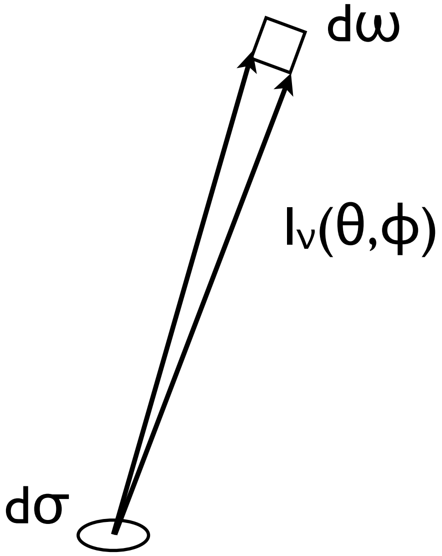
III. Radiative Processes

A. Intensity, flux, brightness temperature

- Specific Intensity I_ν
  - ◇ Consider a point P in a radiation field encompassing area $d\sigma$
  - ◇ Definition: $I_\nu d\nu d\omega d\sigma dt$ is the energy dE_ν flowing through $d\sigma$ in time dt in direction $\hat{n} = (\theta, \phi)$ with frequency within $d\nu$ of ν



  - ◇ Fig
  - ◇ Solid angle
    - ▲ We may consider $d\omega$ to be a fixed quantity
    - ▲ But the projection of $d\sigma$ scales as $\cos\theta$
  - ◇ Therefore

$$I_\nu(\theta,\phi) \equiv \frac{dE_\nu}{dt d\omega d\sigma \cos\theta d\nu} \tag{1}$$

- Specific Flux F_ν
  - ◇ Net energy flowing across a surface $d\sigma$ at $\vec{r}$ from *all directions* per unit time in frequency interval $d\nu$

$$F_\nu \equiv \frac{dE_\nu}{dt d\nu d\sigma} \tag{2}$$

$$F_\nu = \int_{4\pi} I_\nu \cos\theta d\omega \tag{3}$$

$$= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} I_\nu \cos\theta \; d\cos\theta \tag{4}$$

- ◇ Isotropic radiation field (I_ν independent of θ, ϕ)

$$F_\nu = I_\nu \int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} \cos\theta d\cos\theta = 0 \tag{5}$$

  - ▲ No net energy flow
  - ▲ Example: Extragalactic UV background radiation

- ◇ Axisymmetric radiation field: $I_\nu = I_\nu(\theta)$

$$F_\nu = 2\pi \int\limits_{-\pi/2}^{\pi/2} I_\nu \cos\theta d\cos\theta \tag{6}$$

- ◇ I_ν constant in upward direction ($\theta > 0$)

$$F_\nu = 2\pi I_\nu \int\limits_0^{\pi/2} \cos\theta d\cos\theta = \pi I_\nu \tag{7}$$

  - ▲ e.g. Blackbody

$$I_\nu = B_\nu(T) \equiv \frac{2h\nu^3}{c^2} \frac{1}{(e^{h\nu/kT} - 1)} \tag{8}$$

  - ▲ Flux at surface of a Blackbody

$$F_\nu = \pi B_\nu \qquad (\mathrm{ergs/cm^2/s/Hz}) \tag{9}$$

- • Mean Intensity J_ν

$$J_\nu = \int \frac{I_\nu d\omega}{4\pi} \tag{10}$$

- • Brightness temperature T_b
  - ◇ Definition: Temperature T_b where a given radiation field I_ν matches the intensity for a Black body $B_\nu(T)$

$$B_\nu(T_b) = I_\nu \tag{11}$$

  - ◇ If $h\nu/kT \ll 1$, e.g. radio frequencies

$$B_\nu(T_b) = \frac{2\nu^2 k T_b}{c^2} \tag{12}$$

B. Milne's Relation (Recombination cross-sections)

$$e^- + A^{x+1} \rightarrow A^x + \gamma \tag{13}$$

- Key concepts
  - ◇ Detailed balance: Number of transitions upward is precisely equal to the number downard in the reverse direction
  - ◇ Need to assume thermal equilibrium and a phase-space distribution for the electrons. But, the results turn out to be independent of these assumptions!
  - ◇ Need the photoionization cross-section from QM
  - ◇ ⇒ Derive $\sigma^{rec}(v)$ without QM!
- Energy balance

$$h\nu = I_n + \frac{1}{2}mv^2 \tag{14}$$

  - ◇ I_n is the Ionization Potential of the ion from level n
  - ◇ $\frac{1}{2}mv^2$ is the kinetic energy of the electron
- Recombinations: Number per dt per dV due to thermal electrons with speed $(v, v + dv)$

$$n_{x+1} n_e \sigma^{rec}(v) f(v) dv \tag{15}$$

  - ◇ n_{x+1} is the ion number density
  - ◇ $\sigma^{rec}(v)$ is the recombination cross-section
- Photoionizations: Number of ions photoionized per dt per dV per frequency $d\nu$

$$\frac{4\pi J_\nu}{h\nu c} n_x \sigma_\nu^{ph} \left[1 - \mathrm{e}^{-h\nu/kT} \right] d\nu \tag{16}$$

  - ◇ The 2nd term in [] is due to 'stimulated recombinations'
  - ◇ It can be derived from assuming the Einstein coefficients assuming thermal equilibrium is satisfied

$$n_m \left[A_{mn} + B_{mn} u \right] = n_n B_{nm} u \tag{17}$$

$$g_n B_{mn} = g_m B_{nm} \tag{18}$$

- Equating

$$n_{x+1} n_e \sigma^{rec}(v) f(v) dv = \frac{4\pi J_\nu}{h\nu c} n_x \sigma_\nu^{ph} \left[1 - \mathrm{e}^{-h\nu/kT} \right] d\nu \tag{19}$$

