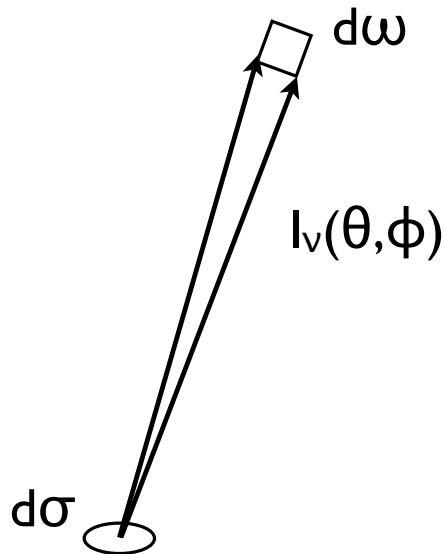


III. Radiative Processes

A. Intensity, flux, brightness temperature

- Specific Intensity I_ν
 - ◇ Consider a point P in a radiation field encompassing area $d\sigma$
 - ◇ Definition: $I_\nu d\nu d\omega d\sigma dt$ is the energy dE_ν flowing through $d\sigma$ in time dt in direction $\hat{n} = (\theta, \phi)$ with frequency within $d\nu$ of ν



- ◇ Fig
- ◇ Solid angle
 - ▲ We may consider $d\omega$ to be a fixed quantity
 - ▲ But the projection of $d\sigma$ scales as $\cos\theta$
- ◇ Therefore

$$I_\nu(\theta, \phi) \equiv \frac{dE_\nu}{dt d\omega d\sigma \cos\theta d\nu} \quad (1)$$

- Specific Flux F_ν
 - ◇ Net energy flowing across a surface $d\sigma$ at \vec{r} from *all directions* per unit time in frequency interval $d\nu$

$$F_\nu \equiv \frac{dE_\nu}{dt d\nu d\sigma} \quad (2)$$

$$F_\nu = \int_{4\pi} I_\nu \cos\theta d\omega \quad (3)$$

$$= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} I_\nu \cos\theta d\cos\theta \quad (4)$$

- ◇ Isotropic radiation field (I_ν independent of θ, ϕ)

$$F_\nu = I_\nu \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cos \theta d\cos \theta = 0 \quad (5)$$

- ▲ No net energy flow

- ▲ Example: Extragalactic UV background radiation

- ◇ Axisymmetric radiation field: $I_\nu = I_\nu(\theta)$

$$F_\nu = 2\pi \int_{-\pi/2}^{\pi/2} I_\nu \cos \theta d\cos \theta \quad (6)$$

- ◇ I_ν constant in upward direction ($\theta > 0$)

$$F_\nu = 2\pi I_\nu \int_0^{\pi/2} \cos \theta d\cos \theta = \pi I_\nu \quad (7)$$

- ▲ e.g. Blackbody

$$I_\nu = B_\nu(T) \equiv \frac{2h\nu^3}{c^2} \frac{1}{(e^{h\nu/kT} - 1)} \quad (8)$$

- ▲ Flux at surface of a Blackbody

$$F_\nu = \pi B_\nu \quad (\text{ergs/cm}^2/\text{s/Hz}) \quad (9)$$

- Mean Intensity J_ν

$$J_\nu = \int \frac{I_\nu d\omega}{4\pi} \quad (10)$$

- Brightness temperature T_b

- ◇ Definition: Temperature T_b where a given radiation field I_ν matches the intensity for a Black body $B_\nu(T)$

$$B_\nu(T_b) = I_\nu \quad (11)$$

- ◇ If $h\nu/kT \ll 1$, e.g. radio frequencies

$$B_\nu(T_b) = \frac{2\nu^2 k T_b}{c^2} \quad (12)$$

B. Milne's Relation (Recombination cross-sections)



- Key concepts
 - ◊ Detailed balance: Number of transitions upward is precisely equal to the number downward in the reverse direction
 - ◊ Need to assume thermal equilibrium and a phase-space distribution for the electrons. But, the results turn out to be independent of these assumptions!
 - ◊ Need the photoionization cross-section from QM
 - ◊ \Rightarrow Derive $\sigma^{rec}(v)$ without QM!

- Energy balance

$$h\nu = I_n + \frac{1}{2}mv^2 \quad (14)$$

- ◊ I_n is the Ionization Potential of the ion from level n
- ◊ $\frac{1}{2}mv^2$ is the kinetic energy of the electron
- Recombinations: Number per dt per dV due to thermal electrons with speed $(v, v + dv)$

$$n_{x+1}n_e\sigma^{rec}(v)f(v)dv \quad (15)$$

- ◊ n_{x+1} is the ion number density
- ◊ $\sigma^{rec}(v)$ is the recombination cross-section
- Photoionizations: Number of ions photoionized per dt per dV per frequency $d\nu$

$$\frac{4\pi J_\nu}{h\nu c} n_x \sigma_\nu^{ph} [1 - e^{-h\nu/kT}] d\nu \quad (16)$$

- ◊ The 2nd term in [] is due to ‘stimulated recombinations’
- ◊ It can be derived from assuming the Einstein coefficients assuming thermal equilibrium is satisfied

$$n_m [A_{mn} + B_{mn}u] = n_n B_{nm}u \quad (17)$$

$$g_n B_{mn} = g_m B_{nm} \quad (18)$$

- Equating

$$n_{x+1}n_e\sigma^{rec}(v)f(v)dv = \frac{4\pi J_\nu}{h\nu c} n_x \sigma_\nu^{ph} [1 - e^{-h\nu/kT}] d\nu \quad (19)$$

