Rendering Vectors:

The Eight Icons

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[This material is a preliminary version of a section to be included in my book Div Grad Curl Are Dead.]

Drawing vectors? How can there be anything to say about this? What is there beyond drawing a line with an arrowhead on one end?

Well, vectors come in eight different geometric types, and each one has a different rendering, a different icon if you will. Only one of these is the line with an arrowhead on it. All of these come up in electrodynamics.

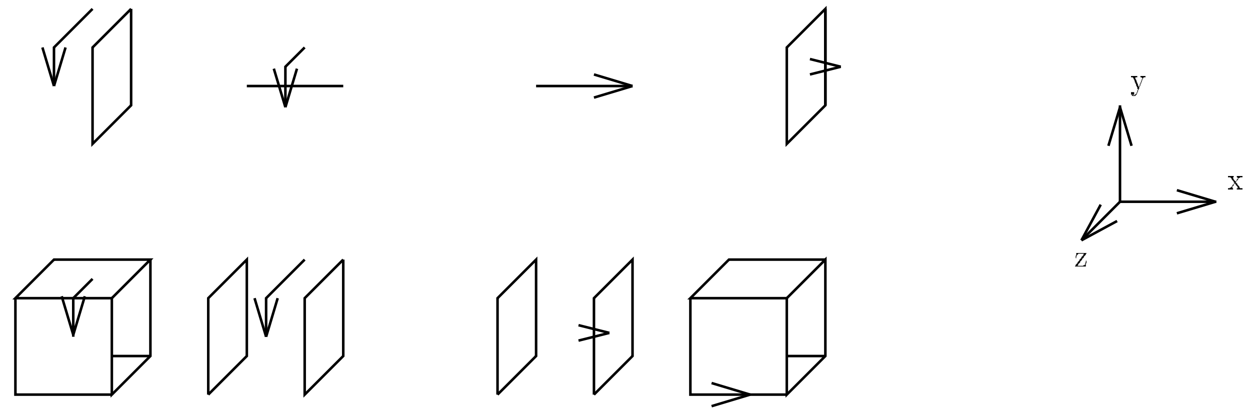


Figure 0-1. The eight icons for vectors. All of these would be conventionally denoted by $\hat{x}$. The electrodynamic fields **B**, **H**, **E**, and **D** would be represented by the icons in the bottom row, in that order.

The objects in all eight classes have three degrees of freedom, and these three components transform in the same way under proper rotations. This justifies calling them all vectors. These objects are stratified into classes by their behavior under other transformations, inversions and dilations. You are probably already familiar with the stratification of vectors and pseudovectors under inversion. By ignoring dilations that stratification missed much of the geometry of the situation, but it was a start.

A reflection about the center line in figure one exchanges an object with one of opposite behavior under inversions. A reflection about a horizontal line connects objects with opposite behavior under dilations. These objects are connected by the usual vector space duality. For example, a field intensity can be integrated along a line. Small pieces of lines look like displacement vectors. Flux densities can be integrated over surfaces, and the dual object looks like a little piece of surface, with a direction through the surface specified. Objects within the right-hand and left-hand squares are connected by the operation called Hodge-duality. Unlike the other two operations, this one involves the metric. In electrodynamics, this is signalled by the presence of the operators ϵ and μ.

Two Dimensional Flow

Let us start by studying the geometrically simpler case of the flow of some conserved stuff (you might think of water) in two dimensions. How would you measure the flow rate? You could find the flux by lowering a container into the flow, and collecting the stuff for some standard time interval.

If the flow were smoothly varying, then over a small enough region, the amount collected would be linear in the length of the region over which the stuff was collected. If you measure the flux for two regions which have different directions, then you can figure out the flux across any other region as well. Just construct a triangular region, with one side the region in question, and the other two parallel to the two measured lines. Using linearity we can find out how much flows in each of these two sides, and use conservation to find out how much flows out the other side. In practice we would take the two regions of measurement parallel to the coordinate lines.

