AY230 2008 – Homework Set #1

(1) Ionization state in a plane-parallel geometry: The goal of this problem is to use the 'on-the-spot' approximation to estimate the physical width of the boundary region for an HII region.

Consider an infinite slab of Hydrogen gas with constant density n_H. Consider an ionizing radiation field with ionizing photon flux

$$\mathcal{J} = \frac{\phi}{4\pi r^2} \qquad (\text{photons/s/cm}^2) \tag{1}$$

(a) Introduce

$$\bar{\sigma} = \frac{\int \mathcal{J} \sigma_\nu^{ph} d\nu}{\int \mathcal{J} d\nu} \tag{2}$$

Derive an expression for the transfer equation of the incident radiation field $d\mathcal{J}/ds$. (Note: for our geometry $ds = dr$)

Solution: We note that we are only interested in the transfer equation of the incident ionizing radiation. Therefore, we can ignore the emissivity term of the transfer equation and focus solely on the opacity. We can suppress the frequency dependence of our opacity (i.e. σ_ν^{ph}) by introducing $\bar{\sigma}$ as defined above. In this fashion, we write

$$\frac{d\mathcal{J}}{ds} = -\kappa \mathcal{J} \tag{3}$$

where $\kappa \equiv n_{HI}\bar{\sigma}$.

(b) Using the on-the-spot approximation, derive an expression for the ionization fraction x.

Solution: Our on-the-spot expression for the stellar intensity is:

$$n_{\rm HI} \int_{\nu_0}^{\infty} \frac{4\pi J_{\nu *}}{h\nu} \sigma_\nu^{ph} d\nu = n_p n_e \alpha_B \tag{4}$$

We can identify the LHS of this equation as

$$n_{\rm HI}\bar{\sigma}\mathcal{J} \tag{5}$$

Finally, replacing $n_{\rm HI} = (1-x)n_H$ and $n_p n_e = x^2 n_H^2$ and rearranging, we have:

$$\frac{x^2}{1-x} = \frac{\bar{\sigma}}{\alpha_B} \mathcal{J} \frac{1}{n_H} \tag{6}$$

(c) Define a dimensionless path ξ, where

$$d\xi = n_H \bar{\sigma} ds \tag{7}$$

Find an expression for $d\xi/dx$ in terms of x.

Solution: We begin by rewriting the Transfer equation in terms of the optical depth:

$$\frac{1}{\mathcal{J}}\frac{d\mathcal{J}}{d\xi} = -(1-x) \tag{8}$$

We can derive an expression for $d\mathcal{J}/dx$ by differentiating Equation 6 with respect to x.

$$\frac{d\mathcal{J}}{dx} = \frac{2xn_H\alpha_B}{(1-x)\bar{\sigma}} + \frac{x^2 n_H \alpha_B}{(1-x)^2\bar{\sigma}} \tag{9}$$

Now multiply by $d\mathcal{J}/\xi$, take the inverse and replace $\mathcal{J}$ using Equation 6. We find:

$$\frac{d\xi}{dx} = \frac{\left[\frac{2}{x} + \frac{1}{1-x}\right]}{x-1} = \frac{(x-2)}{x(1-x)^2} \tag{10}$$

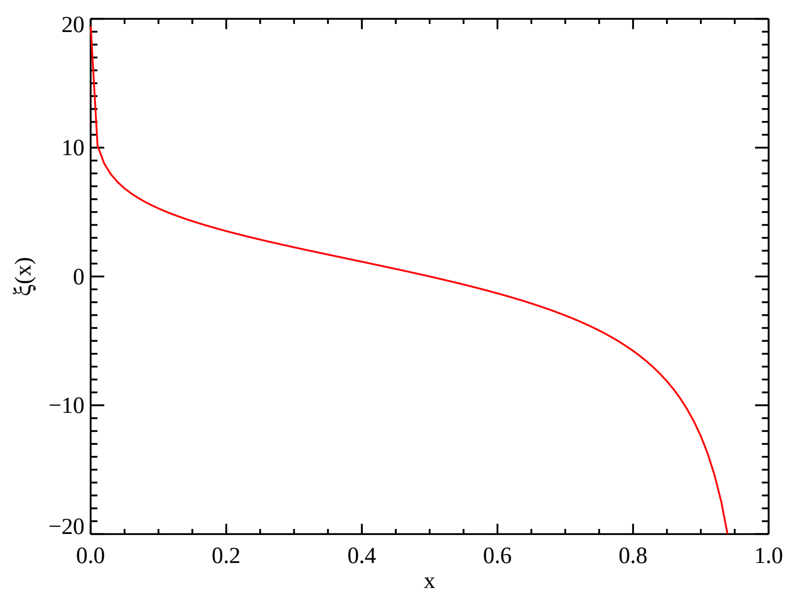
(d) Solve (numerically) for $\xi(x)$ assuming $\xi = 0$ when $x = 0.5$ and plot.

Solution: Integrating

$$2\ln\frac{x}{1-x} + \frac{1}{1-x} = C - \xi \tag{11}$$

For $\xi = 0$ at $x = 0.5$, we have $C = 2$.