- ◊ Note from equation 14

$$mv dv = h d\nu \quad (20)$$

- ◊ And

$$J_\nu = B_\nu = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \quad (21)$$

- ◊ We get

$$\frac{\sigma^{ph}}{\sigma^{rec}} = \frac{n_{x+1}n_e}{n_x} e^{h\nu/kT} \frac{f(v)c^2 h}{8\pi m\nu^2} \quad (22)$$

- Saha's equation

$$\frac{n_{x+1}n_e}{n_x} = 2\frac{g_{x+1}}{g_x} \left(\frac{2\pi m_e kT}{h^2} \right)^{3/2} e^{-I_n/kT} \quad (23)$$

- Massaging

$$\sigma_j^{rec}(v) = 2 \left(\frac{h\nu}{mc} \right)^2 \frac{g_j}{g_e g_{j+1}} \frac{\sigma^{ph}}{v^2} \quad (24)$$

- Hydrogen

$$\diamond g_e = 2; g_{j+1} = g_p = 1; g_j = 2n^2$$

◇ Finally

$$\sigma_n^{rec}(v) = 2 \left(\frac{h\nu}{cm_e} \right)^2 \frac{n^2}{v^2} \sigma_n^{ph}(\nu) \quad (25)$$

C. Radiative Transfer

- Change of I_ν resulting from interaction with matter
- Volume emissivity (emission coefficient): j_ν
 - ◇ Define $j_\nu dV d\nu d\omega dt$ as the energy emitted by the volume element $dV d\omega$ in the intervals $d\nu d\omega dt$
 - ◇ Example: Spontaneous emission of H α photons due to recombination
- Opacity (absorption coefficient): κ_ν
 - ◇ Define $\kappa_\nu I_\nu dV d\nu d\omega dt$ as the energy absorber from a beam of specific intensity I_ν
 - ◇ Inversely proportional to the mean free path

$$\kappa_\nu = \frac{1}{\lambda_\nu} \quad (26)$$

◇ Examples: dust, resonance absorption, photoionization, ...

- Equation of transfer

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \quad (27)$$

- Optical depth: τ_ν

$$d\tau_\nu = -\kappa_\nu ds \quad (28)$$

◇ $\kappa_\nu = \#$ of mean free paths traveled by a photon in ds

◇ At the observer, $\tau_\nu = 0$

◇ If $\kappa_\nu > 0$, τ_ν increases toward the source

- Equation of transfer (Part II)

$$\frac{dI_\nu}{d\tau_\nu} = -S_\nu + I_\nu \quad (29)$$

- ◇ Source function

$$S_\nu \equiv \frac{j_\nu}{\kappa_\nu} \quad (30)$$

- ◇ Consider a cloud of total optical thickness $\tau_{\nu r}$

$$I_\nu = I_\nu(0)e^{-\tau_{\nu r}} + \int_0^{\tau_{\nu r}} \frac{j_\nu}{\kappa_\nu} e^{-\tau_\nu} d\tau_\nu \quad (31)$$

- Kirchoff's Law

- ◇ In strict thermal equilibrium, I_ν is a constant and $dI_\nu/ds = 0$
- ◇ Furthermore, $I_\nu = B_\nu(T)$
- ◇ Therefore,

$$j_\nu = \kappa_\nu B_\nu(T) \quad (32)$$

D. Approximate Recombination Coefficients

- Definition

$$\alpha_n(T) = \langle \sigma_n(v)v \rangle_{max} = \int_0^\infty \sigma_n(v)v f(v) dv \quad (33)$$

- Milne's relation

$$\sigma_n(v) = AZ^4 \frac{\nu_0}{\nu} \frac{h\nu_0}{\frac{1}{2}mv^2} \frac{g_n}{n^3} \quad (34)$$

$$A \equiv \frac{2^5}{3^{\frac{3}{2}}} \alpha^3 \pi a_0^2 = 2.10 \times 10^{-22} \text{ cm}^{-2} \quad (35)$$

- $f(v)$

- ◇ Maxwellian distribution

$$f(v) = \frac{4}{\sqrt{\pi}} L^{\frac{3}{2}} v^2 e^{-Lv^2} \quad (36)$$

- ◇ Replace

$$L \equiv \frac{m}{2kT} \quad (37)$$

$$u \equiv Lv^2 \quad (38)$$

$$\beta_0 \equiv \frac{h\nu_0}{kT} \quad (39)$$

$$\beta \equiv Z^2 \beta_0 \quad (40)$$

- Reducing

$$\alpha_n(T) = \frac{2A}{\sqrt{\pi L}} \frac{\beta^2}{n^3} \int_0^\infty \frac{e^{-u} du}{(\beta/n^2 + u)} \quad (41)$$

◇ Define

$$w \equiv u + \beta/n^2 \quad (42)$$

$$\alpha_n(\beta) = \frac{2A}{\sqrt{\pi L}} \frac{\beta^2}{n^3} e^{\beta/n^2} E_1\left(\frac{\beta}{n^2}\right) \quad (43)$$

$$\alpha_n(\beta) = \frac{2A}{\sqrt{\pi L}} \beta \phi_n(\beta) \quad (44)$$

◇ E_1 : First exponential integral

$$E_m(z) = z^{m-1} \int_z^\infty \frac{e^{-t} dt}{t^m} \quad (45)$$

◇ If $z \gg 1$, express $E_m(z)$ as an asymptotic series

$$E_m(z) \approx \frac{e^{-z}}{z} \left[1 - \frac{m}{z} + \frac{m(m+1)}{z^2} \right] \quad (46)$$

