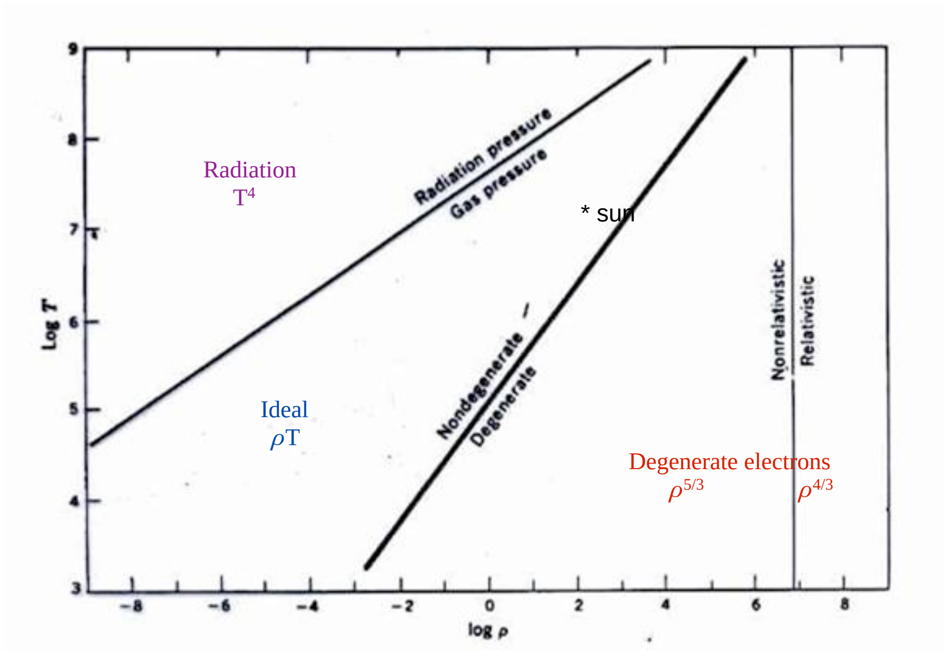
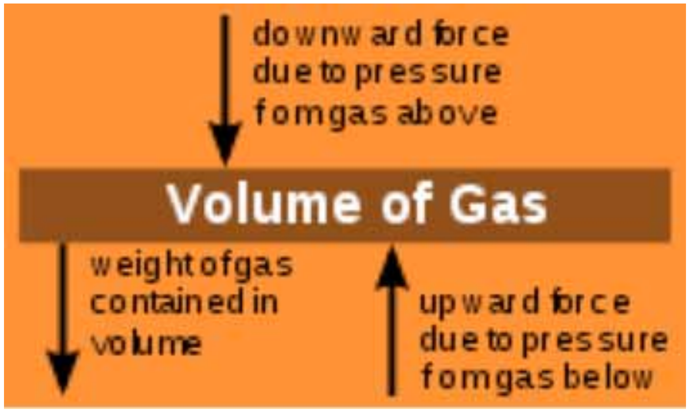
Stellar Interiors -
Hydrostatic Equilibrium and
Ignition on the Main Sequence

http://apod.nasa.gov/apod/astropix.html



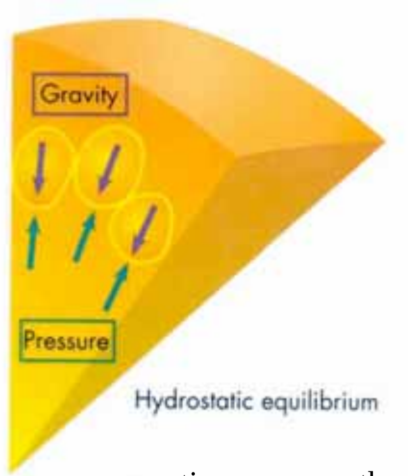
HYDROSTATIC EQUILIBRIUM

forces must balance if nothing is to move

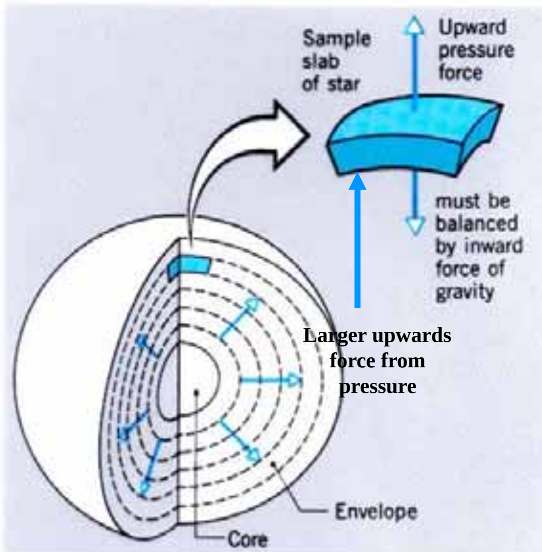


E.g. the earth's atmosphere or a swimming pool

Inside a star the weight of the matter is supported by a *gradient* in the pressure. If the pressure on the top and bottom of a layer were exactly the same, the layer would fall because of its weight.



The difference between pressure times area on the top and the bottom balances the weight



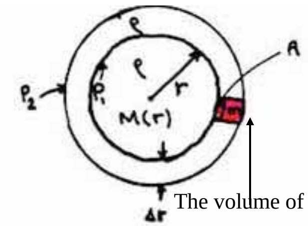
The balance of gas pressure and gravity in a star.

