AY230 – Homework Set #2

(1) Brehmstrahlung emission from HII regions [Stolen from Shu Vol I].
Assume an HII region to have a uniform electron temperature T and density n_e which we would like to determine by observational means.

(a) Since the free-free emission associated with the thermal distribution of electrons occurs under conditions of LTE, show that our radiative transfer equation can be rewritten as:

$$I_\nu = I_\nu(0)\mathrm{e}^{-\tau_\nu} + B_\nu(T)\left(1 - \mathrm{e}^{-\tau_\nu}\right) \tag{1}$$

Solution: Start with the radiative transfer equation

$$\frac{dI_\nu}{d\tau_\nu} = -S_\nu + I_\nu \tag{2}$$

$$S_\nu \equiv \frac{j_\nu}{\kappa_\nu} \tag{3}$$

Integrate this equation through a 'cloud' with optical depth $\tau_{\nu r}$ we have:

$$I_\nu = I_\nu(0)\mathrm{e}^{-\tau_{\nu r}} + \int_0^{\tau_{\nu r}} \frac{j_\nu}{\kappa_\nu}\mathrm{e}^{-\tau_\nu} d\tau_\nu \tag{4}$$

In LTE, $B_\nu(T) = j_\nu/\kappa_\nu$, independent of the optical depth. Therefore:

$$I_\nu = I_\nu(0)\mathrm{e}^{-\tau_{\nu r}} + B_\nu \int_0^{\tau_{\nu r}} \mathrm{e}^{-\tau_\nu} d\tau_\nu \tag{5}$$

$$= I_\nu(0)\mathrm{e}^{-\tau_{\nu r}} + B_\nu\left(1 - \mathrm{e}^{-\tau_{\nu r}}\right) \tag{6}$$

(b) For radio observations spanning $\lambda \sim 100$cm to 1mm, show that $h\nu \ll kT$ for all likely values of T. In this case it is a good approximation to replace $B_\nu(T)$ by its Rayleigh-Jeans limit:

$$B_\nu(T) = \frac{2\nu^2}{c^2}kT \tag{7}$$

Solution: For HII regions, we expect $T > 1000\,\mathrm{K}$. Taking the maximum $h\nu$ value for this range (i.e. $\lambda = 1$mm):

$$h\nu = \frac{hc}{\lambda} = \frac{2\times10^{-16}\mathrm{ergs/cm}}{10^{-1}\mathrm{cm}} = 2\times10^{-15}\mathrm{ergs} \tag{8}$$

$$kT = 1.4\times10^{-13}\ \mathrm{ergs} \tag{9}$$

$$\therefore kT \gg h\nu \tag{10}$$

and the Rayleigh-Jeans limit applies.

(c) Using our definition for the brightness temperature, show that the radiative transfer equation becomes

$$T_b = T_b(0)\mathrm{e}^{-\tau_\nu} + T\left(1 - \mathrm{e}^{-\tau_\nu}\right) \tag{11}$$

Solution: We relate I_ν to the brightness temperature T_b by:

$$I_\nu = \frac{2\nu^2}{c^2} kT_b \tag{12}$$

Applying Equation 6,

$$T_b = T_b(0)\mathrm{e}^{-\tau_\nu} + T\left(1 - \mathrm{e}^{-\tau_\nu}\right) \tag{13}$$

(d) Focusing on free-free emission, the optical depth can be expressed as

$$\tau_\nu = \int \kappa_\nu^{ff}\, ds \tag{14}$$

where κ_ν^{ff} is the free-free opacity given by $j_\nu^{ff}/4\pi B_\nu$. Assuming a pure Hydrogen plasma, show that for small $h\nu/kT$, we have

$$\kappa_\nu^{ff} = C n_e^2 T^{-\frac{3}{2}} \nu^{-2} g_\nu^{ff} \tag{15}$$

where

$$C \equiv \left(\frac{2m_e}{3\pi k}\right)^{\frac{1}{2}} \left[\frac{4\pi e^6}{3m_e^2 ck}\right] \tag{16}$$