• Back to Hydrogen

◇ At $T \sim 10^4$ K, $Z = 1$

$$\frac{\beta}{n^2} = \frac{15.8}{n^2} \quad (47)$$

◇ $\beta/n^2 \gg 1$ (i.e. $n = 1, 2, 3$)

$$\alpha_n(T) \approx \frac{2A}{\sqrt{\pi L}} \frac{\beta^2}{n^3} e^{\beta/n^2} \frac{e^{-\beta/n^2}}{\beta/n^2} \quad (48)$$

$$= \frac{2A}{\sqrt{\pi L}} \frac{\beta}{n} \quad (49)$$

$$\sim \frac{1}{nT^{\frac{1}{2}}} \quad (50)$$

▲ Recombinations to low n are preferred

◇ For $n \gg 1$, $\frac{1}{2}mv^2 \approx h\nu$

$$\sigma_n(v) \propto \frac{1}{\nu} \frac{1}{v^2} \frac{1}{n^3} \quad (51)$$

$$\propto \frac{1}{v^4 n^3} \quad (52)$$

$$\propto \frac{1}{n^3 T^{\frac{3}{2}}} \quad (53)$$

◇ Altogether:

$$\alpha_n(T) \propto T^{-m} \quad \text{with} \quad \frac{1}{2} \leq m \leq \frac{3}{2} \quad (54)$$

- Case A and B coefficients
 - ◊ Approximate expression

$$\alpha_A(T) = \frac{2A}{\sqrt{\pi L}} \beta \sum_{n=1}^{\infty} \phi_n(\beta) \quad (55)$$

- ◊ Attack with Euler-Maclaurin Sum Rule!
- ◊ Summarizing

$$\phi(\beta) \equiv \sum_{n=1}^{\infty} \phi_n(\beta) = \frac{1}{2} \left[1 + \gamma + \ln \beta + 0.1579 + \frac{1}{6\beta} + O\left(\frac{1}{\beta^2}\right) \right] \quad (56)$$

$$\approx \frac{1}{2} \left[1.735 + \ln \beta + \frac{1}{6\beta} \right] \quad (57)$$

- ◊ Therefore

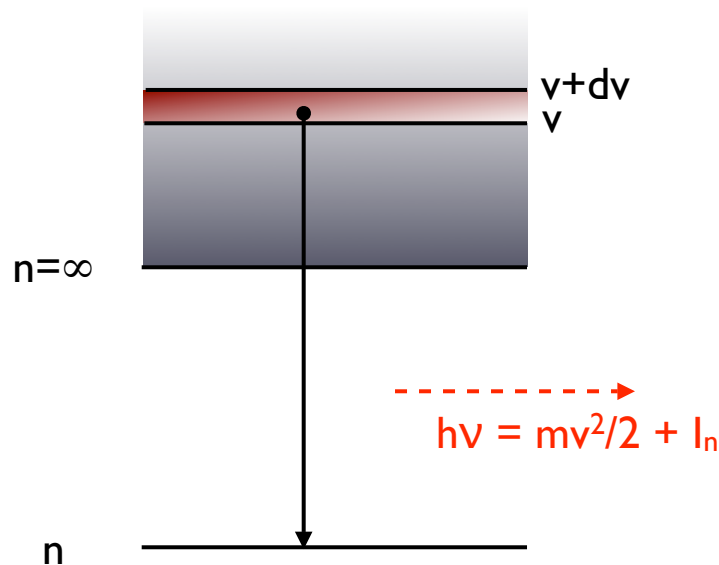
$$\alpha_A(T) \approx \frac{2A}{\sqrt{\pi L}} \beta \frac{1}{2} \left[1.735 + \ln \beta + \frac{1}{6\beta} \right] \quad (58)$$

- ◊ Similarly

$$\alpha_B(T) \approx \frac{2A}{\sqrt{\pi L}} \beta [\phi(\beta) - \phi_1(\beta)] \quad (59)$$

- Emissivity of a free-bound continuum (Hydrogen)

- ◊ Fig



- ▲ $h\nu = \frac{1}{2}mv^2 + I_n$
- ▲ $I_n = Z^2 h\nu_0/n^2$
- ▲ $hd\nu = mv dv$

◇ Emissivity: $j_n(\nu)$

$$j_n(\nu)d\nu = \frac{1}{4\pi}n_p n_e 4\pi \frac{d^3p}{h^3} \sigma_n(v) v h\nu \quad (60)$$

◇ Expressing in terms of ν alone ($\frac{1}{2}mv^2 = h\nu - I_n$)

$$j_n(\nu)d\nu = \frac{1}{4\pi}n_p n_e 4\pi \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} e^{-(h\nu - I_n)/kT} \frac{h}{m} \left[\frac{2}{m}(h\nu - I_n)\right]^{\frac{1}{2}} \times \\ AZ^4 \frac{\nu_0}{\nu} \frac{h\nu_0}{(h\nu - I_n)} \frac{g_n}{n^3} \left[\frac{2}{m}(h\nu - I_n)\right]^{\frac{1}{2}} h\nu$$