Hydrostatic Equilibrium

$\Delta P = P_1 - P_2$

$\text{Force up} = P_1 A - P_2 A$

$\text{Force down} = \frac{GM(r)m}{r^2}$



The volume of the red solid is its area, A, times its thickness, Δr. For example, for a cylinder $A = \pi a^2$ where a is the radius of the cylinder and the volume is $V = \pi a^2 \Delta r$

$$\Delta P.A = -\left(\frac{GM(r)}{r^2}\right)\underbrace{(\rho A \Delta r)}_{m}$$

$$\frac{\Delta P}{\Delta r} = -\frac{GM(r)\rho}{r^2}$$

or more properly

For all kinds of gases – ideal, degenerate, whatever.

If $M(r)$ is the mass *interior* to radius r and $\rho(r)$ is the density *at* r

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

For a sphere of constant density, $M(r) = \frac{4}{3}\pi r^3 \rho$ and one may integrate this equation to obtain

$$P_{central} = \frac{GM\rho}{2R}$$

This (top) equation is one of the fundamental equations of stellar structure. It is called the "equation of hydrostatic equilibrium". Whenever dP/dr differs from this value, matter moves.

Proof

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}$$

Assume density is constant $\quad M(r) = \frac{4\pi r^3 \rho_o}{3}$

$$\frac{dP}{dr} = \frac{-G\rho_o^2 4\pi r^3}{3r^2}$$

$$\int_{P_c}^{0} dP = \frac{-4\pi G\rho_o^2}{3}\int_0^R r\, dr$$

$$P_{central} = \frac{4\pi G\rho_o^2}{3}\frac{R^2}{2}$$

$$P_{central} = \frac{G\rho_o}{2R}\frac{4\pi R^3 \rho_o}{3} = \frac{GM\rho_o}{2R}$$

Note implications for central temperature. *If an ideal gas, 75% H and 25% He, fully ionized*

$$P_c = 1.69\rho N_A k T_c = \frac{GM\rho}{2R}$$

or

$$T_c = \frac{GM}{3.38 N_A k R}$$

For example, for the sun (not really constant in density), so answer is an underestimate

$$T_c \approx \frac{(6.67\times10^{-8})(2.00\times10^{33})}{(3.38)(6.02\times10^{23})(1.38\times10^{-16})(6.96\times10^{10})}$$
$$= 6.8\times10^{6}\ \mathrm{K}$$

$$T_c \propto \frac{M}{R}$$

For stars supported by ideal gas pressure and near uniform structure (not red giants)

Note that as the radius gets smaller, the central temperature *increases*.

What is the time scale for this increase (if gravity is the only source of power)?

$$\tau_{KH} = \frac{3GM^2}{10RL}$$

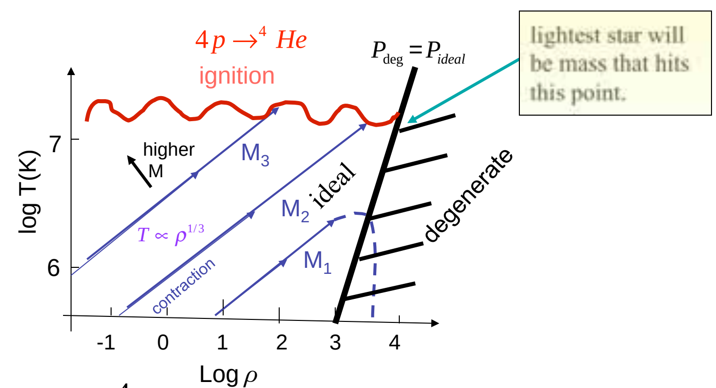
which is ~ 10 million years for the sun.

(20 - 30 My is more accurate)

Ignition happens when the nuclear energy generation rate becomes comparable to the luminosity of the contracting proto-star. As we shall see shortly, nuclear burning rates are very sensitive to the temperature. Almost all main sequence stars burn hydrogen in their middles at temperatures between 1 and 3 x 10^7 K. (The larger stars are hotter in their centers). If ε_{nuc} is the equation for the energy release per second in a gram of matter because of nuclear reactions

$$\varepsilon_{nuc} \propto T^n \qquad n \gg 1$$

on the main sequence n ≈ 4 to 16



$$T \propto \frac{M}{R} \qquad M \sim \frac{4\pi}{3}R^3\rho$$

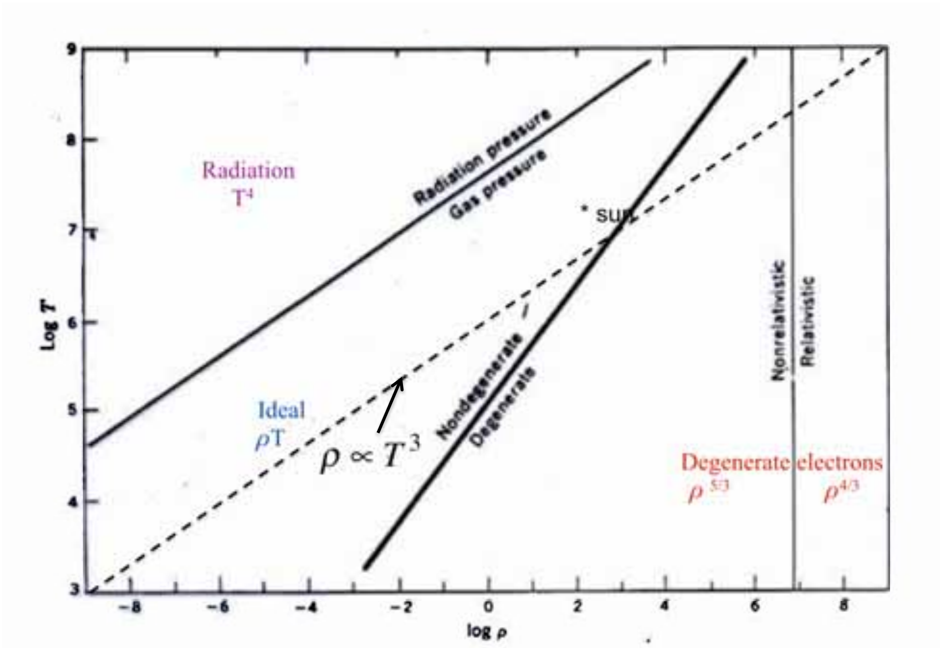
$$\Rightarrow R \propto \left(\frac{M}{\rho}\right)^{1/3}$$

