2.1 Introduction

There are three basic approaches to the theory of plasmas. The first one met by most students of plasma physics is the description in terms of particle orbit theory, and this has been set out in Chapter 1. Each particle is considered individually, and its motion followed under the influence of the Lorentz force

$$\boldsymbol{F} = m\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = q(\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})). \tag{2.1.1}$$

This approach is very valuable for gaining physical insight into plasma behaviour. We discover that particles move on helical trajectories, along field lines; we discover a wide variety of mechanisms causing particles to drift across the magnetic field, and the physical origin is revealed of currents which appear in magnetically confined plasma. But the model tacitly assumes the presence of, for example, an electrical field $\boldsymbol{E}$. It offers no explanation of the origin of such a field, and indeed we have seen in the physics of Debye shielding that such a field cannot simply be imposed by external means. It often arises self-consistently from the cooperative motion of all the plasma charges, and cannot be described by the single-particle approach. An alternative way of presenting the limitations of the single-particle model is to observe that the variables are particle position and velocity: useful concepts, but not measurable or even knowable, and therefore not relatable to experiment. We wish to deal with measurable variables such as fluid velocity and particle density, and these cannot be found from a single-particle model, since they depend in a very complex manner on the motion of the individual particles.

For these reasons, most plasma physics problems are handled using a fluid model, in which the plasma is described by a modified set of fluid equations, incorporating the electromagnetic forces which are so central to plasma behaviour. These fluid equations are not self-evident, however, and must be derived on some firm and reliable basis. That basis is the subject of this chapter.

The kinetic theory of plasma is the most fundamental description of the plasma state; we shall develop the essential core of the approach and show how the fluid equations can be derived, and, very importantly, their reliability assessed. We shall define distribution functions for the particle species, and show how measurable variables like density, velocity and temperature can be obtained by averaging over the distribution.

In case these objectives appear a little dull, however virtuous, we shall then proceed to show that the kinetic theory is much more than a necessary foundation for the practical, and useful, fluid approach. It assumes very great significance and importance due to the low collisionality of many plasmas. Indeed, most of the major phenomena characterizing the plasma state appear in collisionless plasmas, where the cooperative motion causing fluid-like behaviour is not due to collisions (as in a gas) but due to electromagnetic coupling of the particles. It is quite common, therefore, for particle species in a plasma to have distinctly non-Maxwellian distribution functions, the physical effects of which cannot be described using a fluid theory. We shall give some examples of this important feature later in this chapter, and reveal some physics which, when first discovered, was quite startlingly new.

2.2 The distribution function

How do we begin to describe a plasma in detail? If there are N particles, and each particle has a position and velocity $\boldsymbol{x}_i$ and $\boldsymbol{v}_i$ $(i = 0, \ldots, N)$, then the total number of coordinates is $6N$. The actual configuration of the system at any given time is represented by a single point in the $6N$-dimensional space $(\boldsymbol{x}_1 \ldots \boldsymbol{x}_N, \boldsymbol{v}_1 \ldots \boldsymbol{v}_N)$. Some advanced texts (for example, Clemmow and Dougherty, 1969) adopt this as the way into the problem.

The decription has, of course, to be simplified at some stage, and we shall do that now. We may reduce the description down to six dimensions $(\boldsymbol{x}, \boldsymbol{v})$, the coordinate space for a single particle, and define a distribution function $f(\boldsymbol{x}, \boldsymbol{v}, t)$. Using this, we find that

$$f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}v_x\, \mathrm{d}v_y\, \mathrm{d}v_z \tag{2.2.1}$$

is then the number of particles in the volume element $\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ at position $\boldsymbol{x}$, and the element $\mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z$ in velocity space with velocity $\boldsymbol{v}$, at time t. The $(\boldsymbol{x}, \boldsymbol{v})$ space is called phase space.

The spatial density of particles, $n(\boldsymbol{x},t)$, can now be obtained by integrating Eq. (2.2.1) over all velocities:

$$n(\boldsymbol{x}, t) = \int_{-\infty}^{\infty} f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}^3\boldsymbol{v}, \tag{2.2.2}$$

where $\mathrm{d}^3\boldsymbol{v}$ means $\mathrm{d}v_x\,\mathrm{d}v_y\,\mathrm{d}v_z$. The velocity distribution function, normalized to unity, is thus

$$f'(\boldsymbol{x}, \boldsymbol{v}, t) = \frac{f(\boldsymbol{x}, \boldsymbol{v}, t)}{n(\boldsymbol{x}, t)}. \tag{2.2.3}$$

If the system is collisional, with collision frequency ν, then after a time which is long compared to the collision time $1/\nu$, equipartition of energy by collisions (i.e. thermalization) will always cause the system to move towards a Maxwellian velocity distribution:

$$f'_{\mathrm{M}} = \left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{3/2} \exp\left(-v^2/v_{\mathrm{T}}^2\right), \tag{2.2.4}$$

where $v_{\mathrm{T}} = (2k_{\mathrm{B}}T/m)^{1/2}$ is the thermal velocity characterizing the distribution. For any given species, this distribution is defined by one parameter only, the temperature T. It is the only distribution for which a temperature can be properly defined.