  - ◇ Note from equation 14

$$mv dv = h d\nu \tag{20}$$

  - ◇ And

$$J_\nu = B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{\mathrm{e}^{h\nu/kT} - 1} \tag{21}$$

  - ◇ We get

$$\frac{\sigma^{ph}}{\sigma^{rec}} = \frac{n_{x+1} n_e}{n_x} \mathrm{e}^{h\nu/kT} \frac{f(v) c^2 h}{8\pi m \nu^2} \tag{22}$$

- Saha's equation

$$\frac{n_{x+1}n_e}{n_x} = 2\frac{g_{x+1}}{g_x}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \mathrm{e}^{-I_n/kT} \tag{23}$$

- Massaging

$$\sigma_j^{rec}(v) = 2\left(\frac{h\nu}{mc}\right)^2 \frac{g_j}{g_e g_{j+1}}\frac{\sigma^{ph}}{v^2} \tag{24}$$

- Hydrogen
  - ◇ $g_e = 2; g_{j+1} = g_p = 1; g_j = 2n^2$
  - ◇ Finally

$$\sigma_n^{rec}(v) = 2\left(\frac{h\nu}{cm_e}\right)^2 \frac{n^2}{v^2}\sigma_n^{ph}(\nu) \tag{25}$$

C. Radiative Transfer

- Change of I_ν resulting from interaction with matter
- Volume emissivity (emission coefficient): j_ν
  - ◇ Define $j_\nu dV d\nu d\omega dt$ as the energy emitted by the volume element $dsd\sigma$ in the intervals $d\nu d\omega dt$
  - ◇ Example: Spontaneous emission of Hα photons due to recombination
- Opacity (absorption coefficient): κ_ν
  - ◇ Define $\kappa_\nu I_\nu dV d\nu d\omega dt$ as the energy absorber from a beam of specific intensity I_ν
  - ◇ Inversely proportional to the mean free path

$$\kappa_\nu = \frac{1}{\lambda_\nu} \tag{26}$$

  - ◇ Examples: dust, resonance absorption, photoionization, ...
- Equation of transfer

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \tag{27}$$

- Optical depth: τ_ν

$$d\tau_\nu = -\kappa_\nu ds \tag{28}$$

  - ◇ κ_ν = # of mean free paths traveled by a photon in ds
  - ◇ At the observer, $\tau_\nu = 0$
  - ◇ If $\kappa_\nu > 0$, τ_ν increases toward the source
- Equation of transfer (Part II)

$$\frac{dI_\nu}{d\tau_\nu} = -S_\nu + I_\nu \tag{29}$$

- ◇ Source function

$$S_\nu \equiv \frac{j_\nu}{\kappa_\nu} \tag{30}$$

- ◇ Consider a cloud of total optical thickness $\tau_{\nu r}$

$$I_\nu = I_\nu(0)\mathrm{e}^{-\tau_{\nu r}} + \int\limits_0^{\tau_{\nu r}} \frac{j_\nu}{\kappa_\nu}\mathrm{e}^{-\tau_\nu}\, d\tau_\nu \tag{31}$$

- Kirchoff's Law
  - ◇ In strict thermal equilibrium, I_ν is a constant and $dI_\nu/ds = 0$
  - ◇ Furthermore, $I_\nu = B_\nu(T)$
  - ◇ Therefore,

$$j_\nu = \kappa_\nu\, B_\nu(T) \tag{32}$$

D. Approximate Recombination Coefficients

- Definition

$$\alpha_n(T) = < \sigma_n(v)v >_{max} = \int\limits_0^\infty \sigma_n(v)vf(v)dv \tag{33}$$

- Milne's relation

$$\sigma_n(v) = AZ^4\frac{\nu_0}{\nu}\frac{h\nu_0}{\frac{1}{2}mv^2}\frac{g_n}{n^3} \tag{34}$$

$$A \equiv \frac{2^5}{3^{\frac{3}{2}}}\alpha^3\pi a_0^2 = 2.10 \times 10^{-22}\,\mathrm{cm}^{-2} \tag{35}$$

- $f(v)$
  - ◇ Maxwellian distribution

$$f(v) = \frac{4}{\sqrt{\pi}}L^{\frac{3}{2}}v^2\mathrm{e}^{-Lv^2} \tag{36}$$

  - ◇ Replace

$$L \equiv \frac{m}{2kT} \tag{37}$$

$$u \equiv Lv^2 \tag{38}$$

$$\beta_0 \equiv \frac{h\nu_0}{kT} \tag{39}$$

$$\beta \equiv Z^2\beta_0 \tag{40}$$

- Reducing

$$\alpha_n(T) = \frac{2A}{\sqrt{\pi L}}\frac{\beta^2}{n^3}\int\limits_0^\infty \frac{\mathrm{e}^{-u}du}{(\beta/n^2 + u)} \tag{41}$$

- ◇ Define

$$w \equiv u + \beta/n^2 \tag{42}$$

$$\alpha_n(\beta) = \frac{2A}{\sqrt{\pi L}} \frac{\beta^2}{n^3} \mathrm{e}^{\beta/n^2} E_1\left(\frac{\beta}{n^2}\right) \tag{43}$$

$$\alpha_n(\beta) = \frac{2A}{\sqrt{\pi L}} \beta \phi_n(\beta) \tag{44}$$

- ◇ E_1: First exponential integral

$$E_m(z) = z^{m-1} \int_z^\infty \frac{\mathrm{e}^{-t} dt}{t^m} \tag{45}$$

- ◇ If $z \gg 1$, express $E_m(z)$ as an asymptotic series

$$E_m(z) \approx \frac{\mathrm{e}^{-z}}{z}\left[1 - \frac{m}{z} + \frac{m(m+1)}{z^2}\right] \tag{46}$$