Here is a plot:



i. Estimate the width of the boundary region $\Delta\xi$ and Δr taking $n_H = 10\,\mathrm{cm}^{-3}$.
Solution: It is straightforward to show that $\Delta\xi \approx 17.7$ for x changing from 0.9 to 0.1. Using the definition, $d\xi = n_H \bar{\sigma} dr$, we have $\Delta r = 0.09\,\mathrm{pc}$ for $n_H = 10\,\mathrm{cm}^{-3}$.

ii. Evaluate R_S for $\phi = 10^{48.8}$ and $n_H = 10\,\mathrm{cm}^{-3}$. Discuss
Solution: Our equation for the Stromgren radius is:

$$R_S = \left[\frac{3\phi}{4\pi n_e^2 \alpha_B(T)} \right]^{1/3} \tag{12}$$

Taking $T = 10^4$K (this is the temperature of the gas in the HII region, not the star!), and $n_e \approx n_H$ and ϕ as above, we have $R_S = 13\,\mathrm{pc}$. This is much larger than Δr because Δr corresponds to the small region where the gas goes from partially ionized $x = 0.9$ to largely neutral $x = 0.1$. The majority of R_S covers from $x \approx 1$ to $x = 0.9$, where our solution diverges.

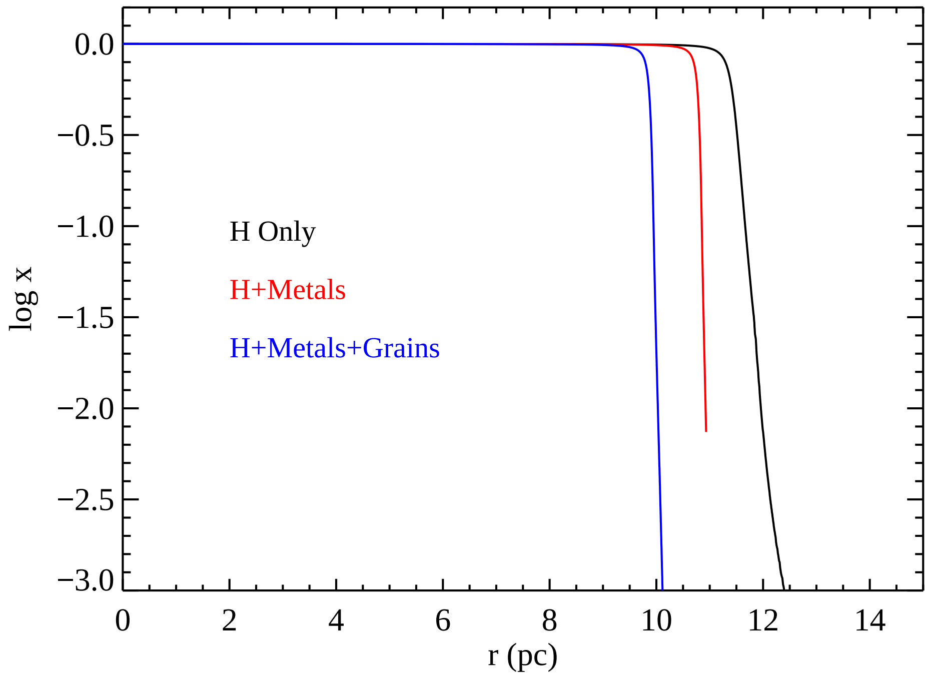
(2) First Cloudy calculation: The goal of this problem is to gain familiarity with the Cloudy software package and use it to investigate the structure of HII regions.

(a) Download, compile and install Cloudy: http://www.nublado.org/

(b) Use Cloudy to generate an idealized HII region ($n_H = 10\,\mathrm{cm}^{-3}$, H gas only, no metals, no dust) with our canonical O7 star ($T_* = 3.5 \times 10^4$K, $\log\phi = 48.84$). Plot the ionization fraction x as a function of distance from the star (pc).

(c) Use your calculations to estimate the width of the boundary region for your HII region. How does this compare with your estimate from Problem 1? Discuss.

(d) Repeat steps (b,c) for a more realistic HII region, e.g. allow for He, a solar abundance of metals and dust.



Solution: The attached figure shows the calculations from Cloudy for (black) H only, (red) H+metals with a solar abundance, and (blue) H+metals+dust.

For H only (black), we see that $R_S \approx 11\,\mathrm{pc}$ which agrees very well with our estimate from Problem 1. The transition region is also shown to be very sharp. I calculate $\Delta r = 0.5\,\mathrm{pc}$ for $x = 0.9$ to 0.1. This is a few times larger than our semi-analytic estimation. Clearly, radiative transfer effects (especially the propagation of higher energy photons) broaden out the transition region.

If we include metals (red), there are two effects that decrease R_S. First, there is a higher opacity to ionizing photons. Second, the metals will cool the region which increases the recombination rate $\alpha_B(T) \propto T^{-0.8}$, and therefore decreases R_S. The transition region is sharper here $\Delta r = 0.25\,\mathrm{pc}$, probably because of the cooler temperature of the gas and the higher opacity to higher energy photons from the metals.

If we include grains (blue), the main effect is an increased opacity which further reduces R_S. I calculate essential the same Δr value.

The following page gives the input file for the H+metals+grains curve.

Cloudy Input File for H+Metals+dust grains

```
c hii_typical.in
title typical HII region
sphere
c 05 star with temperature 35000K = 10**4.5441
blackbody 4.5441
c Ionizing flux
Q(H) = 48.84
c gas density in log of number density of all protons
hden = 1
c log of starting radius r_cm
radius 17
c Solar metallicity
metals 0
grains
stop temperature 2
plot continuum range 0.1
iterations = 3
print last iteration
punch overview last file="Output/hii_hmwk1_2c2.ovr"
```