AY230 – Solutions #5

(1) Dust acceleration: Because of radiation pressure, a dust grain at a distance r_0 from a star with luminosity L_* will be accelerated to a terminal velocity that depends on the grain size a, ρ_s the specific density of the grain material, and Q_{pr} the radiation pressure efficiency.

(a) Derive an analytic expression for the terminal velocity.

Solution:

The dust grains will feel a force f from the radiation pressure of the starlight,

$$f(r) = \frac{F_0(r)}{c}\pi a^2 Q_{pr} \tag{1}$$

where $F_0(r)$ the flux of radiation at distance r,

$$F_0(r) = \frac{L_*}{4\pi r^2} \tag{2}$$

We can express the force as

$$\begin{aligned} f &= m\frac{dv}{dt} && (3) \\ &= m\frac{dv}{dt}\left(v \cdot \frac{dt}{dr}\right) && (4) \\ &= mv\frac{dv}{dr} && (5) \end{aligned}$$

Lastly, the mass m is given by

$$\rho_s = \frac{m}{4\pi a^3/3} \tag{6}$$

Putting it all together, we have:

$$\frac{L_*}{c4\pi r^2}\,\pi a^2\,Q_{pr} = \frac{4}{3}\pi a^3 \rho_s\, v\frac{dv}{dr} \tag{7}$$

which reduces to

$$\frac{3L_*Q_{pr}}{16\pi a\rho_s c}\frac{dr}{r^2} = vdv \tag{8}$$

Integrating from $r = r_0 \to \infty$ and $v = 0 \to v_t$, we get

$$v_t = \left(\frac{3L_*Q_{pr}}{8\pi a\rho_s c r_0}\right)^{\frac{1}{2}} \tag{9}$$

(b) Calculate the terminal velocity for a grain radius of 0.1μm, a specific density of $3\,\mathrm{g\ cm}^{-3}$, $r_0 = 10\,\mathrm{pc}$, a radiation pressure efficiency of 1 and a luminosity of $10^4 L_\odot$.

Solution:

The value is calcualted rather simply to be $v_t = 4.1$km/s. With an intense star-forming region that contains many O and B stars, it is conceivable that dust will get accelerated to near the escape velocity of the galaxy. This requires, however, that the stars act in concert which is not generally the case (see Socrates & Ramirez-Ruiz 2006).

(2) Molecular estimate

Consider an electrically neutral medium of diatomic molecules (nuclear mass M_A and M_B) in thermal equilibrium with temperature T. [**Note**: Portions of this problem are discussed in the Solutions to Rybicky & Lightman. You are on your honor to ignore these solutions.]

(a) Derive an expression for the temperature of the gas where you would expect purely rotational emission lines from a substantial fraction of the molecules.

Solution:

Treating the molecule like a rigid rotator, the rotational energies are

$$E_{rot} = \frac{\hbar^2}{2I} J(J+1) \tag{10}$$

where the moment of inertia is $I \sim M a_0^2$. In order for the gas to be excited rotationally, we require $kT \gg E_{rot}$.

However, if we are demanding that no vibrational modes are excited, then we must have $kT \ll E_{vib}$ where

$$E_{vib} \approx E_{rot} \left(\frac{M}{m_e} \right)^{\frac{1}{2}} \tag{11}$$

Therefore, our constraint on the temperature is

$$\frac{\hbar^2}{M a_0^2} \ll kT \ll \frac{\hbar^2}{M a_0^2} \left(\frac{M}{m_e} \right)^{\frac{1}{2}} \tag{12}$$

(b) What range of temperatures would be appropriate for the CO molecule which has rotational constant $B = 2.77$K?

Solution:

Estimating M/m_e for CO, we get $(M/m_e)^{\frac{1}{2}} \sim 100$.

$$2B \ll T \ll 200B \tag{13}$$

(3) CO rotational emission [Consult Tielens, Chapter 2]:

(a) Assuming optically thin emission in LTE, what is the expected intensity of the $J = 1 \rightarrow 0$ line for a column of $10^{15}\,\mathrm{cm}^{-2}$ CO molecules at $T = 10\mathrm{K}$?