◇ Altogether now

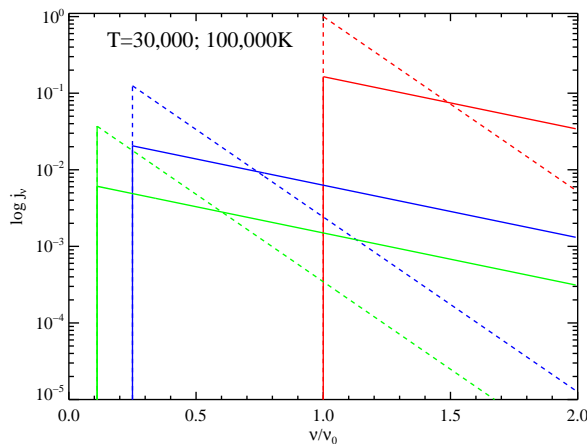
$$j_n(\nu) = \frac{2Ah(h\nu_0)^2}{m_e^2} \left(\frac{m_e}{2\pi k}\right)^{\frac{3}{2}} \frac{n_e n_p Z^4}{T^{\frac{3}{2}}} \frac{g_n}{n^3} e^{h\nu_0 Z^2/n^2 kT} e^{-h\nu/kT} \quad (61)$$

◇ For optically thin gas

$$j_\nu \propto \frac{n_e^2}{n^3 T^{\frac{3}{2}}} e^{-(h\nu - Z^2 h\nu_0/n^2)/kT} \quad (62)$$

Table 1: HYDROGEN EMISSIVITY

n	Continuum	j_n	Edge	λ
1	Lyman	$j_1 \propto e^{-h\nu/kT}/T^{\frac{3}{2}}$	$h\nu > h\nu_0$	912Å
2	Balmer	$j_2 \propto j_1/8$	$h\nu > h\nu_0/4$	3646Å
3	Paschen	$j_3 \propto j_1/27$	$h\nu > h\nu_0/9$	8203Å



- Temperature

◇ Both the Balmer jump and the continuum slope depend on T

- ◇ Therefore, one can use observations of these quantities to determine T observationally
- The ambitious ones can now proceed to derive expressions for α_A and α_B by summing $\langle \sigma_n(v)v \rangle$

E. Saha Equation

- H atoms in thermal equilibrium
 - ◇ Generally, this implies collisions are important
 - ◇ Consider level populations
- Boltzman equation
 - ◇ Describes the density of atoms in the j th principal quantum level relative to level k

$$\frac{n_k}{n_j} = \frac{g_k}{g_j} \exp \left[-\frac{(E_k - E_j)}{kT} \right] \quad (63)$$

- ◇ g_k, g_j are the statistical weights of levels j, k
- ◇ For a hydrogen atom in quantum level n

$$g_n = 2 \sum_{\ell=0}^{n-1} (2\ell + 1) = 2n^2 \quad (64)$$

- ◇ Consider the total density of neutral Hydrogen

$$n_{\text{HI}} = \sum_1^{\infty} n_j \quad (65)$$

$$= \frac{n_1}{g_1} \sum_{j>1} g_j e^{-E_j/kT} \quad (66)$$

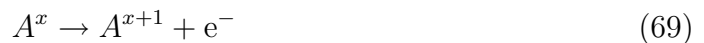
- ▲ We have taken $E_1 = 0$
- ▲ Recognize the partition function

$$f = \sum_j g_j e^{-E_j/kT} \quad (67)$$

- ◇ Alternate Boltzman equation

$$n_j = \frac{n_{\text{HI}}}{f} g_j e^{-E_j/kT} \quad (68)$$

- Photoionization of element A between $x \rightarrow x + 1$ stages of ionization



- ◇ Electron has energy $\frac{1}{2}mv^2$

- ◇ This process requires $\chi_x + \frac{1}{2}mv^2$
- Saha equation
 - ◇ In thermal equilibrium, the relative densities of the particles is the ratio of their partition functions

$$\frac{n_{x+1}n_e}{n_x} = \frac{f_{x+1}f_e}{f_x} \quad (70)$$

- ◇ Partition function of a free per particle per unit volume

$$f_e = \int 2 \frac{d^3p}{h^3} e^{-\beta\varepsilon_i} \quad (71)$$

- ▲ Non-relativistic

$$\varepsilon_i = \frac{1}{2}mv_i^2 \quad (72)$$

$$d^3p = 4\pi p^2 dp = 4\pi m^3 v^2 dv \quad (73)$$

- ◇ Integrating

$$\frac{8\pi m^3}{h^3} \int v^2 e^{-mv^2/2kT} dv \quad (74)$$

- ▲ Let $x^2 = mv^2/2kT$

- ▲ Evaluate

$$\left(\frac{2kT}{m}\right)^{3/2} \int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4} \left(\frac{2kT}{m}\right)^{3/2} \quad (75)$$

- ▲ Altogether

$$f_e = 2 \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \quad (76)$$

$$= \frac{2}{\lambda_e^3} \quad (77)$$

- ◇ Again,

$$f_{x+1} = \sum_i g_{x+1,i} e^{-E_{x+1,i}/kT} \quad (78)$$

- ▲ For low T (e.g. $kT \ll E_{x+1,i}$), the first term dominates

$$f_{x+1} \approx g_{x+1,1} e^{-E_{x+1,1}/kT} \quad (79)$$

- ◇ Alternate Saha Equation

$$\frac{n_{x+1}n_e}{n_x} = 2 \frac{g_{x+1,1}}{g_{x,1}} \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_x/kT} \quad (80)$$