$$T \propto \frac{M\rho^{1/3}}{M^{1/3}}$$

$$T \propto M^{2/3}\rho^{1/3}$$

$T \propto \rho^{1/3}$ for a given M and T at a given ρ is higher for bigger M

This gives the blue lines in the plot



Minimum Mass Star

Solve for condition that ideal gas pressure and degeneracy pressure are equal at 10^7 K.

$$P_{deg} \approx P_{ideal}$$

$$1.69\,\rho N_A kT \approx 1.00\times 10^{13}\,(\rho Y_e)^{5/3}$$

(assuming 75% H, 25% He by mass)

At 10^7 K, this becomes

$$1.40\times 10^8\,\rho(10^7) \approx 8.00\times 10^{12}\,\rho^{5/3}$$

(taking $Y_e = 0.875$)

which may be solved for the density to get $\rho \approx 2300\,\text{gm cm}^{-3}$

The total pressure at this point is

$$P_{tot} \approx \frac{1}{2}\left(P_{deg} + P_{ideal}\right) \approx \frac{1}{2}(2P_{ideal}) \approx P_{ideal}$$

$$\approx 1.40\times 10^8\,(2300)\left(10^7\right) \approx 3.2\times 10^{18}\ \text{dyne cm}^{-2}$$

$$= \left(\frac{GM\rho}{2R}\right)$$

But $R = \left(\dfrac{3M}{4\pi\rho}\right)^{1/3}$ i.e., $\rho = \dfrac{M}{4/3\,\pi R^3}$

Combining terms we have

$$3.2x10^{18} \approx \frac{\left(G\,M\,\rho\right)\left(4\pi\rho\right)^{1/3}}{2\left(3M\right)^{1/3}}$$

$$M^{2/3} \approx \frac{2\left(3.2x10^{18}\right)\left(3^{1/3}\right)}{G\,\rho^{4/3}\left(4\pi\right)^{1/3}}$$

and using again $\rho \approx 2300\,\text{gm cm}^{-3}$

$$M \approx 8.7x10^{31}\ \text{gm}$$

or 0.044 solar masses.

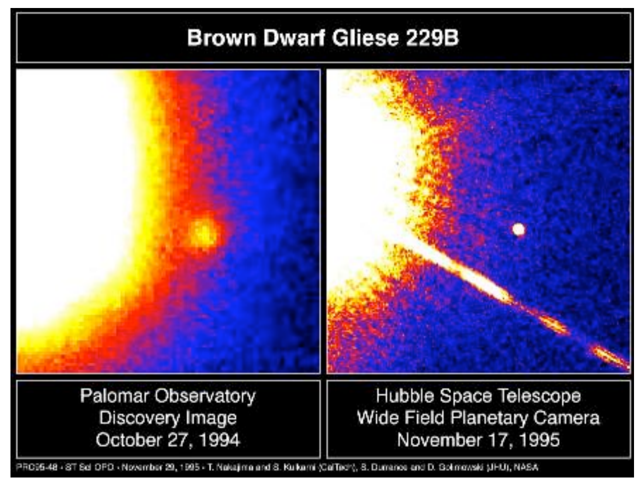
For constant density

$$P = \left(\frac{GM\rho}{2R}\right)$$

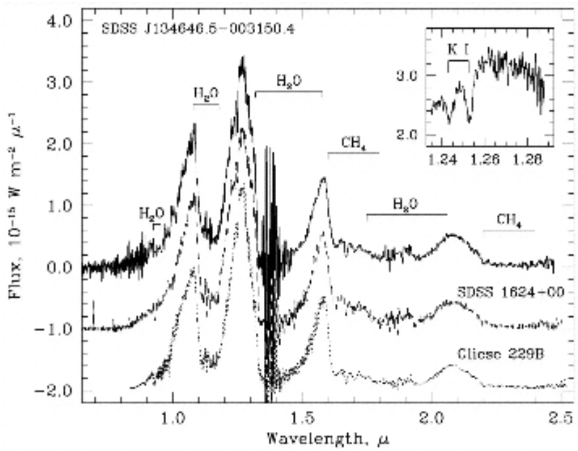
$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$$

A more detailed calculation gives *0.08 solar masses.*
Protostars lighter than this can never ignite nuclear reactions.
They are known as brown dwarfs (or planets if the mass is less than 13 Jupiter masses, or about 0.01 solar masses.
[above 13 Jupiter masses, some minor nuclear reactions occur that do not provide much energy - "deuterium burning"]

Brown Dwarfs - heavier than a planet ($13\ M_{Jupiter}$) and lighter than a star



14 light years away in the constellation Lepus orbiting the low mass red star Gliese 229 is the brown dwarf Gliese 229B. It has a distance comparable to the orbit of Pluto but a mass of 20-50 times that of Jupiter. Actually resolved with the 60" Palomar telescope in 1995 using adaptive optics.