Several 'mean' velocities can be defined: for example

$$\overline{|v|} = \frac{1}{n}\int_{-\infty}^{\infty} vf\,\mathrm{d}^3\boldsymbol{v} = \left(\frac{8k_\mathrm{B}T}{\pi m}\right)^{1/2} \tag{2.2.5}$$

is the mean speed of the particles. Other examples are the mean rms velocity v_{rms}:

$$v_{\mathrm{rms}} = (3k_\mathrm{B}T/m)^{1/2} \tag{2.2.6}$$

and the mean magnitude of one velocity component:

$$\overline{|v_x|} = (2k_\mathrm{B}T/\pi m)^{1/2}. \tag{2.2.7}$$

The function $f(\boldsymbol{x}, \boldsymbol{v}, t)$ describes a velocity distribution in three-dimensional velocity space. We can also define a distribution of velocity magnitude, $g(v)$, by averaging the Maxwellian distribution over the velocity directions. The result obtained is:

$$g(v) = 4\pi n\left(\frac{m}{2\pi k_\mathrm{B}T}\right)^{1/2} v^2 \exp\left(-v^2/v_{\mathrm{th}}^2\right). \tag{2.2.8}$$

The functions f and g for a Maxwellian distribution are shown in Fig. 2.1.

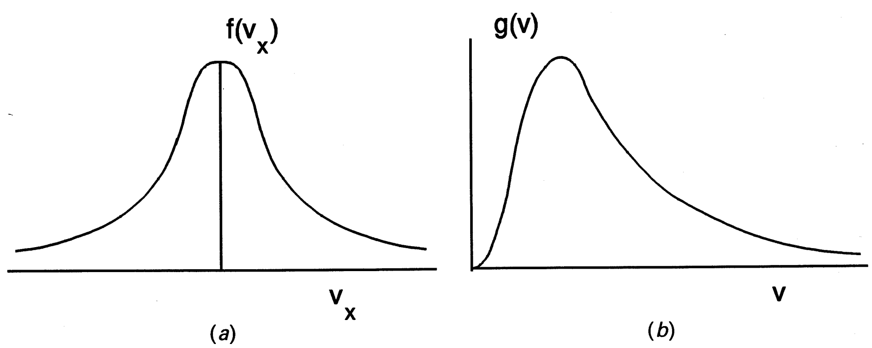

Fig. 2.1. The Maxwellian distribution: (*a*) the function $f(v_x)$; (*b*) the function $g(v)$.

Now it is clear that for any system in which a *true* Maxwellian distribution exists, the interesting physics has disappeared! The plasma would be uniform in space, in general equilibrium, and the free energy would be minimized. In practice, therefore, we work with the concept of a *local* Maxwellian: the plasma is considered to be Maxwellian in velocity distribution, with temperature T, over a region that is small compared to the system but large enough to contain many particles, and on a correspondingly small time scale. The distribution, and hence the density and temperature, are then allowed to vary on larger scales of length and time, and real plasmas can be effectively described in this way.

We shall be mainly concerned with the distribution function $f(\boldsymbol{x}, \boldsymbol{v}, t)$, defined in Eq. (2.2.1). The six-dimensional phase space is difficult to visualize, and illustrations are often given in two-dimensional (x, v_x) space.

It is important to realize that x and v_x (and the corresponding y- and z-components) are *independent* variables, although $v_x = \mathrm{d}x/\mathrm{d}t$ for any particle. The instantaneous velocity of a particle is not a function of its position: a particle can have any velocity at any position. Thus $(\boldsymbol{x}, \boldsymbol{v})$ is merely a coordinate space.

2.3 A reminder: the convective derivative

The time derivative of any property of a fluid, for example the density, has to describe two effects. There can, of course, be an *explicit* variation of the property with time; but there is also the possibility that the property being considered may vary in time because the fluid is moving, and the particular element in question moves to a different location where that property has a different value.