- • Back to Hydrogen
  - ◇ At $T \sim 10^4\,\mathrm{K}$, $Z = 1$

$$\frac{\beta}{n^2} = \frac{15.8}{n^2} \tag{47}$$

  - ◇ $\beta/n^2 \gg 1$ (i.e. $n = 1, 2, 3$)

$$\alpha_n(T) \approx \frac{2A}{\sqrt{\pi L}} \frac{\beta^2}{n^3} \mathrm{e}^{\beta/n^2} \frac{\mathrm{e}^{-\beta/n^2}}{\beta/n^2} \tag{48}$$

$$= \frac{2A}{\sqrt{\pi L}} \frac{\beta}{n} \tag{49}$$

$$\sim \frac{1}{nT^{\frac{1}{2}}} \tag{50}$$

    - ▲ Recombinations to low n are preferred
  - ◇ For $n \gg 1$, $\frac{1}{2}mv^2 \approx h\nu$

$$\sigma_n(v) \propto \frac{1}{\nu} \frac{1}{v^2} \frac{1}{n^3} \tag{51}$$

$$\propto \frac{1}{v^4 n^3} \tag{52}$$

$$\propto \frac{1}{n^3 T^{\frac{3}{2}}} \tag{53}$$

  - ◇ Altogether:

$$\alpha_n(T) \propto T^{-m} \qquad \text{with} \quad \frac{1}{2} \leq m \leq \frac{3}{2} \tag{54}$$

- Case A and B coefficients
    - ◇ Approximate expression

$$\alpha_A(T) = \frac{2A}{\sqrt{\pi}L}\beta \sum_{n=1}^{\infty} \phi_n(\beta) \tag{55}$$

    - ◇ Attack with Euler-Maclaurin Sum Rule!
    - ◇ Summarizing

$$\phi(\beta) \equiv \sum_{n=1}^{\infty} \phi_n(\beta) = \frac{1}{2}\left[1 + \gamma + \ln\beta + 0.1579 + \frac{1}{6\beta} + O\left(\frac{1}{\beta^2}\right)\right] \tag{56}$$

$$\approx \frac{1}{2}\left[1.735 + \ln\beta + \frac{1}{6\beta}\right] \tag{57}$$

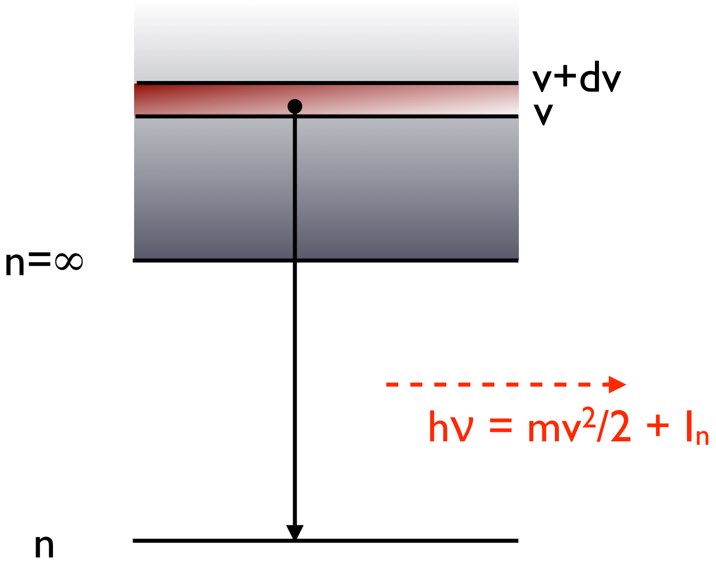
    - ◇ Therefore

$$\alpha_A(T) \approx \frac{2A}{\sqrt{\pi}L}\beta\frac{1}{2}\left[1.735 + \ln\beta + \frac{1}{6\beta}\right] \tag{58}$$

    - ◇ Similarly

$$\alpha_B(T) \approx \frac{2A}{\sqrt{\pi}L}\beta\left[\phi(\beta) - \phi_1(\beta)\right] \tag{59}$$

- Emissivity of a free-bound continuum (Hydrogen)
    - ◇ Fig



- ▲ $h\nu = \frac{1}{2}mv^2 + I_n$
- ▲ $I_n = Z^2 h\nu_0/n^2$
- ▲ $hd\nu = mvdv$

- Emissivity: $j_n(\nu)$

$$j_n(\nu)d\nu = \frac{1}{4\pi} n_p n_e 4\pi \frac{d^3p}{h^3} \sigma_n(v) v\, h\nu \tag{60}$$

- Expressing in terms of ν alone $\quad (\frac{1}{2}mv^2 = h\nu - I_n)$

$$j_n(\nu)d\nu = \frac{1}{4\pi} n_p n_e 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \mathrm{e}^{-(h\nu - I_n)/kT} \frac{h}{m} \left[\frac{2}{m}(h\nu - I_n)\right]^{\frac{1}{2}} \times$$
$$AZ^4 \frac{\nu_0}{\nu} \frac{h\nu_0}{(h\nu - I_n)} \frac{g_n}{n^3} \left[\frac{2}{m}(h\nu - I_n)\right]^{\frac{1}{2}} h\nu$$