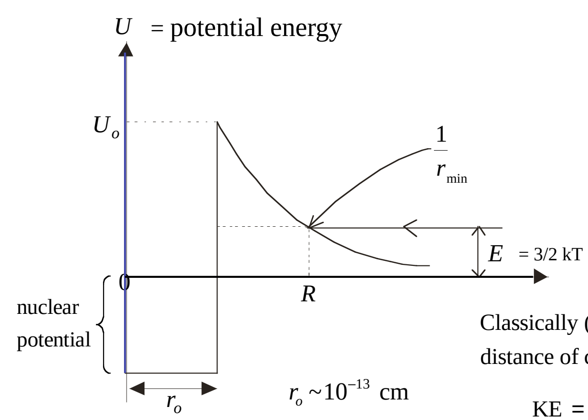
Spectrum of Gliese 229B

Nuclear Fusion Reactions

Main Sequence Evolution
(i.e., Hydrogen burning)

The basis of energy generation by nuclear fusion is that two reactants come together with sufficient collisional energy to get close enough to experience the strong force. This force has a range ~10^{-13} cm, i.e., about 1/100,000 the size of the hydrogen atom.

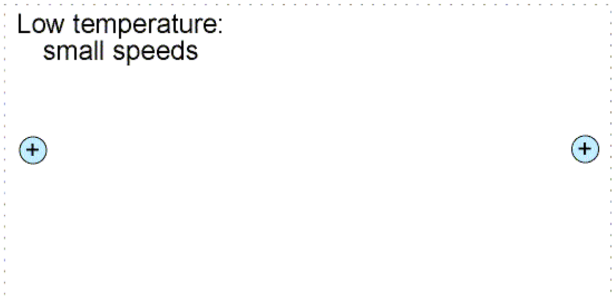
- Before 2 protons can come close enough to form a bound state, they have to overcome their electrical repulsion.



For two hydrogen nuclei
$Z_1 = Z_2 = +1$

Classically (before QM), the distance of closest approach, r_{min}, was given by

$$KE = \frac{1}{2}mv^2 = \frac{3}{2}kT = \frac{e^2}{r_{min}}$$



$$\frac{3}{2}kT = \frac{e^2}{r_{\min}} \Rightarrow r_{\min} = \frac{2}{3}\frac{e^2}{kT}$$

At 10^7 K this gives $r \sim 10^{-10}$ cm, i.e., far too distant for the strong force to have any effect or for a nuclear reaction to occur

The range of the nuclear force is about 10^{-13} cm

Two solutions to this quandry

- Use protons far out on tail of velocity distribution ($\sim 10\,kT$)
- Quantum mechanical barrier penetration

$\exp(x) \equiv e^x$

$e = 2.71828...$

$$P_0 \propto \exp(-(4\pi e^2)/(hv))$$

($Z_1 Z_2$)

is the probability that two protons with center of mass velocity v approach to zero separation. Note that if either v or h goes to zero, this probability is zero, but for finite values of both, it is non-zero.

QM Barrier Penetration

The *classical turning radius* is given by energy conservation

http://storytellingbox.tumblr.com/

$$\frac{1}{2}mv^2 = \frac{3}{2}kT = \frac{Z_1 Z_2 e^2}{r} \Rightarrow r_{classical} = \frac{2Z_1Z_2e^2}{mv^2} = \frac{2}{3}\frac{Z_1Z_2e^2}{kT}$$

The De Broglie wavelenth of the particle with mass *m* is,

$$\lambda = \frac{h}{p} = \frac{h}{mv}$$

So the ratio

$$\frac{r_{classical}}{\lambda} = \frac{2Z_1Z_2e^2}{mv^2}\frac{mv}{h} = \frac{2Z_1Z_2e^2}{hv} \qquad v \propto \sqrt{T}$$

Which is approximately the factor in the penetration function

$$P \sim \exp(-r_{classical}/\lambda) \propto \exp\left(\frac{-\text{const}\, Z_1Z_2}{\sqrt{T}}\right) \qquad note: r_{classical} >> \lambda$$

Note that as the charges become big or T gets small, P gets very small.

But when two protons do get close enough to (briefly) feel the strong force, they almost always end up flying apart again. The nuclear force is strong but the "diproton", ^{2}He, is not sufficiently bound to be stable.

One must also have, while the protons are briefly together, a ***weak*** *interaction.*

$$p + p \rightarrow (^2He) \rightarrow {}^2H + e^+ + \nu_e$$

That is, a proton turns into a neutron, a positron, and a neutrino. The nucleus ^{2}H, deuterium, is permanently bound.

The rate of hydrogen burning in the sun is thus quite slow because:

- The protons that fuse are rare, only the ones with about ten times the average thermal energy
- Even these rare protons must penetrate a barrier to go from 10^{-10} cm to 10^{-13} cm and the probability of doing that is exponentially small
- Even the protons that do get together generally fly apart unless a weak interaction occurs turning one to a neutron while they are briefly togther

and that is all quite good because if two protons fused every time they ran into each other, the sun would explode.