Solution:

For an optically thin gas ($\kappa_\nu \approx 0$) our radiative transfer equation reduces to:

$$I_\nu = \int j_\nu ds \tag{14}$$

The emissivity j_ν of the gas in LTE is simply

$$j_\nu = \frac{A_{10} n_{\mathrm{CO}}^{J=1} h\nu_{10}}{4\pi} \tag{15}$$

where $A_{10} = 7.2 \times 10^{-8}\,\mathrm{s}^{-1}$ and $\nu_{10} = 115.3\,\mathrm{GHz}$. Note that this is the total energy emitted by the gas due to the $J = 1 \rightarrow 0$ transition, i.e. we are *not* considering the frequency depedence due to the line-profile ψ_ν.

And, in LTE, the excitation of the rotational levels are populated according to the Boltzman distribution:

$$\frac{n_{\mathrm{CO}}^{J}}{n_{\mathrm{CO}}} = \frac{g_J \exp(-E_J/kT)}{\sum_J g_J \exp(-E_J/KT)} \tag{16}$$

We can perform the sum from $J = 0, 20$ noting that $E_J = J(J+1)B$ with $B = 2.77\mathrm{K}$ and find that $n_{J=1} \approx 0.435 n_{\mathrm{CO}}$.

Finally, $\int n_{\mathrm{CO}}\, ds = N_{\mathrm{CO}} = 10^{15}\,\mathrm{cm}^{-2}$. Putting it all together we have:

$$I_{10} = \frac{A_{10} N_{\mathrm{CO}}^{J=1} h\nu_{10}}{4\pi} = 1.90 \times 10^{-9}\ \ \mathrm{erg\,cm^{-2}\,s^{-1}} \tag{17}$$

Note, this calculation has ignored the opacity altogether which includes a term due to stimulated emission. We will see below that this means we have somewhat underestimated I_{10}.

(b) For a line width of 2 km/s, what is the corresponding peak brightness temperature?

Solution:

My first solution takes a different (simpler yet not quite exact) path than what the rest of you considered. Namely, adopting the total energy emitted in the line from part (a), the only missing piece is the emissivity line-profile, ψ_ν. It is reasonable to assume it has the form of our Doppler profile, i.e.

$$I_\nu = I_{10}\psi_\nu = I_{10}\phi_\nu \tag{18}$$

where

$$\phi_\nu = \frac{c}{\nu_{10} b \sqrt{\pi}} \exp\left[-\left(\frac{\Delta\nu}{\Delta\nu_D} \right)^2 \right] \tag{19}$$

Evaluated at line center, the line-profile reduces to $c/(\nu_{10} b\sqrt{\pi})$. Finally, in the low frequency limit, we know

$$T_b = I_\nu \frac{c^2}{2\nu^2 k} \tag{20}$$

Evaluated at the line center (peak brightness), we have

$$\begin{aligned} T_b^0 &= I_\nu^0 \frac{c^2}{2\nu^2 k} && (21) \\ &= \frac{I_{10} c}{\nu_{10} b \sqrt{\pi}} \frac{c^2}{2\nu_{10}^2 k} && (22) \\ &= \frac{I_{10} c^3}{2\nu_{10}^3 b k \sqrt{\pi}} && (23) \\ &= 0.34 \ \ \mathrm{K} && (24) \end{aligned}$$

Most of you chose to take the following approach: First, you noted the radiative transfer equation (without a background source) can be written as

$$T_b = T\left(1 - \mathrm{e}^{-\tau_\nu} \right) \tag{25}$$

which for an optically thin gas reduces to $T_b \approx T\tau_\nu$. Many then grabbed an equation out of Tielens for τ_ν that is *wrong* by a factor of $\sqrt{\pi}$ because his is the line averaged opacity not the peak opacity. Allow me to derive the proper expression, starting with $\tau_\nu = \int \kappa_\nu \, ds$ The opacity, including the stimulated emission term in LTE, is:

$$\kappa_\nu = \frac{h\nu_{10} B_{01} n_{\mathrm{CO}}^{J=0} \phi_\nu}{4\pi} \left(1 - \mathrm{e}^{h\nu_{10}/kT} \right) \tag{26}$$