The usual vector representation of this flux comes from a representation of the line across which the flow is to be measured by a vector perpendicular to the line, pointed across the surface in the direction of measurement, and with a length proportional to the length of the line. Call this vector $\mathbf{S}$. The flux vector is the vector $\mathbf{J}$ such that the flux through a surface is given by the dot product $\mathbf{J} \cdot \mathbf{S}$.

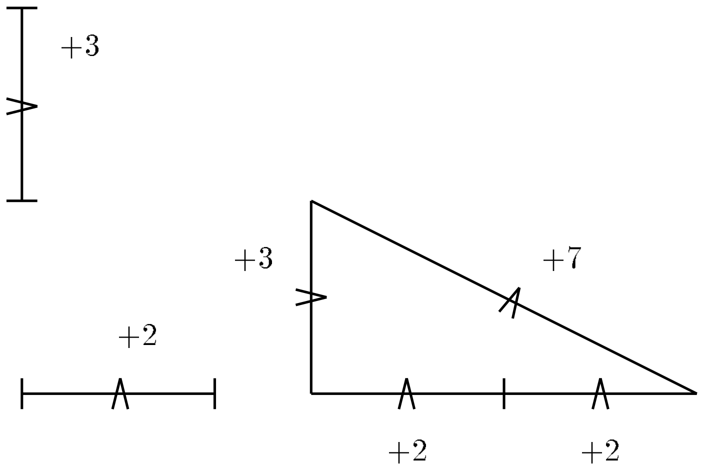


Figure 0-2. Calculating flux across a line in terms of two measurements.

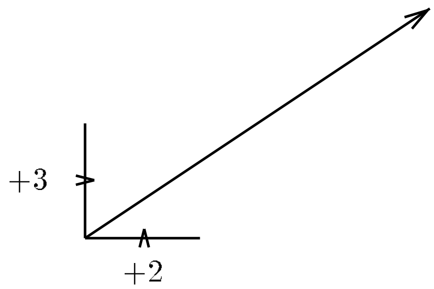


Figure 0-3. Drawing the flux vector corresponding to two orthogonal flow measurements.

Now I show how to draw a picture of this flux vector that we will see is more geometrically correct. The icon will be a pair of parallel lines forming a rectangle, with a direction specified along the lines. Such a picture can be specified by two perpendicular vectors, **a** and **b**. These vectors must be related to **J** by the following equations. The vector **b** does not have a unique length.

$$\begin{aligned} \mathbf{a}\cdot\mathbf{b} &= 0, \\ \mathbf{a}\cdot\mathbf{J} &= 0, \\ \mathbf{b}\cdot\mathbf{J} &> 0, \\ \mid \mathbf{a} \mid\mid \mathbf{J} \mid &= 1. \end{aligned}$$

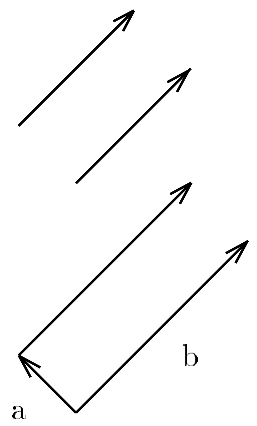


Figure 0-4. The geometrically correct icon for the flux vector, and its representation by two ordinary vectors.

Why do I consider this icon more geometrically natural? Look at the next figure, which shows two representations of a flow question. In the first case, you can immediately see that there are two units flowing across the given line. In the second, you can't do anything until I also tell you what a unit length vector is. Now we are going to do the same geometrical constructions for three dimensions.

Three Dimensional Vector Icons

The most familiar objects are in the class containing dispacement vectors and velocities. Under an inversion, taking

$$\begin{aligned} x &\mapsto -x, \\ y &\mapsto -y, \\ z &\mapsto -z. \end{aligned}$$

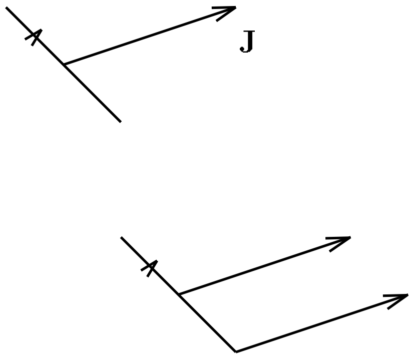


Figure 0-5. Two representations of the flux vector used to compute the flux across a line segment.

these dispacement vectors change sign. Under a dilation

$$\begin{aligned} x \mapsto &\alpha x, \\ y \mapsto &\alpha y, \\ z \mapsto &\alpha z. \end{aligned}$$

the magnitude of the vector scales by the same factor of α. This is the same scaling behavior as the arrow icon.