Proton-proton fusion chain process

1st step: In two separate reactions, 2 protons in each reaction fuse

THE PP 1 CYCLE

Bethe 1939
(Nobel 1967)
MeV = 1.602 × 10^{-6} erg

$$p + p \rightleftarrows (2p) \rightarrow {}^2H + e^+ + \nu_e + 0.42\,\text{MeV}$$

nb. strong and weak interaction

where 1 MeV = 1.602 × 10^{-6} erg and the energy comes off in the form of radiation and the kinetic energy of the products.

In shorthand this can also be written $p(p,e^+\nu)^2H$.

So now we have protons, ^{4}He, and ^{2}H. Next

$$^2H + p \rightarrow {}^3He + \gamma + 5.49\,\text{MeV}$$

or $^2H(p,\gamma)^3He$. This may be followed by either

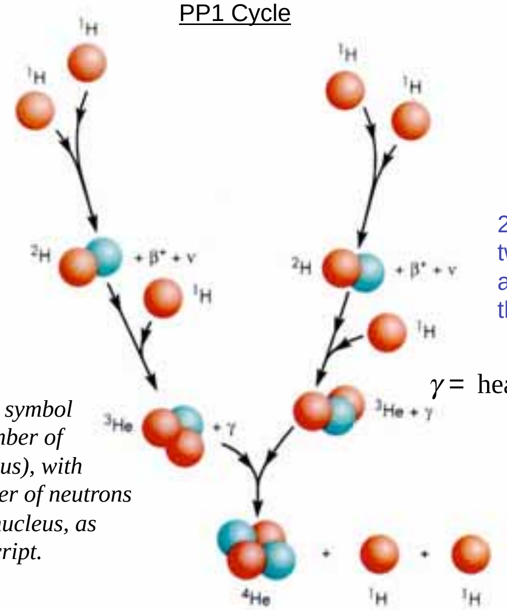
$$a)\ {}^3He + {}^3He \rightarrow {}^4He + 2p + 12.86\,\text{MeV}$$ pp1

or

$$b)\ {}^3He + {}^4He \rightarrow {}^7Be + \gamma + 1.59\,MeV$$ pp2,3

PP1 Cycle

Here $\beta^+ = e^+$



2 neutrinos and two positrons are made along the way

$\gamma =$ heat and light

Write the element's symbol (given by Z, the number of protons in the nucleus), with A = Z+N, the number of neutrons and protons in the nucleus, as a preceding superscript.

Lifetimes against various reactions

Reaction	Lifetime (years)
$^1H(p,e^+\nu)^2H$	7.9×10^9
$^2H(p,\gamma)^3He$	4.4×10^{-8}
$^3He(^3He,2p)^4He$	2.4×10^5
$^3He(^4He,\gamma)^7B$	9.7×10^5

For 50% H, 50% He at a density of 100 g cm^{-3} and a temperature of 15 million K

The time between proton collisions is about a hundred millionth of a second.

or in shorthand $^3He(^3He,2p)^4He$ or $^3He(\alpha,\gamma)^7Be$. In the sun the former is much more likely than the latter.

Neglecting for now reaction b), the total effect is

$$4p \rightarrow {}^4He + 2e^+ + 2\nu_e + 24.68 \text{ MeV}$$

The neutrinos carry away 0.26 MeV (on the average each) and are lost (more about these later). The positrons annihilate with two electrons to give an additional 2 x 0.511 MeV x 2 = 2.04 MeV. Thus

$$4p \rightarrow {}^4He + 26.20 \text{ MeV}$$

How many ergs per gram is this?

$$Q_{pp} = (26.20)(1.602 \times 10^{-6})(6.02 \times 10^{23})/4$$
$$= 6.4 \times 10^{18} \text{ erg g}^{-1}$$

i.e. per gram of H burned

Implication for the sun

$$\tau_{nuc} \approx \frac{(6.4 \times 10^{18})(1.99 \times 10^{33})(0.15)(0.70)}{(3.93 \times 10^{33})}$$
$$= 3.4 \times 10^{17} \text{ sec}$$

or 10.4 *billion* years!

Nuclear reaction shorthand:

I(j,k)L

I = Target nucleus j = incident particle
L = Product nucleus k = outgoing particle or energy

E.g., pp1

$p(p,e^+\nu)^2H(p,\gamma)^3He(^3He, 2p)^4He$

E.g., the main CNO cycle (later)

$^{12}C(p,\gamma)^{13}N(e^+\nu)^{13}C(p,\gamma)^{14}N(p,\gamma)^{15}O(e^+\nu)^{15}N(p,^4He)^{12}C$

H. A. Bethe (b 1906)
Nobel 1967

NUCLEAR REACTION RATES

Because of the exponential dependence of barrier penetration on energy and charge, the rates of nuclear fusion reactions are extremely temperature sensitive. It also obviously takes a higher temperature to fuse heavier isotopes that have larger charge.

For the pp 1 cycle

$$\epsilon_{pp} = 0.076\rho X_H^2 (T_c/10^7)^4 \text{ erg g}^{-1}\text{ s}^{-1}$$

where X_H is the mass fraction of hydrogen.