- Saha's equation for Hydrogen

$$\frac{n_p n_e}{n_{\text{HI}}} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} \frac{2}{f_{\text{HI}}} e^{-h\nu_0/kT} \quad (81)$$

◇ Replace f_{HI}

◇ Finally

$$n_x = \frac{1}{2} n_p n_e g_x \left(\frac{h^2}{2\pi m_e kT} \right)^{3/2} e^{I_x/kT} \quad (82)$$

◇ $I_x \equiv h\nu_0 - E_x$

F. Rate Coefficients and Cross-sections for Collisional Excitations by Free Electrons

- Consider two levels in an ion i, j with $E_j > E_i$
- Define a collisional rate coefficient for excitation: q_{ij}

$$\frac{\# \text{ excitations}}{\text{cm}^3 \text{s}} = n_e n_i q_{ij}(T) \equiv n_e n_i \int_{v_0}^{\infty} \sigma_{ij}(v_i) v_i f(v_i) dv_i \quad (83)$$

◇ $\frac{1}{2} m v_0^2 = E_j - E_i$

◇ σ_{ij} : Excitation cross-section
(often units of $a_0 = 2.8010 \times 10^{-17} \text{ cm}^2$)

◇ $f(v_i)$: Maxwellian

- Relate q_{ij} and q_{ji} via detailed balance in Thermal Equil.

$$n_e n_i q_{ij} = n_e n_j q_{ji} \quad (84)$$

◇ In thermal equilibrium, we can use the Boltzman equation

$$\frac{n_j}{n_i} = \frac{g_j}{g_i} e^{-(E_j - E_i)/kT} \quad (85)$$

◇ Therefore,

$$q_{ij} = q_{ji} \frac{g_j}{g_i} e^{-(E_j - E_i)/kT} \quad (86)$$

◇ This holds outside of TE provided the electrons have a Maxwellian distribution at temperature T

- Similarly, we can relate σ_{ij} and σ_{ji}

◇ Energy conservation

$$\frac{1}{2} m v_i^2 + E_i = \frac{1}{2} m v_j^2 + E_j \quad (87)$$

◇ Again, assume TE and invoke detailed balancing

$$n_e f(v_i) dv_i \cdot n_i v_i \sigma_{ij} = n_e f(v_j) dv_j \cdot n_j v_j \sigma_{ji} \quad (88)$$

◇ And the Boltzman equation (Eq. 85)

◇ Finally, we have (which holds outside TE!)

$$g_i v_i^2 \sigma_{ij}(v_i) = g_j v_j^2 \sigma_{ji}(v_j) \quad (89)$$

• Collision Strength: Define

$$\Omega_{ij}(v_i) = \left(\frac{m_e v_i}{\hbar} \right)^2 \frac{g_i}{\pi} \sigma_{ij}(v_i) \quad (90)$$

◇ In most of the literature, Ω_{ij} values are reported instead of σ_{ij}

◇ Consider a simple (semi-classical) relationship between Ω and σ

▲ Suppose an electron with velocity v_i approaches an ion (at rest) with impact parameter R

▲ Fig



▲ Express the cross-section for $i \rightarrow j$ excitation by an electron with velocity v_i to be:

$$\sigma_{ij}(v_i) = \int_0^{\infty} P_{ij}(R, v_i) 2\pi R dR \quad (91)$$

○ P_{ij} is the probability for $i \rightarrow j$

○ $P_{ij} \leq 1$

▲ The angular momentum of the electron is

○ $L = m v_i R$

○ Quantize: $L = \hbar \sqrt{\ell(\ell + 1)}$, $\ell = 0, 1, 2, \dots$

▲ Define

$$R_\ell = \frac{L}{m v_i} = \frac{\hbar \sqrt{\ell(\ell + 1)}}{m v_i} \quad (92)$$

$$\equiv \frac{\sqrt{\ell(\ell + 1)}}{k_i} \quad (93)$$

▲ De Broglie wave number

$$k_i \equiv \frac{m v_i}{\hbar} \quad (\text{cm}^{-1}) \quad (94)$$

▲ Our integral (Equation 91) becomes a sum

$$\sigma_{ij}(v_i) = \frac{\pi}{k_i^2} \sum_{\ell=0}^{\infty} (2\ell + 1) P_{ij}(R_\ell, v_i) \quad (95)$$

$$\equiv \frac{\pi}{k_i^2} \sum_{\ell=0}^{\infty} \Omega_{ij}(\ell) \quad (96)$$

$$\equiv \frac{\pi}{k_i^2} \Omega_{ij}(v_i) \quad (97)$$

▲ If the ion has a degeneracy g_i , then we simply average over the initial states

$$\sigma_{ij} = \frac{\pi}{k_i^2} \frac{\Omega_{ij}}{g_i} \quad (98)$$

▲ We note that

$$\Omega_{ij}(v_i) \propto v_i^2 g_i \sigma_{ij} = v_j^2 g_j \sigma_{ji} \quad (99)$$

▲ Therefore

$$\Omega_{ij}(v_i) = \Omega_{ji}(v_j) \quad (100)$$

- This is convenient, because recall that $\sigma \propto 1/v^2$
- Therefore, Ω is a weak function of v

$$\Omega_{ij} \approx \Omega_{ji} \equiv \Omega \quad (101)$$

◇ Typically

$$\begin{aligned} \Omega &\sim 10 && \text{for dipole permitted lines} \\ \Omega &\sim 1 && \text{for forbidden optical lines} \\ \Omega &\sim .1 && \text{for forbidden IR lines} \end{aligned}$$

