# Water Wave Reflection Paradox

Appropo the discussion in class of the reflection of water waves from a moving "mirror." The paradox arose when we discovered that for some combinations of speeds there appeared to be no possible reflected mode.

To clarify this, I give the following example. Use the dispersion relations, two branches of the square root:

*In[1]:=*

```
OmegaR[k_] := Sqrt[-k]
```

*In[2]:=*

```
OmegaL[k_] := Sqrt[k]
```

You can use the same techniques as in the problem solutions to find the wave-crest envelopes corresponding to these dispersion relations.

*In[3]:=*

```
fnR[k_] := k x + OmegaR[k] t
```

*In[7]:=*

```
dfnR[k_] := x + OmegaR'[k] t
```

*In[11]:=*

```
gp = Solve[{fnR[k]==1,dfnR[k]==0}, {x,t}][[1]]
```

*Out[11]=*

```
          1           -2 Sqrt[-k]
{x -> -(-), t -> -------------}
          k                k
```

These are parametric equations for the envelope. In this simple case we can give an explicit equation for the envelope:

*In[12]:=*

```
(t^2 - 4 x) /. gp
```

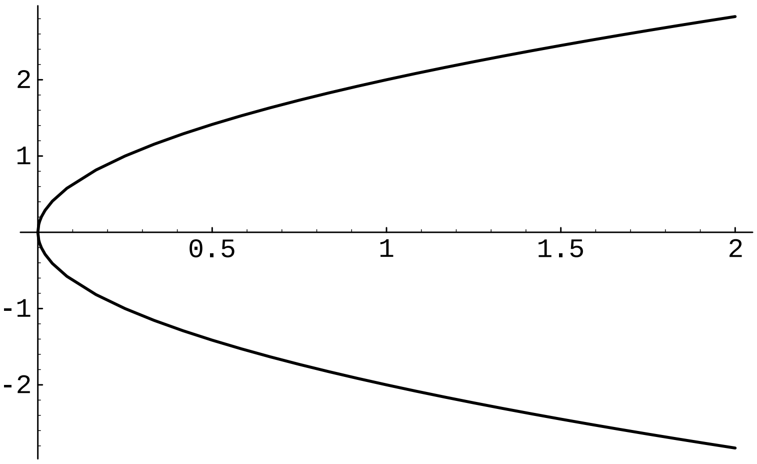
*Out[12]=*

```
0
```

You can see that the envelope is a parabola opening off to the right.

*In[13]:=*

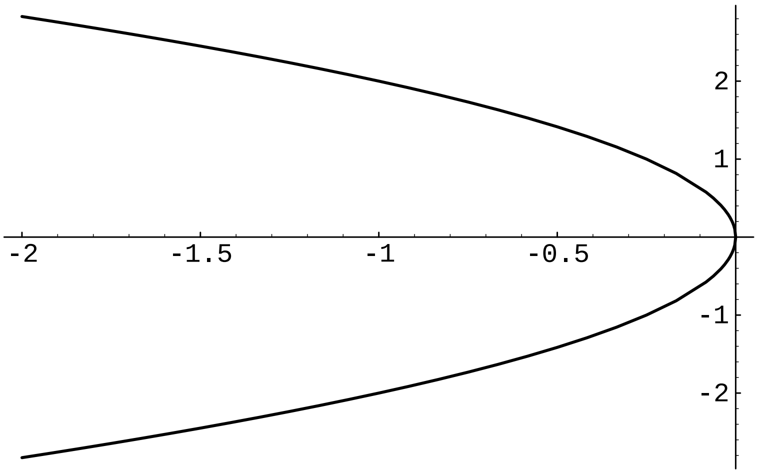
```
Plot[{Sqrt[4 x], -Sqrt[4 x]}, {x, 0, 2}]
```




The other branch is the parabola facing the other way

*In[14]:=*

```
Plot[{Sqrt[-4 x], -Sqrt[-4 x]}, {x, -2, 0}];
```




Now let us take an explicit wave for the incoming wavetrain. Since all waves are equivalent, this entails no loss of generality.