The next class of vectors are what I call field intensity vectors, such as the electric field in electrodynamics, and gradients in general. Under the above dilation the magnitude of a gradient scales like $1/\alpha$. Stretching a function decreases it gradient. The icon for a field intensity vector that has this scaling behavior is to take a pair of parallel planes. The shape and extent of the planes is irrelevant. Only the spacing matters. Furthermore, there is a mark on one of the planes to denote the "uphill" direction.

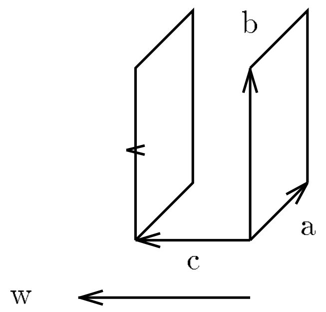
If we describe the icon by three ordinary vectors, $\mathbf{a}, \mathbf{b}, \mathbf{c}$, then the icon is related to the usual gradient vector $\mathbf{w}$ by the relations

$$\begin{aligned} \mathbf{w} \cdot \mathbf{c} =&1, \\ \mathbf{w} \times (\mathbf{a} \times \mathbf{b}) =&0. \end{aligned}$$

The lengths of $\mathbf{a}$ and $\mathbf{b}$ are irrelevant, as are changes of shape such as

$$\mathbf{a} \mapsto \mathbf{a} + \lambda \mathbf{b}.$$

Figure 0-6. The field intensity icon.



The next class of vectors are what I call flux density vectors. The prototypes in electrodynamics are the **D** field and the current density. These measure how much of a given stuff passes through some particular surface. If we dilate by a factor of α, the flux changes by $1/\alpha^2$. The icon with this invariance is a rectangular prism of indefinite length and shape, but with a specific direction and cross-sectional area. If we represent the icon by three vectors, the the relation between the usual vector **w** and the icon is given by

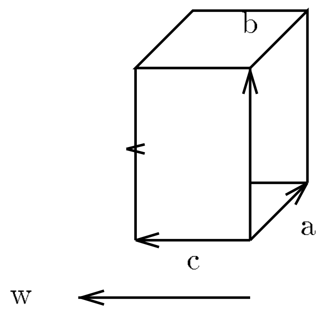
$$\mathbf{w} \cdot (\mathbf{a} \times \mathbf{b}) = \mathrm{sgn}(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}),$$
$$\mathbf{w} \times \mathbf{c} = 0.$$

The term involving the sign of the triple product is there to make the direction of **c** agree with the direction of **w**, and to make it independent of an interchange of **a** and **b**.

If we leave out the factor involving the sign of the triple product, then we have a similar icon, but with a direction of circulation specified around the prism, rather than a direction along the prism. This is the class of vector fields that the magnetic field **B** belongs to.

Finally, if we add a sign term to the field intensity vector, we find the icon for the **H** field in electrodynamics. I call this a twisted field intensity, and the **B** field a twisted flux density. These are shown in figure one.

Figure 0-7. The flux density icon.



Discussion

There are five reasons to use the calculus of differential forms rather than the usual vector calculus: ease of computations, need to go to dimension other than three, need to use a metric other than Euclidean, need to use oblique coordinates, and desire to draw geometrically correct pictures. By geometrically correct, I mean that the pictures have the same invariances as the concepts themselves. This work here has addressed only the last point.

Mathematicians call field intensity vectors 1-forms, and twisted flux density vectors 2-forms. The flux density vectors are called twisted 2-forms, and the twisted field intensities are called twisted 1-forms. If you want to work in a manner that is not specific to three dimensions, it makes more sense to call something twisted when it as the sign of the triple product in its definition.