- Altogether now

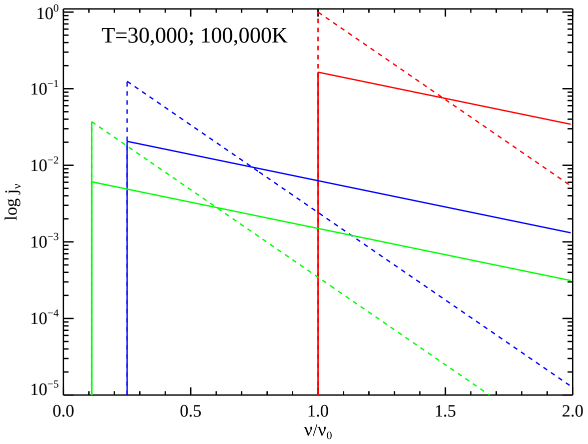
$$j_n(\nu) = \frac{2Ah(h\nu_0)^2}{m_e^2} \left(\frac{m_e}{2\pi k}\right)^{\frac{3}{2}} \frac{n_e n_p Z^4}{T^{\frac{3}{2}}} \frac{g_n}{n^3} \mathrm{e}^{h\nu_0 Z^2/n^2 kT} \mathrm{e}^{-h\nu/kT} \tag{61}$$

- For optically thin gas

$$j_\nu \propto \frac{n_e^2}{n^3 T^{\frac{3}{2}}} \mathrm{e}^{-(h\nu - Z^2 h\nu_0/n^2)/kT} \tag{62}$$

Table 1: Hydrogen Emissivity

n	Continuum	j_n	Edge	λ
1	Lyman	$j_1 \propto \mathrm{e}^{-h\nu/kT}/T^{\frac{3}{2}}$	$h\nu > h\nu_0$	912Å
2	Balmer	$j_2 \propto j_1/8$	$h\nu > h\nu_0/4$	3646Å
3	Paschen	$j_3 \propto j_1/27$	$h\nu > h\nu_0/9$	8203Å



- Temperature
  - Both the Balmer jump and the continuum slope depend on T

  - ◇ Therefore, one can use observations of these quantities to determine T observationally

- The ambitious ones can now proceed to derive expressions for α_A and α_B by summing $< \sigma_n(v)v >$

E. Saha Equation

- H atoms in thermal equilibrium
  - ◇ Generally, this implies collisions are important
  - ◇ Consider level populations
- Boltzman equation
  - ◇ Describes the density of atoms in the jth principal quantum level relative to level k

$$\frac{n_k}{n_j} = \frac{g_k}{g_j} \exp\left[-\frac{(E_k - E_j)}{kT} \right] \tag{63}$$

  - ◇ g_k, g_j are the statistical weights of levels j, k
  - ◇ For a hydrogen atom in quantum level n

$$g_n = 2\sum_{\ell=0}^{n-1}(2\ell+1) \ = 2n^2 \tag{64}$$

  - ◇ Consider the total density of neutral Hydrogen

$$n_{\rm HI} = \sum_1^\infty n_j \tag{65}$$

$$= \frac{n_1}{g_1}\sum_{j>1} g_j {\rm e}^{-E_j/kT} \tag{66}$$

    - ▲ We have taken $E_1 = 0$
    - ▲ Recognize the partition function

$$f = \sum_j g_j {\rm e}^{-E_j/kT} \tag{67}$$

  - ◇ Alternate Boltzman equation

$$n_j = \frac{n_{\rm HI}}{f} g_j {\rm e}^{-E_j/kT} \tag{68}$$

- Photoionization of element A between $x \to x+1$ stages of ionization

$$A^x \to A^{x+1} + {\rm e}^- \tag{69}$$

  - ◇ Electron has energy $\frac{1}{2}mv^2$

  - ◇ This process requires $\chi_x + \frac{1}{2}mv^2$
- Saha equation
  - ◇ In thermal equilibrium, the relative densities of the particles is the ratio of their partition functions

$$\frac{n_{x+1}n_e}{n_x} = \frac{f_{x+1}f_e}{f_x} \tag{70}$$

  - ◇ Partition function of a free per particle per unit volume

$$f_e = \int 2\frac{d^3p}{h^3}\mathrm{e}^{-\beta\varepsilon_i} \tag{71}$$

    - ▲ Non-relativistic

$$\varepsilon_i = \frac{1}{2}mv_i^2 \tag{72}$$

$$d^3p = 4\pi p^2 dp = 4\pi m^3 v^2 dv \tag{73}$$

  - ◇ Integrating

$$\frac{8\pi m^3}{h^3}\int v^2 \mathrm{e}^{-mv^2/2kT}dv \tag{74}$$

    - ▲ Let $x^2 = mv^2/2kT$
    - ▲ Evaluate

$$\left(\frac{2kT}{m}\right)^{3/2}\int\limits_0^\infty x^2\mathrm{e}^{-x^2}dx = \frac{\sqrt{\pi}}{4}\left(\frac{2kT}{m}\right)^{3/2} \tag{75}$$

    - ▲ Altogether

$$f_e = 2\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \tag{76}$$

$$= \frac{2}{\lambda_e^3} \tag{77}$$

  - ◇ Again,

$$f_{x+1} = \sum_i g_{x+1,i}\mathrm{e}^{-E_{x+1,i}/kT} \tag{78}$$

    - ▲ For low T (e.g. $kT \ll E_{x+1,i}$), the first term dominates

$$f_{x+1} \approx g_{x+1,1}\mathrm{e}^{-E_{x+1,1}/kT} \tag{79}$$

  - ◇ Alternate Saha Equation

$$\frac{n_{x+1}n_e}{n_x} = 2\frac{g_{x+1,1}}{g_{x,1}}\left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\mathrm{e}^{-\chi_x/kT} \tag{80}$$

- Saha's equation for Hydrogen

$$\frac{n_p n_e}{n_{\mathrm{HI}}} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2}\frac{2}{f_{HI}}\mathrm{e}^{-h\nu_0/kT} \tag{81}$$