*In[16]:=*

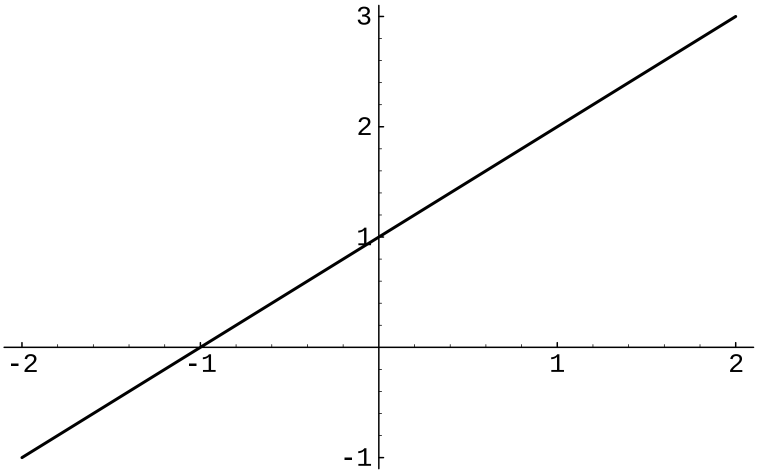
```
ka = -1; wa = OmegaR[ka]
```

*Out[16]=*

```
1
```

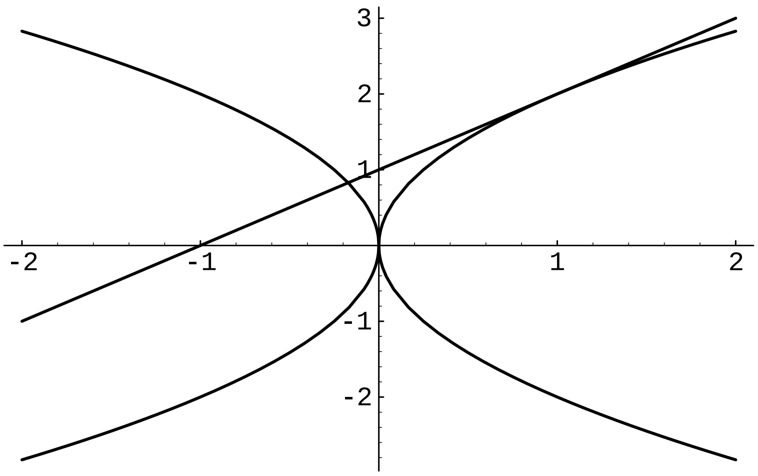
*In[17]:=*

```
Plot[ Evaluate[(1 - ka x)/wa], {x, -2, 2}];
```



*In[20]:=*

```
Show[%13, %14, %17];
```



To find the paradox, we need to consider a motion of the mirror so that a line moving with the speed of the mirror through the origin hits the wavecrest line above inside of the lefthand parabola. A wave mode that will do this will be

*In[24]:=*

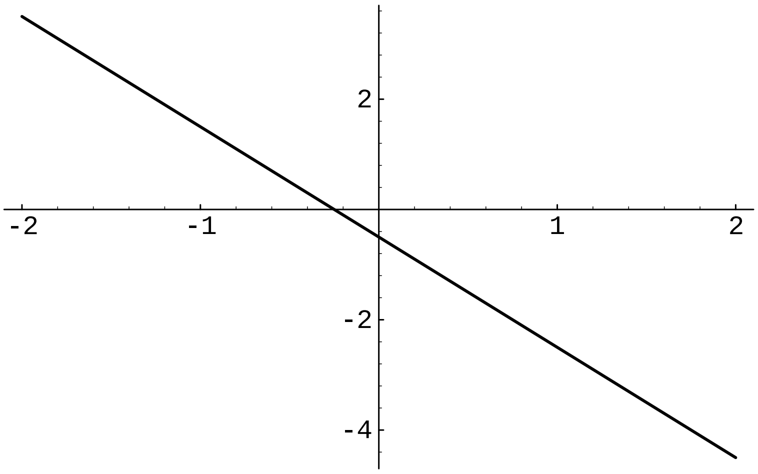
```
kb = -4; wb = -OmegaR[kb]
```

*Out[24]=*

```
-2
```

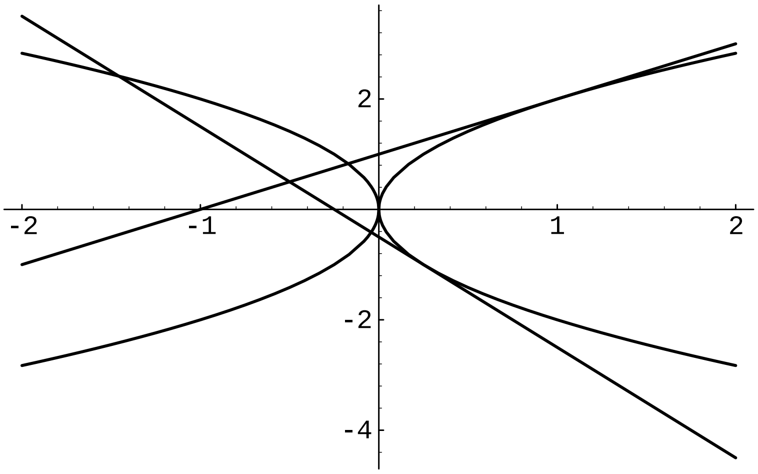
*In[25]:=*

```
Plot[ Evaluate[(1 - kb x)/wb], {x, -2, 2}];
```




*In[28]:=*

```
Show[ %13, %14, %17, %25];
```




The intersection of these two wavecrests will give a point which, together with the origin, defines a line with slope such that a mirror carried along this line has both of the phase gradients pulling back to the same quantity. I.e. these waves can couple through linear boundary conditions on the moving line since they have the same relative phase. Clearly no other solutions can be found. The intersection point is the soution to

*In[29]:=*

```
Solve[{ka x + wa t == 1, kb x + wb t == 1},{x,t}]
```

*Out[29]=*

$$\{\{x \to -(\frac{1}{2}),\ t \to \frac{1}{2}\}\}$$

The mirror will be moving to the left with a speed

*In[30]:=*

```
(x/t) /. %
```

*Out[30]=*

```
{-1}
```

The second wave mode, however, is moving to the left with a speed

*In[32]:=*

```
OmegaR'[kb]
```

*Out[32]=*

$$-\left(\frac{1}{4}\right)$$

Thus this wave will be overrun by the mirror, and cannot propagate away from the mirror to the left.  That is the way it is.