We shall consider a simple fluid, and identify some property, for example the density $n(\boldsymbol{x}, t)$. The time derivative is thus given by

$$\begin{aligned}\frac{\mathrm{d}n}{\mathrm{d}t} &= \frac{\partial n}{\partial t} + \frac{\partial n}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial n}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\partial n}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}t} \\ &= \frac{\partial n}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})n. \qquad (2.3.1)\end{aligned}$$

We have written it in this particular vector form to retain generality, for n could be replaced by a vector property $\boldsymbol{G}$. Note that $(\boldsymbol{v}\cdot\boldsymbol{\nabla})$ is a scalar operator.

In Eq. (2.3.1) $\partial n/\partial t$ is the *explicit* variation with time; $(\boldsymbol{v}\cdot\boldsymbol{\nabla})n$ is the variation because the fluid is moving. At a point moving with the fluid, n can change simply because of the *spatial* variation of n as the fluid element moves to a new region of space.

We call $\mathrm{d}n/\mathrm{d}t$ the convective derivative, and it is taken in the frame of the fluid element. As an illustration of the physical meaning of the two terms in the convective derivative, consider a waterfall. Let the property being considered be the fluid velocity $\boldsymbol{v}$. The water is moving, but the form of the waterfall is not varying with time: the term $\partial \boldsymbol{v}/\partial t$ is in this case zero. An element of the water has a rapidly varying velocity, however, as it moves through the waterfall, and this is described by the term $(\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}$. In the particular case of Niagara, the water is diverted at night through turbine generators and the falls diminish markedly: $\partial \boldsymbol{v}/\partial t$ is finite under those circumstances, as there is an explicit time variation of the system.

2.4 Comparison with the kinetic theory of gases

In gases, the length scale is frequently much larger than the mean free path of the particles, and the time scale much longer than the collision time. Under those circumstances, the system averages itself on a much finer scale of length and time than the macroscopic variables, and so the kinetic theory can readily be used to establish those variables – such as pressure and temperature – by assuming a Maxwellian distribution. The gas behaviour can then be described by fluid theory. Kinetic theory is essential only for low pressures and molecular flow conditions, when the mean free path becomes comparable with the vessel dimensions.

In plasmas, the interaction is via Coulomb forces – long range and weak, by comparison with direct collisions. A single particle moves in the average field due to many others, and actual short range collisional interactions can be rare. The long range interaction actually allows the particles to *avoid* collisions, in loose analogy with the way that we do not bump into other shoppers in the busy High Street because we can see them coming. Relaxation to a Maxwellian distribution is therefore slow, and many processes occur on a faster time scale than the relaxation process. Thus non-Maxwellian distributions are often encountered, and under those conditions kinetic theory is essential. However, for large scale, slow phenomena such as occur in fusion confinement studies, the fluid description is normal and very effective.

2.5 'Moments' of the distribution

We have defined a distribution function $f(\boldsymbol{x}, \boldsymbol{v}, t)$, but it is not very useful in itself, being difficult to obtain experimentally. However, measurable macroscopic variables can be obtained from it as velocity moments. For example, the particle density is obtained by integrating over the velocity distribution:

$$n(\boldsymbol{x}, t) = \int f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}^3\boldsymbol{v}. \tag{2.5.1}$$

The velocity of the fluid, $\boldsymbol{u}(\boldsymbol{x}, t)$ is obtained as the average value of the particle velocity:

$$\boldsymbol{u}(\boldsymbol{x}, t) = \frac{\int \boldsymbol{v} f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}^3\boldsymbol{v}}{\int f(\boldsymbol{x}, \boldsymbol{v}, t)\, \mathrm{d}^3\boldsymbol{v}} = \frac{1}{n} \int \boldsymbol{v} f\, \mathrm{d}^3\boldsymbol{v}. \tag{2.5.2}$$

Note that $\boldsymbol{u}$ and $\boldsymbol{v}$ are quite different quantities: $\boldsymbol{u}$, the fluid velocity, is quite clearly a function of position $\boldsymbol{x}$, whereas $\boldsymbol{v}$, the particle velocity, is not.

The density $n(\boldsymbol{x}, t)$ is called the zeroth order moment of f, abbreviated to $(f\boldsymbol{v}^0)$; $\boldsymbol{u}(\boldsymbol{x}, t)$ is the first order moment $(f\boldsymbol{v})$. The mean particle energy $\frac{1}{2}mv^2$, and hence pressure and temperature, are obtained from the second order moment $(f\boldsymbol{v}\boldsymbol{v})$, and so on.

2.6 The Boltzmann equation

Let us, for now, ignore all particle interactions and consider an assembly of identical, noninteracting particles with distribution $f(\boldsymbol{x}, \boldsymbol{v}, t)$. This may appear to be a worrying and unreasonable assumption, for is not the very essence of a plasma the interaction of the particles? We shall see that the long range Coulomb interactions, which are the cause of cooperative behaviour, can in fact be elegantly accommodated within this model, although short range collisions have to be described separately.