- ◇ Replace f_{HI}
- ◇ Finally

$$n_x = \frac{1}{2} n_p n_e g_x \left(\frac{h^2}{2\pi m_e kT} \right)^{3/2} \mathrm{e}^{I_x/kT} \tag{82}$$

- ◇ $I_x \equiv h\nu_0 - E_x$

F. Rate Coefficients and Cross-sections for Collisional Excitations by Free Electrons

- Consider two levels in an ion i, j with $E_j > E_i$
- Define a collisional rate coefficient for excitation: q_{ij}

$$\frac{\#\text{ excitations}}{\text{cm}^3\text{s}} = n_e n_i q_{ij}(T) \equiv n_e n_i \int_{v_0}^{\infty} \sigma_{ij}(v_i) v_i f(v_i) dv_i \tag{83}$$

  - ◇ $\frac{1}{2} m v_0^2 = E_j - E_i$
  - ◇ σ_{ij}: Excitation cross-section
    (often units of $a_0 = 2.8010 \times 10^{-17}\,\text{cm}^2$
  - ◇ $f(v_i)$: Maxwellian

- Relate q_{ij} and q_{ji} via detailed balance in Thermal Equil.

$$n_e n_i q_{ij} = n_e n_j q_{ji} \tag{84}$$

  - ◇ In thermal equilibrium, we can use the Boltzman equation

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} \mathrm{e}^{-(E_j - E_i)/kT} \tag{85}$$

  - ◇ Therefore,

$$q_{ij} = q_{ji} \frac{g_j}{g_i} \mathrm{e}^{-(E_j - E_i)/kT} \tag{86}$$

  - ◇ This holds outside of TE provided the electrons have a Maxwellian distribution at temperature T

- Similarly, we can relate σ_{ij} and σ_{ji}
  - ◇ Energy conservation

$$\frac{1}{2} m v_i^2 + E_i = \frac{1}{2} m v_j^2 + E_j \tag{87}$$

  - ◇ Again, assume TE and invoke detailed balancing

$$n_e f(v_i) dv_i \cdot n_i v_i \sigma_{ij} = n_e f(v_j) dv_j \cdot n_j v_j \sigma_{ji} \tag{88}$$

  - ◇ And the Boltzman equation (Eq. 85)

  - ◇ Finally, we have (which holds outside TE!)

$$g_i v_i^2 \sigma_{ij}(v_i) = g_j v_j^2 \sigma_{ji}(v_j) \tag{89}$$

- Collision Strength: Define

$$\Omega_{ij}(v_i) = \left(\frac{m_e v_i}{\hbar}\right)^2 \frac{g_i}{\pi} \sigma_{ij}(v_i) \tag{90}$$

  - ◇ In most of the literature, Ω_{ij} values are reported instead of σ_{ij}
  - ◇ Consider a simple (semi-classical) relationship between Ω and σ
    - ▲ Suppose an electron with velocity v_i approaches an ion (at rest) with impact parameter R
    - ▲ Fig

    - ▲ Express the cross-section for $i \rightarrow j$ excitation by an electron with velocity v_i to be:

$$\sigma_{ij}(v_i) = \int_0^\infty P_{ij}(R, v_i) 2\pi R dR \tag{91}$$

      - ○ P_{ij} is the probability for $i \rightarrow j$
      - ○ $P_{ij} \leq 1$
    - ▲ The angular momentum of the electron is
      - ○ $L = mv_i R$
      - ○ Quantize: $L = \hbar\sqrt{\ell(\ell+1)}, \qquad \ell = 0, 1, 2, \ldots$
    - ▲ Define

$$R_\ell = \frac{L}{mv_i} = \frac{\hbar\sqrt{\ell(\ell+1)}}{mv_i} \tag{92}$$

$$\equiv \frac{\sqrt{\ell(\ell+1)}}{k_i} \tag{93}$$

    - ▲ De Broglie wave number

$$k_i \equiv \frac{mv_i}{\hbar} \qquad (\mathrm{cm}^{-1}) \tag{94}$$

- ▲ Our integral (Equation 91) becomes a sum

$$\sigma_{ij}(v_i) = \frac{\pi}{k_i^2}\sum_{\ell=0}^{\infty}(2\ell+1)P_{ij}(R_\ell, v_i) \tag{95}$$

$$\equiv \frac{\pi}{k_i^2}\sum_{\ell=0}^{\infty}\Omega_{ij}(\ell) \tag{96}$$

$$\equiv \frac{\pi}{k_i^2}\Omega_{ij}(v_i) \tag{97}$$

- ▲ If the ion has a degeneracy g_i, then we simply average over the initial states

$$\sigma_{ij} = \frac{\pi}{k_i^2}\frac{\Omega_{ij}}{g_i} \tag{98}$$

- ▲ We note that

$$\Omega_{ij}(v_i) \propto v_i^2 g_i \sigma_{ij} = v_j^2 g_j \sigma_{ji} \tag{99}$$

- ▲ Therefore

$$\Omega_{ij}(v_i) = \Omega_{ji}(v_j) \tag{100}$$

  - ○ This is convient, because recall that $\sigma \propto 1/v^2$
  - ○ Therefore, Ω is a weak function of v

$$\Omega_{ij} \approx \Omega_{ji} \equiv \Omega \tag{101}$$

- ◇ Typically

$\Omega \sim 10$ for dipole permitted lines
$\Omega \sim 1$ for forbidden optical lines
$\Omega \sim .1$ for forbidden IR lines

- ◇ Note, Ω values are generally known to only $\sim 10-15\%$ and are a significant source of systematic error