Assume that the sun is hot enough to run the pp 1 cycle only in the inner 10% of its mass

$$\begin{aligned} L &\approx 0.1 M_\odot \epsilon_{pp} \\ &\approx 7.4 \times 10^{30} \rho (T_c/10^7)^4 \\ &\approx 7.4 \times 10^{30} (100)(1.5)^4 \\ &\approx 3.7 \times 10^{33} \text{ erg s}^{-1} \end{aligned}$$

$X_H \sim 0.7$ actually less

which is pretty close to correct

How much mass burned per second?

$$\dot{M} = \frac{L_\odot}{q} = \frac{3.8x10^{33} \text{ erg s}^{-1}}{6.4\ x10^{18} \text{ erg gm}^{-1}}$$

$$= 5.9\,x10^{14} \text{ gm s}^{-1} = 650 \text{ million tons per second}$$

How much mass energy does the sun lose each year?

$$\dot{M} = \frac{L_\odot}{c^2} = \frac{3.83\times10^{33}\text{ erg/s}}{(3.0\times10^{10})^2 \text{ erg/gm}} = 4.3\times10^{12} \text{ gm s}^{-1}$$

or about 7×10^{-14} $M_\odot$ per year

Now…

Mass ($M_\odot$)	Radius ($R_\odot$)	Luminosity ($L_\odot$)	Temperature (10^6 °K)	Density (g cm^{-3})
0.0000	0.000	0.0000	15.513	147.74
0.0001	0.010	0.0009	15.48	146.66
0.001	0.022	0.009	15.36	142.73
0.020	0.061	0.154	14.404	116.10
0.057	0.090	0.365	13.37	93.35
0.115	0.120	0.594	12.25	72.73
0.235	0.166	0.845	10.53	48.19
0.341	0.202	0.940	9.30	34.28
0.470	0.246	0.985	8.035	21.958
0.562	0.281	0.997	7.214	15.157
0.647	0.317	0.992	6.461	10.157
0.748	0.370	0.9996	5.531	5.566
0.854	0.453	1.000	4.426	2.259
0.951	0.611	1.000	2.981	0.4483
0.9809	0.7304	1.0000	2.035	0.1528
0.9964	0.862	1.0000	0.884	0.042
0.9999	0.965	1.0000	0.1818	0.00361
1.0000	1.0000	1.0000	0.005770	1.99×10^{-7}

Radiative

Convective

*Adapted from Turck-Chiéze et al. (1988).
Composition X = 0.7046, Y = 0.2757, Z = 0.0197

ENERGY TRANSPORT IN STELLAR INTERIORS

1) Diffusion of Radiation:

Light diffuses slowly out of the star

$$\tau_{dif} = \frac{R^2}{l_{mfp}c} \qquad \text{i..e., } \tau_{diff} = \left(\frac{R}{l}\right)^2\left(\frac{l}{c}\right)$$

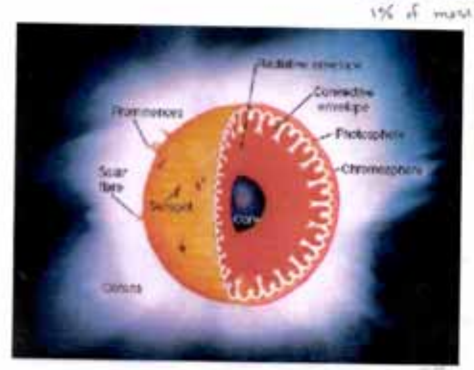
$$l_{mfp} = \frac{1}{\kappa\rho} \qquad \text{k is the “opacity” (cm}^2\text{ gm}^{-1}\text{)}$$

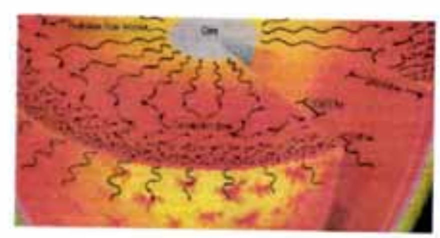
2) Convection:

Mass moves, carrying energy with it.

Happens in regions where the temperature gradient or opacity is large.

Convection
25% of radius
1% of mass





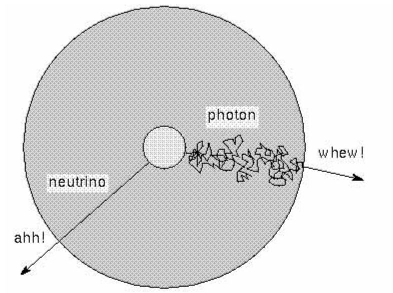
Diffusion time for the sun

On the average, $\rho \sim 1$ gm cm^{-3}

$\kappa \sim 1 \text{cm}^2 \text{gm}^{-1}$

$l \sim 1/\kappa\rho = 1\text{cm}$ (less in center, more farther out)

R = 6.96 x 10^{10} cm



Photons take tortuous paths out of the Sun's interior. Neutrinos pass right on through in just two seconds.

$$= \frac{R^2}{lc} = \frac{(6.96x10^{10})^2}{(1)(3.0x10^{10})} = 1.6\,x10^{11}\,\text{sec}$$

= 5100 years

3) Conduction:

Heat carried by **electrons like** in a metal.

Important in white dwarfs.

