The minimum mass for a main sequence star is about $\boxed{0.08\,M_\odot}$. A protostar with a smaller mass is unable to ignite hydrogen burning in its core, and therefore cannot become a bona fide star. Instead it will continue to contract and radiate away energy as a *brown dwarf*. Ultimately contraction is halted by electron degeneracy pressure, and the brown dwarf cools and fades away...

4. The number of rotations N of a given point of the Sun is given by $N = t/P$, where t is the length of time we are interested in, and P is the period of rotation. At the equator, $P = 26.8$ d. The number of rotations after 10 years is thus

$$N_{\mathrm{eq}} = \frac{t}{P} = \frac{10 \text{ yr}}{26.8 \text{ d} \times \frac{1 \text{ yr}}{365 \text{ d}}} = 136 \text{ rotations.}$$

A similar calculation for a point located at 75° longitude ($P = 31.8$ d) gives 114 rotations. Thus the Sun's equator rotates about $136 - 114 = \boxed{\text{22 times more}}$ after 10 years.

5. The three kinds of pressure that can act within a star are ideal gas pressure, radiation pressure, and degeneracy pressure. Each depends on temperature and/or density as follows:

Pressure	Dependency
ideal gas	$P_{\mathrm{ideal}} \propto \rho T$
radiation	$P_{\mathrm{rad}} \propto T^4$
degeneracy	$P_{\mathrm{deg}} \propto \rho^{5/3}$ or $\propto \rho^{4/3}$

6. Energy can be transported by $\boxed{\text{convection, radiative diffusion, and conduction}}$. Radiative diffusion dominates at the center of the Sun, while convection dominates at its surface. For the $10M_\odot$ star, convection dominates at the center, while radiative diffusion dominates at the surface.

7. a) The Little Ice Age was a period of a few hundred years when the Earth experienced below average temperatures, possibly related to a period of low solar activity. b) The umbra is the darkest region of a sunspot. c) The Babcock model describes how the Sun's magnetic field works and explains the origin of sunspots and the solar cycle.

8. The present sun on the main sequence is growing $\boxed{\text{more luminous}}$, getting $\boxed{\text{larger}}$ in radius, and getting $\boxed{\text{hotter}}$ in its center.

9. Horizontal branch stars are burning helium in their cores.