- Relating q_{ij} to Ω_{ij}
  - ◇ Consider the de-excitation rate coefficient

$$q_{ji} = \int_0^\infty \sigma_{ji}(v_j) v_j f(v_j) dv_j \tag{102}$$

  - ◇ Express

$$\sigma_{ji} = \frac{\pi}{k_j^2}\frac{\Omega_{ji}}{g_j} = \frac{\pi\hbar^2}{m_e v_j^2}\frac{\Omega_{ji}}{g_j} \tag{103}$$

- ◇ Altogether (with $L \equiv m/2kT$)

$$q_{ji} = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT} \right)^{\frac{3}{2}} \frac{\pi \hbar^2}{m^2 g_j} \int_0^\infty \frac{\Omega_{ij}}{v_j^2} v_j v_j^2 \mathrm{e}^{-L v_j^2} dv_j \tag{104}$$

- ◇ Substitute $\varepsilon/kT = L v_j^2$

$$q_{ji} = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT} \right)^{\frac{3}{2}} \frac{\pi \hbar^2}{m^2 g_j} \frac{kT}{m} \int_0^\infty \Omega_{ji} \mathrm{e}^{-\varepsilon/kT} d\left(\frac{\varepsilon}{kT} \right) \tag{105}$$

$$= \frac{C}{g_j T^{\frac{1}{2}}} \overline{\Omega_{ji}} \tag{106}$$

$$C = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k} \right)^{\frac{3}{2}} \frac{k \pi \hbar^2}{m^3} = 8.63 \times 10^{-6} \quad \mathrm{cm^3\,K^{\frac{1}{2}}\,s^{-1}} \tag{107}$$

- ◇ Consider $\overline{\Omega_{ji}}$
  - ▲ It is a slow function of T, ε
  - ▲ Therefore,

$$\overline{\Omega_{ji}} \approx \Omega_{ji} \tag{108}$$

- ◇ Detailed balance implies $\Omega_{ij} = \Omega_{ji}$
- ◇ Applying the Boltzman equation and substituting

$$q_{ij} = \frac{C \Omega_{ij}}{g_i T^{\frac{1}{2}}} \mathrm{e}^{-(E_j - E_i)/kT} \tag{109}$$

- Critical Density :: Density required to establish (roughly) a Boltzman population
  - ◇ Consider a 2-level ion (1,2)
  - ◇ Total collision de-excitation rate

$$n_e n_2 q_{21} = n_e n_2 \int_0^\infty v \sigma_{21} f(v) dv \tag{110}$$

  - ◇ Boltzman tells us

$$q_{21} = \frac{g_2}{g_1} q_{12} \mathrm{e}^{-(E_2 - E_1)/kT} \tag{111}$$

  - ◇ Detailed balance of excitation and de-excitation

$$n_e n_1 q_{12} = n_e n_2 q_{21} + n_2 A_{21} \tag{112}$$

  - ◇ The emission in the line is

$$\varepsilon_{21} = n_2 A_{21} h\nu_{21} \tag{113}$$

$$= n_e n_1 q_{12} h\nu_{21} \left[\frac{A_{21}}{A_{21} + n_e q_{21}} \right] \tag{114}$$

$$= n_e n_1 q_{12} h\nu_{21} \left[\frac{1}{1 + \frac{n_e q_{21}}{A_{21}}} \right] \tag{115}$$

- ◇ As $n_e \to 0$,

$$\varepsilon_{21} \to n_e n_1 q_{12} h\nu_{21} \propto n_e^2 \tag{116}$$

  - ▲ Every collisional excitation produces a line photon
  - ▲ i.e. there is no collisional de-excitation
- ◇ As $n_e \to \infty$,

$$\begin{aligned}\varepsilon_{21} &\to n_1 h\nu_{21} A_{21} \frac{q_{12}}{q_{21}} && (117)\\ &= n_1 h\nu_{12} A_{21} \cdot \frac{g_2}{g_1} \mathrm{e}^{-h\nu_{12}/kT} && (118)\\ &= (n_2)_{TE} h\nu_{12} A_{21} && (119)\\ &\propto n_e && (120)\end{aligned}$$

  - ▲ This is the Thermal equilibrium cooling rate
- ◇ Define: Critical density – Transition from low density to T.E.

$$(n_e)_C = \frac{A_{21}}{q_{21}} \tag{121}$$

  - ▲ Example point: Hβ emission is always $\propto n_e^2$
  - ▲ Therefore, when $n_e > (n_e)_C$, the forbidden lines weaken relative to Hβ

G. Atomic Notation

- Electrostatic Interactions: Russel-Saunders Coupling
  - ◇ Multiple electrons only
  - ◇ Recall our central field Hamiltonian for the hydrogenic ions

$$H = \sum_j \frac{p_j^2}{2m} - \frac{Ze^2}{r}$$

    - ▲ Degenerate in ℓ, m, m_s
    - ▲ Additional terms for multiple electrons break some of this degeneracy
  - ◇ Express the multi-electron Hamiltonian as

$$H = \sum_j \frac{p_j^2}{2m} - Ze^2 \sum_j \frac{1}{r_j} + \sum_i V(r_i) + H_1 \equiv H_0 + H_1 \tag{122}$$

    - ▲ Identify $\sum V(r_i)$ as the central field potential due to the 'smeared out' electrons
    - ▲ H_1 is a perturbation to our spherically symmetric Hamiltonian (H_0)
  - ◇ Explicitly

$$H_1 = e^2 \sum_{i>j} \frac{1}{r_{ij}} - \sum_i V(r_i) + H_{SO} + \dots \tag{123}$$