We need to massage this a bit using:

$$B_{01} = \frac{g_1}{g_0} \frac{A_{10} c^2}{2h\nu_{10}^3} \tag{27}$$

and (in LTE)

$$n_{\mathrm{CO}}^{J=0} = n_{\mathrm{CO}}^{J=1} \frac{g_0}{g_1} \exp\left(-h\nu_{10}/kT \right) \tag{28}$$

and our expression for ϕ_ν from above. Taking $\int n_{\mathrm{CO}}^{J=1} \, ds = N_{\mathrm{CO}}^{J=1}$ and evaluating at line-center, we arrive at

$$\tau_\nu^0 = \frac{A_{10} c^3}{8\pi \nu^3 b \sqrt{\pi}} N_{\rm CO}^{J=1} \left[\exp\left(\frac{h\nu_{10}}{kT} \right) - 1 \right] \tag{29}$$

This differs by a factor of $\sqrt{\pi}$ from the one given by Tielens. In the limit that $h\nu_{10} \ll kT$, the expression in the brackets reduces to $h\nu_{10}/kT$ and we recover an expression for T_b^0 that exactly matches the formulation from above (Equation 24). In this limit, there is no stimulated emission.

In our example, however, $kT \approx 0.5 h\nu_{10}$ so that it isn't really proper to ignore stimulated emission. Continuing on with our values, I find $\tau_\nu^0 = 0.0457$. Even plugging into our 'full' expression for T_b (Equation 25), we find $T_b = 0.45$K which is a bit higher than my value from above (Equation 24). Again, our estimate from part (a) is actually a bit too low because we neglected stimulated emission.

(c) What is the expected optically thick peak brightnes temperature for the line?

Solution:

This one is rather straightforward. Our equation for the brightness temperature (Equation 25) shows us that $T_b = T$ when $\tau_\nu \gg 1$. Therefore, $T_b = 10$K in the optcially thick limit.

(d) The measured peak brightness temperatures for the two main isotopes of CO are $T_B(^{12}\mathrm{C}^{16}\mathrm{O}) = 12.4$K and $T_B(^{13}\mathrm{C}^{16}\mathrm{O}) = 4.9$K. Assuming that $^{12}\mathrm{C}^{16}\mathrm{O}$K is optically thick and $^{13}\mathrm{C}^{16}\mathrm{O}$ is optically thin and adopting $^{12}\mathrm{C}^{16}\mathrm{O}/^{13}\mathrm{C}^{16}\mathrm{O} = 65$, what is the column density of $^{12}\mathrm{C}^{16}\mathrm{O}$?

Solution:

We have learned above that for an optically thick gas (e.g. $^{12}\mathrm{C}^{16}\mathrm{O}$) that the peak brightness temperature provides an estimate of the gas temperature, i.e. $T_b = T$. Therefore, we infer $T = 12.4$K. The downside of being optically thick, is that one observes only a skin layer of that gas from a cloud that could be extremely massive. To infer the mass, one must observe an optically thin tracer, e.g. $^{13}\mathrm{C}^{16}\mathrm{O}$.

For an optically thin gas, we can repeat our calculations from above for $^{13}\mathrm{C}^{16}\mathrm{O}$ and we shall assume $b = 2$km/s (this should have been given). Because $^{13}\mathrm{C}^{16}\mathrm{O}$ has an extra neutron, the moment of intertia $I = \sum_i m_i r_i^2$ will be larger implying a smaller rotational constant $B \propto I^{-1}$. Morton & Noreau 1994 give a rotational constant $B^{13} = 2.64$K implying $\nu^{13} = 110.2$GHz. And, because $A \propto \nu^3$, we expect its A value to be $A^{13} \approx A^{12}(\nu^{13}/\nu^{12}) \approx 6.3 \times 10^{-8}\,\mathrm{s}^{-1}$. Repeating the calculation from part (b) but solving for $N(^{13}\mathrm{C}^{16}\mathrm{O})$, I find:

$$\tau^0 = -\ln\left[1 - T_b^{13}/T\right] = 0.52 \tag{30}$$

$$N_{\rm C^{13}O}^{J=1} = 6.9 \times 10^{15}\ \mathrm{cm}^{-2} \tag{31}$$

$$N_{\rm C^{13}O} = 1.8 \times 10^{16}\ \mathrm{cm}^{-2} \tag{32}$$

$$N_{\rm C^{12}O} = 1.2 \times 10^{18}\ \mathrm{cm}^{-2} \tag{33}$$

where the last relation assumes $^{12}\mathrm{C}^{16}\mathrm{O} = 65 ^{13}\mathrm{C}^{16}\mathrm{O}$. Note that this gas is marginally, optically thin but that the formalism still applies just fine.