Solution: Using $\kappa_\nu^{ff} = j_\nu^{ff}/4\pi B_\nu$ and our expression for the free-free emission in the small $h\nu/kT$ limit:

$$j_\nu^{ff} = n_p n_e \left(\frac{2m_e}{3\pi kT}\right)^{\frac{1}{2}} \left[\frac{32\pi^2 e^6}{3m_e^2 c^3}\right] \bar{g}_{ff} \tag{17}$$

and the definition for B_ν, we have:

$$\kappa_\nu^{ff} = \frac{n_e^2}{4\pi} \frac{c^2}{2\nu^2} \left(\frac{2m_e}{3\pi}\right)^{\frac{1}{2}} \left(\frac{1}{kT}\right)^{\frac{3}{2}} \left(\frac{32\pi^2 e^6}{3m_e^2 c^3}\right) \bar{g}_{ff} \tag{18}$$

$$= n_e^2 T^{-3/2} \nu^{-2} \bar{g}_{ff} \cdot \left(\frac{2m_e}{3\pi k}\right)^{\frac{1}{2}} \frac{4\pi e^6}{3m_e^2 ck} \tag{19}$$

(e) An expression for the Gaunt factor (somewhat different from the one in the notes) is:

$$\bar{g}_{ff} = \frac{\sqrt{3}}{2\pi}\left[\ln\left(\frac{8k^3T^3}{\pi^2 e^4 m_e \nu^2}\right) - 5\gamma\right] \tag{20}$$

with $\gamma = 0.5772$. Compute $\bar{g}_{ff}$ for $\nu = 10^9$Hz and $T = 10^4$K and show that $\bar{g}_{ff}$ should not be approximated by unity here (unlike at optical wavelengths).

Solution: Plug and chug...

$$\bar{g}_{ff}(\nu = 10^9\text{Hz}, T = 10^4\text{K}) = 6.0 \tag{21}$$

(f) Define the emission measure as the integral

$$\text{EM} \equiv \int n_e^2 \, ds \tag{22}$$

and show that τ_ν can be expressed as

$$\tau_\nu = (\text{EM})CT^{-\frac{3}{2}}\nu^{-2}\bar{g}_{ff} \tag{23}$$

Solution: By definition

$$\tau_\nu = \int \kappa_\nu ds \tag{24}$$

$$= \int C n_e^2 T^{-\frac{3}{2}}\nu^{-2}\bar{g}_{ff} ds \tag{25}$$

$$= (EM)CT^{-\frac{3}{2}}\nu^{-2}\bar{g}_{ff} \tag{26}$$

(g) At low frequencies $\tau_\nu \gg 1$, whereas at high frequencies $\tau_\nu \ll 1$. With no background source, show that this implies $T_b \approx T$ at low frequencies, while $T_b \approx T\tau_\nu$ at high ν.

Solution: Examining Equation 13 with $T_b(0) = 0$,

$$T_b = \begin{cases} T, & \tau_\nu \gg 1(\text{low frequency}) \\ T\tau_\nu, & \tau_\nu \ll 1(\text{high frequency}) \end{cases} \tag{27}$$

(h) For a spherical HII region with radius R_S, show that the observed flux (measured in Janksys = 10^{-26} watts m^{-2} Hz^{-1})

$$F_\nu = \pi I_\nu \left(\frac{R_S}{r}\right)^2 \tag{28}$$

where r is the distance to the source.

Solution: Expressing the flux at the surface of the HII region as F_{HII}, the luminosity is

$$L_{HII} = 4\pi R_S^2 F_{HII} \tag{29}$$

Noting, that the flux is related to the intensity by $F_{HII} = \pi I_\nu$, the flux at a distance r is:

$$F_\nu = \pi I_\nu \left(\frac{R_S}{r}\right)^2 \tag{30}$$

(i) The size R_S can be determined if the source is angularly resolved and its distance is known. Show that F_ν is proportional to ν^2 at low frequencies and $\bar{g}_{ff}$ (i.e. $\nu^{-0.1}$) at high (radio) frequencies.

Solution: Using Equation 27 we adopt these relations for I_ν.

Low frequency:

$$I_\nu = \frac{2\nu^2 kT_b}{c^2} \propto \nu^2 T \tag{31}$$

High frequency:

$$I_\nu = \frac{2\nu^2 kT_b}{c^2} \propto \nu^2 T \tau_\nu \propto \frac{\bar{g}_{ff}}{T^{\frac{1}{2}}}(EM) \tag{32}$$

(j) Describe qualitatively how this information could be used to deduce T and EM if the spectrum on both sides of the turnover frequency ν_c (where $\tau_\nu = 1$) can be measured.

Solution: Because I_ν is nearly independent of ν at high frequency, a measurement of F_ν at high ν sets the ratio $(EM)/T^{\frac{1}{2}}$. At low frequency, a fit to $F_\nu \propto \nu^2 T$ yields the temperature alone. Therefore, we can solve for both (EM) and T.