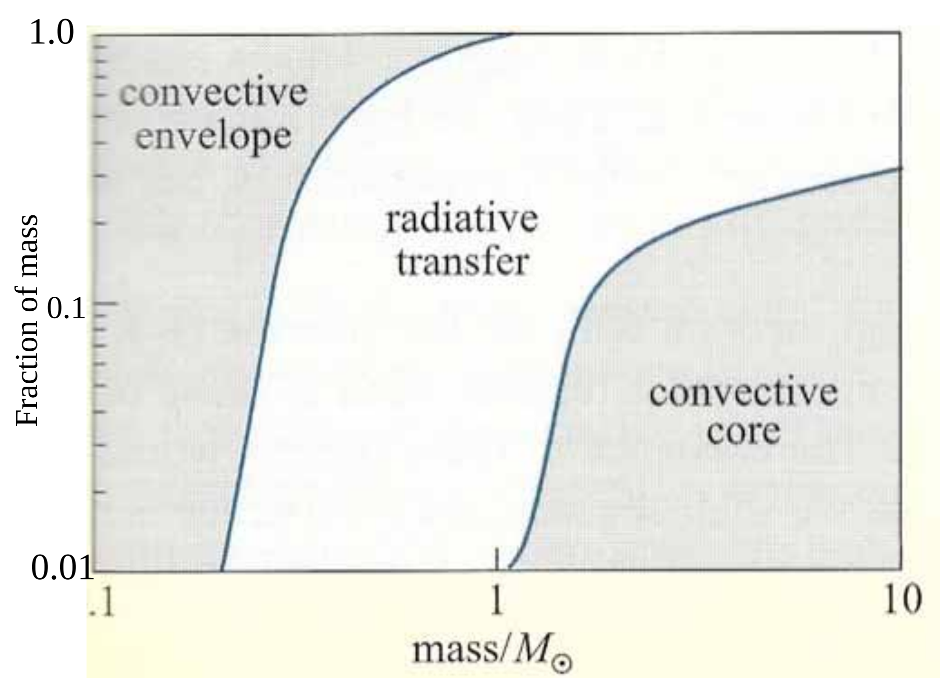
ON THE MAIN SEQUENCE

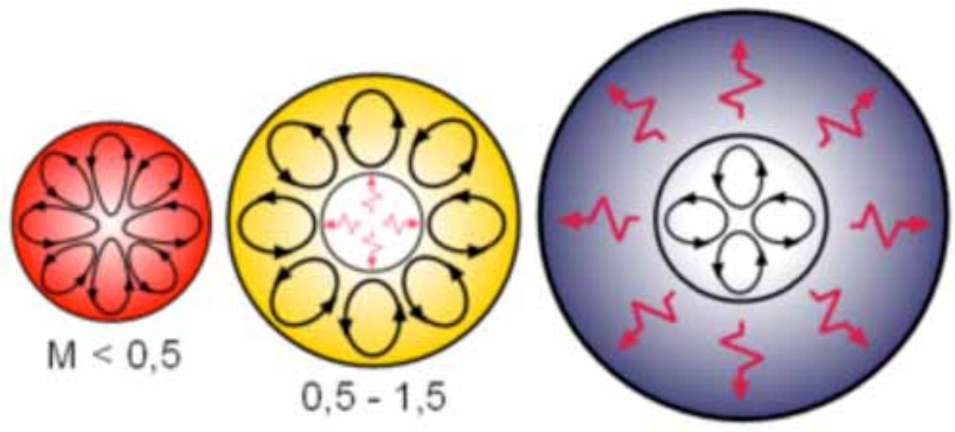
$\leq 0.30\ M_\odot$ - Star completely convective

$1.0\ M_\odot$ – Only outer layers convective

$1.5\,M_\odot$ - Whole star radiative

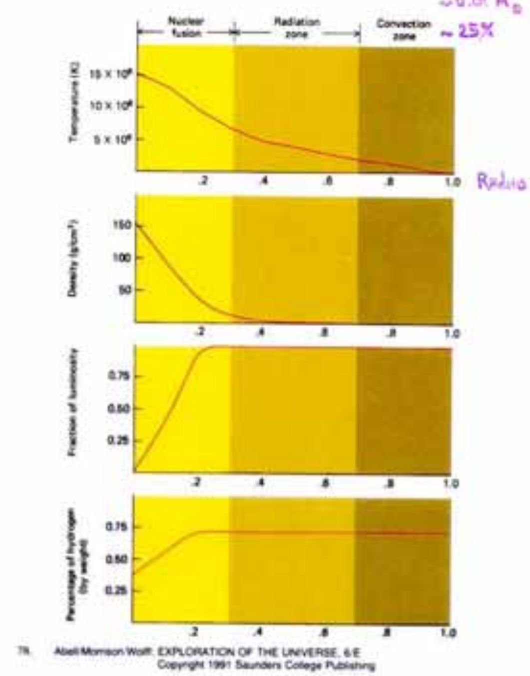
$\geq 2.0\,M_\odot$ – Surface stable; core convective