◇ Note, Ω values are generally known to only $\sim 10 - 15\%$ and are a significant source of systematic error

● Relating q_{ij} to Ω_{ij}

◇ Consider the de-excitation rate coefficient

$$q_{ji} = \int_0^{\infty} \sigma_{ji}(v_j) v_j f(v_j) dv_j \quad (102)$$

◇ Express

$$\sigma_{ji} = \frac{\pi}{k_j^2} \frac{\Omega_{ji}}{g_j} = \frac{\pi \hbar^2}{m_e v_j^2} \frac{\Omega_{ji}}{g_j} \quad (103)$$

- ◇ Altogether (with $L \equiv m/2kT$)

$$q_{ji} = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT} \right)^{\frac{3}{2}} \frac{\pi \hbar^2}{m^2 g_j} \int_0^{\infty} \frac{\Omega_{ij}}{v_j^2} v_j v_j^2 e^{-Lv_j^2} dv_j \quad (104)$$

- ◇ Substitute $\varepsilon/kT = Lv_j^2$

$$q_{ji} = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2kT} \right)^{\frac{3}{2}} \frac{\pi \hbar^2}{m^2 g_j} \frac{kT}{m} \int_0^{\infty} \Omega_{ji} e^{-\varepsilon/kT} d\left(\frac{\varepsilon}{kT}\right) \quad (105)$$

$$= \frac{C}{g_j T^{\frac{1}{2}}} \overline{\Omega_{ji}} \quad (106)$$

$$C = \frac{4}{\sqrt{\pi}} \left(\frac{m}{2k} \right)^{\frac{3}{2}} \frac{k\pi \hbar^2}{m^3} = 8.63 \times 10^{-6} \text{ cm}^3 \text{ K}^{\frac{1}{2}} \text{ s}^{-1} \quad (107)$$

- ◇ Consider $\overline{\Omega_{ji}}$

- ▲ It is a slow function of T, ε

- ▲ Therefore,

$$\overline{\Omega_{ji}} \approx \Omega_{ji} \quad (108)$$

- ◇ Detailed balance implies $\Omega_{ij} = \Omega_{ji}$

- ◇ Applying the Boltzman equation and substituting

$$q_{ij} = \frac{C \Omega_{ij}}{g_i T^{\frac{1}{2}}} e^{-(E_j - E_i)/kT} \quad (109)$$

- Critical Density :: Density required to establish (roughly) a Boltzman population

- ◇ Consider a 2-level ion (1,2)

- ◇ Total collision de-excitation rate

$$n_e n_2 q_{21} = n_e n_2 \int_0^{\infty} v \sigma_{21} f(v) dv \quad (110)$$

- ◇ Boltzman tells us

$$q_{21} = \frac{g_2}{g_1} q_{12} e^{-(E_2 - E_1)/kT} \quad (111)$$

- ◇ Detailed balance of excitation and de-excitation

$$n_e n_1 q_{12} = n_e n_2 q_{21} + n_2 A_{21} \quad (112)$$

- ◇ The emission in the line is

$$\varepsilon_{21} = n_2 A_{21} h \nu_{21} \quad (113)$$

$$= n_e n_1 q_{12} h \nu_{21} \left[\frac{A_{21}}{A_{21} + n_e q_{21}} \right] \quad (114)$$

$$= n_e n_1 q_{12} h \nu_{21} \left[\frac{1}{1 + \frac{n_e q_{21}}{A_{21}}} \right] \quad (115)$$

◇ As $n_e \rightarrow 0$,

$$\varepsilon_{21} \rightarrow n_e n_1 q_{12} h\nu_{21} \propto n_e^2 \quad (116)$$

▲ Every collisional excitation produces a line photon

▲ i.e. there is no collisional de-excitation

◇ As $n_e \rightarrow \infty$,

$$\varepsilon_{21} \rightarrow n_1 h\nu_{21} A_{21} \frac{q_{12}}{q_{21}} \quad (117)$$

$$= n_1 h\nu_{12} A_{21} \cdot \frac{g_2}{g_1} e^{-h\nu_{12}/kT} \quad (118)$$

$$= (n_2)_{TE} h\nu_{12} A_{21} \quad (119)$$

$$\propto n_e \quad (120)$$

▲ This is the Thermal equilibrium cooling rate

◇ Define: Critical density – Transition from low density to T.E.

$$(n_e)_C = \frac{A_{21}}{q_{21}} \quad (121)$$

▲ Example point: H β emission is always $\propto n_e^2$

▲ Therefore, when $n_e > (n_e)_C$, the forbidden lines weaken relative to H β

G. Atomic Notation

• Electrostatic Interactions: Russel-Saunders Coupling

◇ Multiple electrons only

◇ Recall our central field Hamiltonian for the hydrogenic ions

$$H = \sum_j \frac{p_j^2}{2m} - \frac{Ze^2}{r}$$

▲ Degenerate in ℓ, m, m_s

▲ Additional terms for multiple electrons break some of this degeneracy

◇ Express the multi-electron Hamiltonian as

$$H = \sum_j \frac{p_j^2}{2m} - Ze^2 \sum_j \frac{1}{r_j} + \sum_i V(r_i) + H_1 \equiv H_0 + H_1 \quad (122)$$

▲ Identify $\sum V(r_i)$ as the central field potential due to the ‘smeared out’ electrons

▲ H_1 is a perturbation to our spherically symmetric Hamiltonian (H_0)