(k) Observations of an HII region in the Orion nebula [see Shu Figure P4.1] show F_ν =50Jy at $\nu = 10^2$MHz and 300Jy at $\nu = 10^4$MHz where $\nu = 10^3$MHz corresponds to the transition from low to high frequency. Compute approximate values for T and n_e assuming the region has size $R_S \approx 0.6$pc and is at a distance $r \approx 500$pc.

Quoted values in the literature are $T \approx 8000$K and $n_e \approx 2000\,\text{cm}^{-3}$. Check to see at what frequency $\tau_\nu = 1$ for your results. Surprisingly, your T value will not agree with the value of 8000 K because the low-frequency measurements have larger effective beam sizes than the high-frequency measurements. See the discussion in Osterbrock (p. 128-130) for more.

Solution:

$$F_\nu = \frac{2\pi k}{c^2}\left(\frac{R_S}{r}\right)^2 \nu^2 T \tag{33}$$

$$50\text{Jy} = 50 \cdot 10^{-23}\text{cgs} \tag{34}$$

$$= \frac{2\pi k}{c^2}\left(\frac{0.6}{500}\right)^2 (10^8\text{Hz})^2 T \tag{35}$$

$$= 1.4 \times 10^{-26} T \tag{36}$$

$$\therefore T = 36000\text{K} \tag{37}$$

Measuring $(EM), n_e$ and ν for $\tau_\nu = 1$:

$$300\text{Jy} = \frac{2\pi k}{c^2}\left(\frac{R_S}{r}\right)^2 \frac{(EM)C}{T^{\frac{1}{2}}}\bar{g}_{ff} \tag{38}$$

Evaluating $\bar{g}_{ff}$ at $T = 36000\text{K}$ and $\nu = 10^9\text{Hz}$, $\bar{g}_{ff} = 7$, and $C = 0.0177$. Equating

$$300 \cdot 10^{-23} = 9 \times 10^{-46}\,(EM) \tag{39}$$

Therefore,

$$(EM) = \int n_e^2 ds = 2 \times 10^{24} \tag{40}$$

Assuming a constant density (or that we will calculate an average n_e^2 quantity),

$$n_e \approx \left[\frac{(EM)}{R_S}\right]^{\frac{1}{2}} \approx 1500\,\text{cm}^{-3} \tag{41}$$

Finally, we can use Equation 26 to solve for $\tau_\nu = 1$. I get it is unity at $\nu = 245\,\text{MHz}$.

(l) Is the size $R_S = 0.6\text{pc}$ consistent with our estimate of the Stromgren radius for an O5 star? Discuss.

Solution: Recall our definition for R_S and substitute for n_e, ϕ and $\alpha_B(T)$, using $T = 8000\text{K}$ and $n_e = 2000\,\text{cm}^{-3}$:

$$R_S = \left(\frac{3\phi_{O5}}{4\pi n_e^2 \alpha_B(T)}\right)^{1/3} \tag{42}$$

$$= \left[\frac{3 \cdot 10^{49.67}}{4\pi \cdot 2 \times 10^6 \cdot 3 \times 10^{-13}}\right]^{1/3} \tag{43}$$

$$= 2.5 \times 10^{18}\text{cm} = 0.8\text{pc} \tag{44}$$

Indeed this is consistent with an ionization bounded HII region.

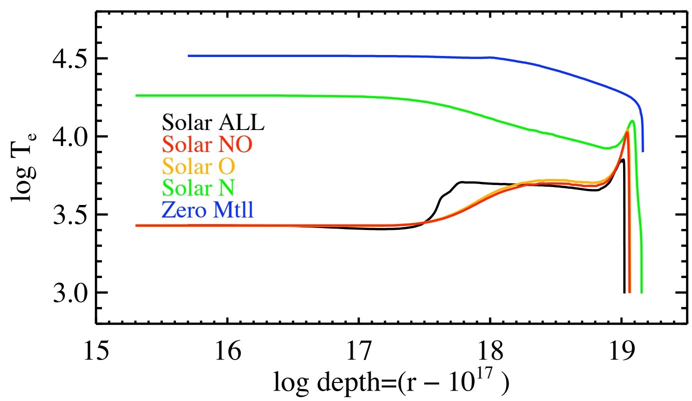
(2) More FUN with CLOUDY: Temperature structure