  - ▲ Electrostatic Interaction: H_{es}

$$H_{es} = e^2 \sum_{i>j} \frac{1}{r_{ij}} - \sum_i V(r_i) \tag{124}$$

  - ▲ H_{SO} = Spin-orbit term
  - ▲ $\ldots$ = Higher order terms
- ◇ Note ℓ_i is not a constant of the motion: $[H_{es}, \ell_i] \neq 0$
- ◇ But, H_{es} does commute with the total angular momentum and spin

$$[H_{es}, \vec{L}] = 0 \qquad [H_{es}, \vec{S}] = 0$$

  - ▲ $\vec{L} = \sum_i \vec{\ell}_i; \quad \vec{S} = \sum_i \vec{s}_i$
  - ▲ So, the total angular momentum and spin are constants of the motion
- ◇ Qualitatively assessing the H_{es} perturbation
  - ▲ Electrons repel each other
  - ▲ $\Rightarrow$ Lower energy if they are farther apart

- Hund's Rules
  - (a) Large Spin $\Rightarrow$ Lower Energy
    - ◇ Large spin $\Rightarrow$ individual $\vec{s}_i$ are aligned $\Rightarrow$ $s_1 = s_2 = s_3 = \cdots = \pm\frac{1}{2}$
    - ◇ Pauli's exclusion $\Rightarrow$ electrons with same spin will be farther apart
    - ◇ This is tyipcally the dominant RS Coupling effect
  - (b) For a given spin, larger angular momentum $\Rightarrow$ Lower Energy
    - ◇ Large L $\Rightarrow$ individual $\vec{l}_i$ are aligned
    - ◇ Pauli $\Rightarrow$ electrons with similar orbits will be farther apart
  - (c) Higher J $\Rightarrow$ Higher energy for a level less than half filled

- Equivalent and Non-equivalent Electrons
  - ◇ Closed shells
    - ▲ Spherically symmetric

$$\sum_m \left|Y_{\ell m}\right|^2 = \frac{2\ell + 1}{4\pi} \tag{125}$$

    - ▲ These electrons interact minimally with outer electrons
    - ▲ Ignore them altogether
  - ◇ Definition
    - ▲ Equivalent electrons have the same n, ℓ
    - ▲ Non-Equivalent electrons have different n, ℓ

- ◇ Spectroscopic labeling

$$^{(2S+1)}L_J$$

  - ▲ L: Total orbital angular momentum
  - ▲ S: Total spin
  - ▲ J: Total angular momentum
- ◇ Example 1: Two ‘s’ electrons
  - (a) Non-equivalent, e.g. 1s2s

$$\vec{L} = \sum_i \vec{\ell_i} = 0$$
$$\vec{S} = \sum_i \vec{s_i} = 0, 1$$

    - ▲ Two terms allowed
    - ▲ ^{1}S (singlet) and ^{3}S (triplet)
    - ▲ By Hund’s rule, ^{3}S should have lower energy
  - (b) Equivalent (e.g. 2s^2)
    - ▲ $\vec{L} = 0$ again
    - ▲ Pauli $\Rightarrow$ $\vec{S} = 0$ only
- ◇ Example 2: Two ‘p’ electrons
  - (a) Non-equivalent

$$\vec{L} = \sum_i \vec{\ell_i} = 0, 1, 2$$
$$\vec{S} = \sum_i \vec{s_i} = 0, 1$$

    - ▲ States:

Term	^{1}S	^{1}P	^{1}D	^{3}S	^{3}P	^{3}D
Number	1	3	5	3	9	15

    - ▲ Each electron had 2 spin x $(2\ \ell + 1) = 6$ states
    - ▲ Total had to be $6 \times 6 = 36$
  - (b) Equivalent
    - ▲ Some states are excluded by Pauli
    - ▲ Some states are indistinguishable
      - ○ $\Rightarrow$ Less than 36 states total (15)
      - ○ Some are easily identified (named), others are not

m_1	m_2	m_{s_1}	m_{s_2}	Label	State
+1	+1	+	+	^{3}D	Pauli
+1	+1	+	-	^{1}D	1
+1	+1	-	+	^{1}D	1
+1	+1	-	-	^{3}D	Pauli
+1	0	+	+	^{3}P	2
+1	0	+	-		3
+1	0	-	+		4
+1	0	-	-	^{3}P	5
+1	−1	+	+		6
+1	−1	+	-		7
+1	−1	-	+		8
+1	−1	-	-		9
0	+1	+	+		2
0	+1	+	-		3
0	+1	-	+		4
0	+1	-	-		5
0	0	+	+	^{3}S	Pauli
0	0	+	-	^{1}S	10
0	0	-	+		10
0	0	-	-	^{3}S	Pauli
0	−1	+	+	^{3}P	11
0	−1	+	-		12
0	−1	-	+		13
0	−1	-	-	^{3}P	14
−1	+1	+	+		6
−1	+1	+	-		7
−1	+1	-	+		8
−1	+1	-	-		9
−1	−1	+	+	^{3}D	Pauli
−1	−1	+	-	^{1}D	15
−1	−1	-	+	^{1}D	15
−1	−1	-	-	^{3}D	Pauli

- ◇ Useful shortcut
  - ▲ Holes in a closed shell give same results as corresponding electrons
  - ▲ e.g. Terms corresponding to p^5 are the same as p^1

H. Einstein Constants

- Assume a set two levels (j, k) in an atomic system
    - ◇ k is the upper level
    - ◇ Energy is $h\nu_{jk}$
- Absorption-line profile:
    - ◇ Let $\phi_\nu d\nu =$ the probability that an absorber in state j will absorb radiation with frequency in the interval $\nu, \nu + d\nu$.
    - ◇ Normalization

$$\int_0^\infty \phi_\nu \, d\nu = 1 \tag{126}$$

    - ◇ Clearly, ϕ_ν has units of Hz^{-1}
- Transfer Equation

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \tag{127}$$

    - ◇ Express the opacity term as

$$\kappa_\nu I_\nu \equiv n_j \phi_\nu h\nu_{jk} I_\nu \frac{B_{jk}}{4\pi} \tag{128}$$