◇ Explicitly

$$H_1 = e^2 \sum_{i>j} \frac{1}{r_{ij}} - \sum_i V(r_i) + H_{SO} + \dots \quad (123)$$

- ▲ Electrostatic Interaction: H_{es}

$$H_{es} = e^2 \sum_{i>j} \frac{1}{r_{ij}} - \sum_i V(r_i) \quad (124)$$

- ▲ H_{SO} = Spin-orbit term

- ▲ ... = Higher order terms

- ◇ Note ℓ_i is not a constant of the motion: $[H_{es}, \ell_i] \neq 0$
- ◇ But, H_{es} does commute with the total angular momentum and spin

$$[H_{es}, \vec{L}] = 0 \quad [H_{es}, \vec{S}] = 0$$

- ▲ $\vec{L} = \sum_i \vec{\ell}_i; \quad \vec{S} = \sum_i \vec{s}_i$

- ▲ So, the total angular momentum and spin are constants of the motion

- ◇ Qualitatively assessing the H_{es} perturbation

- ▲ Electrons repel each other

- ▲ \Rightarrow Lower energy if they are farther apart

- Hund's Rules

- (a) Large Spin \Rightarrow Lower Energy

- ◇ Large spin \Rightarrow individual \vec{s}_i are aligned $\Rightarrow s_1 = s_2 = s_3 = \dots = \pm \frac{1}{2}$
- ◇ Pauli's exclusion \Rightarrow electrons with same spin will be farther apart
- ◇ This is typically the dominant RS Coupling effect

- (b) For a given spin, larger angular momentum \Rightarrow Lower Energy

- ◇ Large L \Rightarrow individual $\vec{\ell}_i$ are aligned
- ◇ Pauli \Rightarrow electrons with similar orbits will be farther apart

- (c) Higher $J \Rightarrow$ Higher energy for a level less than half filled

- Equivalent and Non-equivalent Electrons

- ◇ Closed shells

- ▲ Spherically symmetric

$$\sum_m |Y_{\ell m}|^2 = \frac{2\ell + 1}{4\pi} \quad (125)$$

- ▲ These electrons interact minimally with outer electrons

- ▲ Ignore them altogether

- ◇ Definition

- ▲ Equivalent electrons have the same n, ℓ

- ▲ Non-Equivalent electrons have different n, ℓ

◇ Spectroscopic labeling

$$^{(2S+1)}L_J$$

- ▲ L: Total orbital angular momentum
- ▲ S: Total spin
- ▲ J: Total angular momentum

◇ Example 1: Two 's' electrons

(a) Non-equivalent, e.g. 1s2s

$$\vec{L} = \sum_i \vec{\ell}_i = 0$$

$$\vec{S} = \sum_i \vec{s}_i = 0, 1$$

- ▲ Two terms allowed
- ▲ 1S (singlet) and 3S (triplet)
- ▲ By Hund's rule, 3S should have lower energy

(b) Equivalent (e.g. $2s^2$)

- ▲ $\vec{L} = 0$ again
- ▲ Pauli $\Rightarrow \vec{S} = 0$ only

◇ Example 2: Two 'p' electrons

(a) Non-equivalent

$$\vec{L} = \sum_i \vec{\ell}_i = 0, 1, 2$$

$$\vec{S} = \sum_i \vec{s}_i = 0, 1$$

▲ States:	Term	1S	1P	1D	3S	3P	3D
	Number	1	3	5	3	9	15

- ▲ Each electron had 2 spin x $(2\ell + 1) = 6$ states
- ▲ Total had to be $6 \times 6 = 36$

(b) Equivalent

- ▲ Some states are excluded by Pauli
- ▲ Some states are indistinguishable
 - \Rightarrow Less than 36 states total (15)
 - Some are easily identified (named), others are not

m_1	m_2	m_{s_1}	m_{s_2}	Label	State
+1	+1	+	+	3D	Pauli
+1	+1	+	-	1D	1
+1	+1	-	+	1D	1
+1	+1	-	-	3D	Pauli
+1	0	+	+	3P	2
+1	0	+	-		3
+1	0	-	+		4
+1	0	-	-	3P	5
+1	-1	+	+		6
+1	-1	+	-		7
+1	-1	-	+		8
+1	-1	-	-		9
0	+1	+	+		2
0	+1	+	-		3
0	+1	-	+		4
0	+1	-	-		5
0	0	+	+	3S	Pauli
0	0	+	-	1S	10
0	0	-	+		10
0	0	-	-	3S	Pauli
0	-1	+	+	3P	11
0	-1	+	-		12
0	-1	-	+		13
0	-1	-	-	3P	14
-1	+1	+	+		6
-1	+1	+	-		7
-1	+1	-	+		8
-1	+1	-	-		9
-1	-1	+	+	3D	Pauli
-1	-1	+	-	1D	15
-1	-1	-	+	1D	15
-1	-1	-	-	3D	Pauli

◇ Useful shortcut

▲ Holes in a closed shell give same results as corresponding electrons

▲ e.g. Terms corresponding to p^5 are the same as p^1

H. Einstein Constants

- Assume a set two levels (j, k) in an atomic system
 - ◊ k is the upper level
 - ◊ Energy is $h\nu_{jk}$
- Absorption-line profile:
 - ◊ Let $\phi_\nu d\nu$ = the probability that an absorber in state j will absorb radiation with frequency in the interval $\nu, \nu + d\nu$.
 - ◊ Normalization

$$\int_0^\infty \phi_\nu d\nu = 1 \quad (126)$$

- ◊ Clearly, ϕ_ν has units of Hz^{-1}
- Transfer Equation

$$\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu \quad (127)$$

- ◊ Express the opacity term as

$$\kappa_\nu I_\nu \equiv n_j \phi_\nu h\nu_{jk} I_\nu \frac{B_{jk}}{4\pi} \quad (128)$$