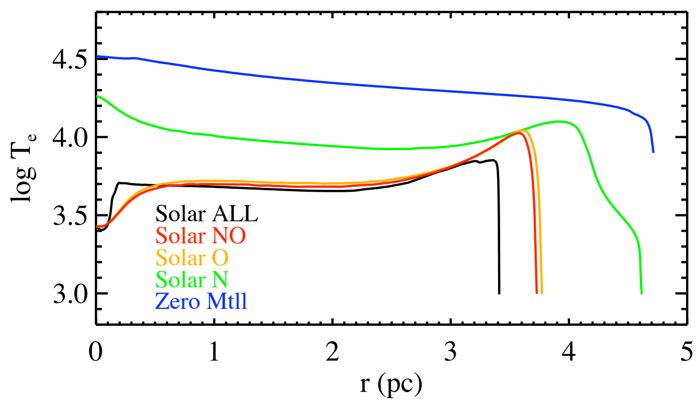
(a) Repeat the calculation for the temperature profile of an HII region as done in the notes. This time consider these scenarios:

i. Solar metallicity gas
ii. Zero metallicity gas
iii. H+O only (solar metallicity)
iv. H+N only (solar metallicity)
v. H+N+O only (solar metallicity)

Plot T against r for all of these cases.

Solution: Cloudy says –

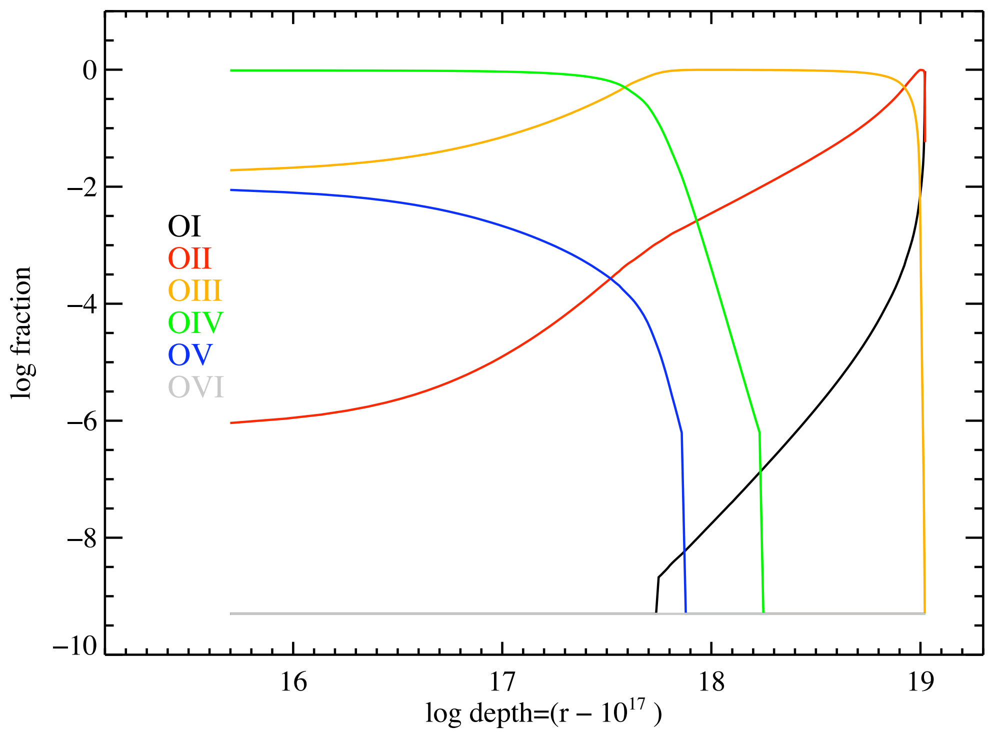




It is clear that Oxygen is the dominant coolant in HII regions and that N is not a major coolant. **Note that this kind of log plot is a bit deceiving.** Most of the gas is in the last few points plotted (i.e. near $10^{19}\,\mathrm{cm}^{-2}$). We see that the coolest HII region is the full, solar abundance as expected. The primordial HII region is qualitatively different than the rest. Clearly, HII regions in the early universe will have rather different properties.

(b) Plot the ionization fraction of the various O ions as a function of radius for the Solar metallicity HII region. Discuss.

Solution: Cloudy says –



OIII is the dominant ion for most of the cloud on the log-scale, but most of the O (by volume and mass) is OII. At the very edge of the cloud, we see the gas finally get neutral as O→OI.

(c) Using CLOUDY, determine the flux (or luminosity) of the [OIII] and [OII] forbidden lines for the Solar metallicity case. Discuss.

Solution: This is accomplished by using the *print last iteration line collisions* command. I find:

Table 1: LINE EMISSION

Ion	Line (Å)	log(L) (ergs/s)
[OII]	3726	36.509
[OII]	3729	36.614
[OII]	4651	34.584
[OIII]	4959	36.340
[OIII]	5007	36.819