        - ▲ This has defined the first Einstein constant, B_{jk} which describes the transition strength independent of ν
        - ▲ But, this expression has ignored stimulated emission!
    - ◇ Consider the emissivity for spontaneous emission

$$j_\nu^{(s)} = n_k \psi_\nu^s \frac{A_{kj}}{4\pi} h\nu_{jk} \tag{129}$$

        - ▲ ψ_ν^s is the fraction of atoms/ions in the upper state that can emit into $\nu, \nu + d\nu$
        - ▲ Similar to ϕ_ν, ψ_ν^s is normalized by

$$\int_0^\infty \psi_\nu^s d\nu = 1 \tag{130}$$

    - ◇ Einstein also realized that there was emissivity due to induced (stimulated) emission

$$j_\nu^{(i)} = n_k \psi_\nu^i \frac{B_{kj}}{4\pi} h\nu_{jk} \tag{131}$$

        - ▲ This is a 2nd B value, and $B_{jk} \neq B_{kj}$
        - ▲ Dirac showed that

$$\psi_\nu^s = \psi_\nu^i = \psi_\nu \tag{132}$$

- ◇ Altogether now,

$$\frac{dI_\nu}{ds} = -n_j \phi_\nu h\nu_{jk} I_\nu \frac{B_{jk}}{4\pi} + \frac{n_k}{4\pi}\psi_\nu \left[A_{kj} + B_{kj} I_\nu \right] h\nu_{jk} \tag{133}$$

  - ▲ In general, $\phi_\nu \neq \psi_\nu$
  - ▲ As such, it is often incorrect to regard stimulated emission as a negative absorption
- ◇ Consider the transfer equation under Thermal Equilibrium
  - ▲ $dI_\nu/ds = 0$
  - ▲ $I_\nu = B_\nu$

$$\begin{aligned} I_\nu &= \frac{A_{kj}}{B_{kj}} \frac{1}{\left(\frac{n_j B_{jk} \phi_\nu}{n_k B_{kj} \psi_\nu} - 1 \right)} && (134) \\ &= B_\nu && (135) \\ &= \frac{2h\nu^3}{c^2} \frac{1}{\mathrm{e}^{h\nu/kT} - 1} && (136) \end{aligned}$$

  - ▲ Consider two limits
    - (a) $T \to \infty$
      - ○ Then we have

$$\frac{g_j}{g_k} \frac{B_{jk}}{B_{kj}} \frac{\phi_\nu}{\psi_\nu} \, \mathrm{e}^{h\nu_{jk}/kT} - 1 \;=\; \mathrm{e}^{h\nu/kT} - 1 \tag{137}$$

      - ○ Implying

$$g_j B_{jk} \phi_\nu = g_k B_{kj} \phi_\nu \tag{138}$$

      - ○ Integrating over all frequencies, we have the First Einstein relation

$$\frac{B_{jk}}{B_{kj}} = \frac{g_k}{g_j} \tag{139}$$

    - (b) $T \to 0$: Gives Einstein's second relation

$$\frac{A_{kj}}{B_{kj}} = \frac{2h\nu^3}{c^2} \tag{140}$$

  - ▲ Although we dereived these under the assumption of Thermal Equilibrium, they hold in all physical conditions
  - ▲ Maintaining the assumption of T.E., where $B_\nu = j_\nu/\kappa_\nu$ we can show

$$\kappa_\nu = \frac{h\nu_{jk}}{4\pi} \left(n_j B_{jk} - n_k B_{kj} \right) \psi_\nu \quad \{ \text{T.E.} \} \tag{141}$$

- ◇ Outside of T.E., our transfer equation is now

$$\frac{dI_\nu}{ds} = \frac{h\nu_{jk}}{4\pi} \left[n_k A_{kj} \psi_\nu - \left(n_j B_{jk} \phi_\nu - n_k B_{kj} \psi_\nu \right) I_\nu \right] \tag{142}$$

- ▲ We can therefore identify the opacity κ_ν as:

$$\begin{aligned}\kappa_\nu &= \frac{h\nu_{jk}}{4\pi}\left(\, n_j B_{jk}\phi_\nu - n_k B_{kj}\psi_\nu \,\right) && (143)\\ &= \frac{n_j B_{jk} h\nu_{jk}}{4\pi}\left(1 - \frac{n_k B_{kj}\psi_\nu}{n_j B_{jk}\phi_\nu}\right) && (144)\\ &= \frac{n_j B_{jk} h\nu_{jk}}{4\pi}\left(1 - \frac{b_k\psi_\nu}{b_j\phi_\nu}\mathrm{e}^{-h\nu_{jk}/kT}\right) && (145)\end{aligned}$$

- ▲ Where b_j is the departure coefficient $n_j = b_j n_j^{(\mathrm{T.E.})}$
- ▲ The stimulated term is important when $b_k \approx b_j$ and $\exp(-h\nu_{jk}/kT) \approx 1$ or $h\nu_{jk} \ll kT$ (e.g. 21cm radiation)
- ▲ In most astrophysical cases, the stimulated term is small and

$$\begin{aligned}\kappa_\nu &\approx n_j B_{jk}\phi_\nu \frac{h\nu_{jk}}{4\pi} && (146)\\ &\equiv n_j s_\nu && (147)\\ &= n_j s_{jk}\phi_\nu && (148)\end{aligned}$$

- ▲ with

$$s_{jk} = \frac{h\nu_{jk}}{4\pi} B_{jk} \ \ (\mathrm{cm^2\,Hz}) \qquad (149)$$