- ▲ This has defined the first Einstein constant, B_{jk} which describes the transition strength independent of ν
- ▲ But, this expression has ignored stimulated emission!
- ◊ Consider the emissivity for spontaneous emission

$$j_\nu^{(s)} = n_k \psi_\nu^s \frac{A_{kj}}{4\pi} h\nu_{jk} \quad (129)$$

- ▲ ψ_ν^s is the fraction of atoms/ions in the upper state that can emit into $\nu, \nu + d\nu$
- ▲ Similar to ϕ_ν , ψ_ν^s is normalized by

$$\int_0^\infty \psi_\nu^s d\nu = 1 \quad (130)$$

- ◊ Einstein also realized that there was emissivity due to induced (stimulated) emission

$$j_\nu^{(i)} = n_k \psi_\nu^i \frac{B_{kj}}{4\pi} h\nu_{jk} \quad (131)$$

- ▲ This is a 2nd B value, and $B_{jk} \neq B_{kj}$
- ▲ Dirac showed that

$$\psi_\nu^s = \psi_\nu^i = \psi_\nu \quad (132)$$

◇ Altogether now,

$$\frac{dI_\nu}{ds} = -n_j \phi_\nu h\nu_{jk} I_\nu \frac{B_{jk}}{4\pi} + \frac{n_k}{4\pi} \psi_\nu [A_{kj} + B_{kj} I_\nu] h\nu_{jk} \quad (133)$$

▲ In general, $\phi_\nu \neq \psi_\nu$

▲ As such, it is often incorrect to regard stimulated emission as a negative absorption

◇ Consider the transfer equation under Thermal Equilibrium

▲ $dI_\nu/ds = 0$

▲ $I_\nu = B_\nu$

$$I_\nu = \frac{A_{kj}}{B_{kj}} \frac{1}{\left(\frac{n_j B_{jk} \phi_\nu}{n_k B_{kj} \psi_\nu} - 1 \right)} \quad (134)$$

$$= B_\nu \quad (135)$$

$$= \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} \quad (136)$$

▲ Consider two limits

(a) $T \rightarrow \infty$

○ Then we have

$$\frac{g_j}{g_k} \frac{B_{jk} \phi_\nu}{B_{kj} \psi_\nu} e^{h\nu_{jk}/kT} - 1 = e^{h\nu/kT} - 1 \quad (137)$$

○ Implying

$$g_j B_{jk} \phi_\nu = g_k B_{kj} \psi_\nu \quad (138)$$

○ Integrating over all frequencies, we have the First Einstein relation

$$\frac{B_{jk}}{B_{kj}} = \frac{g_k}{g_j} \quad (139)$$

(b) $T \rightarrow 0$: Gives Einstein's second relation

$$\frac{A_{kj}}{B_{kj}} = \frac{2h\nu^3}{c^2} \quad (140)$$

▲ Although we derived these under the assumption of Thermal Equilibrium, they hold in all physical conditions

▲ Maintaining the assumption of T.E., where $B_\nu = j_\nu/\kappa_\nu$ we can show

$$\kappa_\nu = \frac{h\nu_{jk}}{4\pi} (n_j B_{jk} - n_k B_{kj}) \psi_\nu \quad \{ \text{T.E.} \} \quad (141)$$

◇ Outside of T.E., our transfer equation is now

$$\frac{dI_\nu}{ds} = \frac{h\nu_{jk}}{4\pi} [n_k A_{kj} \psi_\nu - (n_j B_{jk} \phi_\nu - n_k B_{kj} \psi_\nu) I_\nu] \quad (142)$$

- ▲ We can therefore identify the opacity κ_ν as:

$$\kappa_\nu = \frac{h\nu_{jk}}{4\pi} (n_j B_{jk} \phi_\nu - n_k B_{kj} \psi_\nu) \quad (143)$$

$$= \frac{n_j B_{jk} h\nu_{jk}}{4\pi} \left(1 - \frac{n_k B_{kj} \psi_\nu}{n_j B_{jk} \phi_\nu} \right) \quad (144)$$

$$= \frac{n_j B_{jk} h\nu_{jk}}{4\pi} \left(1 - \frac{b_k \psi_\nu}{b_j \phi_\nu} e^{-h\nu_{jk}/kT} \right) \quad (145)$$

- ▲ Where b_j is the departure coefficient $n_j = b_j n_j^{(\text{T.E.})}$
 ▲ The stimulated term is important when $b_k \approx b_j$ and $\exp(-h\nu_{jk}/kT) \approx 1$ or $h\nu_{jk} \ll kT$ (e.g. 21cm radiation)
 ▲ In most astrophysical cases, the stimulated term is small and

$$\kappa_\nu \approx n_j B_{jk} \phi_\nu \frac{h\nu_{jk}}{4\pi} \quad (146)$$

$$\equiv n_j s_\nu \quad (147)$$

$$= n_j s_{jk} \phi_\nu \quad (148)$$

- ▲ with

$$s_{jk} = \frac{h\nu_{jk}}{4\pi} B_{jk} \quad (\text{cm}^2 \text{ Hz}) \quad (149)$$