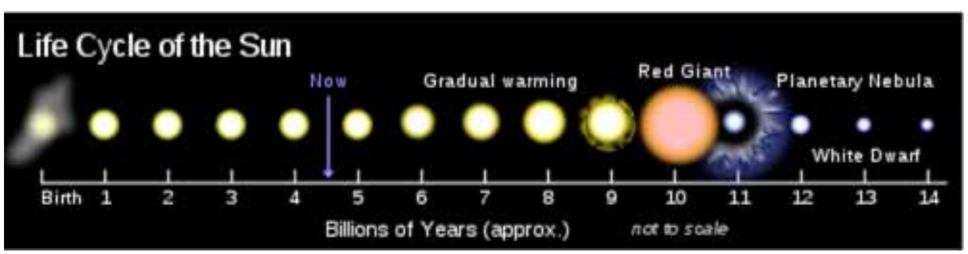




M > 2.0

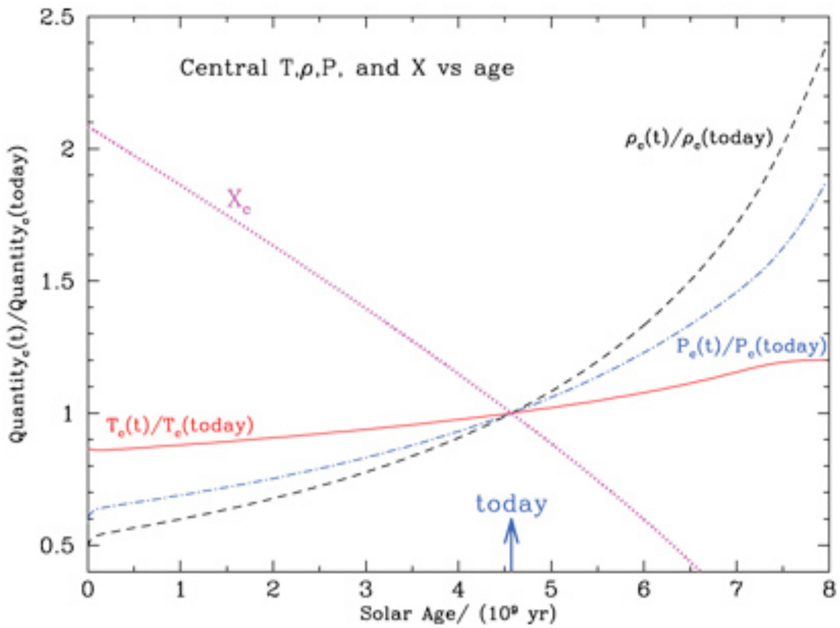
STRUCTURE OF THE SUN





Time (10^9 years)	Luminosity ($L_\odot$)	Radius ($R_\odot$)	$T_{central}$ (10^6 °K)	
Past				
0	0.7688	0.872	13.35	Zero age main sequence
0.143	0.7248	0.885	13.46	
0.856	0.7621	0.902	13.68	
1.863	0.8156	0.924	14.08	
2.193	0.8352	0.932	14.22	
3.020	0.8855	0.953	14.60	
3.977	0.9522	0.981	15.12	
Now				
4.587	1.000	1.000	15.51	
Future				
5.506	1.079	1.035	16.18	
6.074	1.133	1.059	16.65	
6.577	1.186	1.082	17.13	
7.027	1.238	1.105	17.62	
7.728	1.318	1.143	18.42	
8.258	1.399	1.180	18.74	
8.7566	1.494	1.224	18.81	
9.805	1.760	1.361	19.25	Red Giant

*Adapted from Turck-Chièze et al. (1988).
Composition X = 0.7046, Y = 0.2757, Z = 0.0197.
Present values are $R_\odot$ and $L_\odot$.
**For time t before the present age $t_\odot = 4.6 \times 10^9$ years,
$L/L_\odot \approx 1/[1+0.4(1-t/t_\odot)]$



STELLAR STABILITY

$$P_{cent} \propto T$$

$$P_{cent} \sim \frac{GM\rho}{R} \Rightarrow T \propto \frac{1}{R}$$

$$\epsilon_{nuc} \propto T^n \; (n \approx 4)$$

So if start to run reactions faster

$T\uparrow \Rightarrow P\uparrow$
$\Rightarrow R\uparrow$ because an ideal gas expands when heated
$\Rightarrow T\downarrow \Rightarrow \epsilon\downarrow$

Similarly if the rate of reactions declines for some reason,

$T\downarrow \Rightarrow P\downarrow$
$\Rightarrow R\downarrow$
$\Rightarrow T\uparrow \Rightarrow \epsilon\uparrow$

Thus a tight equilibrium is maintained

Important exception:
Degenerate matter

$+ \; \tau_{nuc} \lesssim \tau_{diff}$

Note that the units of pressure and energy density are the same

$$\frac{\text{erg}}{\text{cm}^3} = \frac{\text{dyne cm}}{\text{cm}^3} = \frac{\text{dyne}}{\text{cm}^2}$$

Ideal gas   n k T   is the pressure but
3/2 n k T is the energy density

Radiation $\frac{1}{3}aT^4$ is the pressure
aT^4 is the energy density

Why is $L \propto M^3$?

$$\text{Luminosity} \approx \frac{\text{Heat content in radiation}}{\text{Time for heat to leak out}} = \frac{E_{radiation}}{\tau_{diffusion}}$$

True even if star is not supported by P_{rad}
Note this is not the total heat content, just the radiation.

$$E_{radiation} \approx \frac{4}{3}\pi R^3 aT^4 \propto R^3T^4 \propto \frac{R^3M^4}{R^4} = \frac{M^4}{R}$$

$$\tau_{diffusion} \approx \frac{R^2}{l_{mfp}\, c} \qquad l_{mfp} = \frac{1}{\kappa\rho} \qquad \kappa \text{ is the "opacity" in cm}^2 \text{ gm}^{-1}$$

Assume κ is a constant

$$M \approx \frac{4}{3}\pi R^3 \rho \Rightarrow \rho \approx \frac{3M}{4\pi R^3}$$

$$l_{mfp} \propto \frac{R^3}{M} \qquad \tau_{diffusion} \propto \frac{R^2 M}{R^3} = \frac{M}{R}$$

$$L \propto \frac{M^4}{R} \Big/ \frac{M}{R} = M^3$$

Other powers of M possible when κ is not a constant but varies with temperature and density