We consider a small volume of six-dimensional phase space $\int \mathrm{d}^3\boldsymbol{x}\, \mathrm{d}^3\boldsymbol{v}$. The surface area in real space is $\int \mathrm{d}\boldsymbol{S}$, and in velocity space is $\int \mathrm{d}\boldsymbol{S}_v$. We then assume conservation of particles: the rate of change of the number of particles in the volume is equal to the net flux of particles into the volume. In $\boldsymbol{x}$ space, the flux is

$$\int f\, \dot{\boldsymbol{x}} \cdot \mathrm{d}\boldsymbol{S} = \int f\, \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{S}. \tag{2.6.1}$$

In $\boldsymbol{v}$ space, the flux is

$$\int f\,\dot{\boldsymbol{v}}\cdot \mathrm{d}\boldsymbol{S}_{\mathrm{v}} = \int f\,\boldsymbol{a}\cdot \mathrm{d}\boldsymbol{S}_{\mathrm{v}}. \tag{2.6.2}$$

Hence,

$$\frac{\partial}{\partial t}\int f\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{v} = -\int f\,\boldsymbol{v}\cdot \mathrm{d}\boldsymbol{S}\,\mathrm{d}^3\boldsymbol{v} - \int f\,\boldsymbol{a}\cdot \mathrm{d}\boldsymbol{S}_{\mathrm{v}}\,\mathrm{d}^3\boldsymbol{x}. \tag{2.6.3}$$

Using Gauss's theorem, we can write:

$$\frac{\partial}{\partial t}\int f\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{v} = -\int \frac{\partial}{\partial \boldsymbol{x}}\cdot(\boldsymbol{v}f)\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{v} - \int \frac{\partial}{\partial \boldsymbol{v}}\cdot(\boldsymbol{a}f)\,\mathrm{d}^3\boldsymbol{x}\,\mathrm{d}^3\boldsymbol{v}, \tag{2.6.4}$$

where we have used the notation $\partial/\partial\boldsymbol{x}$ and $\partial/\partial\boldsymbol{v}$ for divergence, in real space and velocity space, respectively. Since the volume element can be made arbitrarily small, so that all the integrands are constant within the volume, we can write

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \boldsymbol{x}}\cdot(\boldsymbol{v}f) + \frac{\partial}{\partial \boldsymbol{v}}\cdot(\boldsymbol{a}f) = 0. \tag{2.6.5}$$

But we remember that $\boldsymbol{x}$ and $\boldsymbol{v}$ are independent variables: the second term in Eq. (2.6.5) becomes $\boldsymbol{v}\cdot(\partial f/\partial\boldsymbol{x})$. The third term can be similarly simplified if the condition is satisfied that the acceleration $\boldsymbol{a}$, and by implication the associated force $\boldsymbol{F}$, is not a function of $\boldsymbol{v}$: thus

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{a}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0. \tag{2.6.6}$$

Hence we can write

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{\boldsymbol{F}}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0. \tag{2.6.7}$$

This is usually called the collisionless Boltzmann equation.

Note that this equation, Eq. (2.6.7), can also be written

$$\frac{\mathrm{d}f}{\mathrm{d}t} = 0 \tag{2.6.8}$$

and simply states that the convective derivative of f is always zero for a collisionless assembly of particles. A corollary is that, as a particle moves around, it sees a constant f in its local frame. Its local distribution is invariant. This fundamental result is known as Liouville's theorem. A further corollary is that particles move on contours of constant f in phase space, a result that can be useful in visualizing such trajectories (see, for example, Chen, 1984).

The Boltzmann equation lies at the heart of all kinetic theory, and is not specific to plasmas. We begin to introduce plasma physics by the choice of the force $\boldsymbol{F}$. The force we should insert into Eq. (2.6.7) to describe a plasma, in which the particles are each moving under the local electric and magnetic fields, is the Lorentz force

$$\boldsymbol{F} = m\boldsymbol{a} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}). \tag{2.6.9}$$

At first sight, the Lorentz force would appear *not* to satisfy the required condition of being independent of $\boldsymbol{v}$. The x-component of $(\boldsymbol{v}\times\boldsymbol{B})$, however, is not a function of v_x; a_x is thus independent of v_x, and similarly for the y- and z-components. Since the only derivatives of $\boldsymbol{a}$ appearing in the expansion of Eq. (2.6.5) are of the form $\partial a_x/\partial v_x$ and so on, the introduction of the Lorentz force is, happily, quite in order.

2.7 The Vlasov equation

Inserting the Lorentz force explicitly into Eq. (2.6.7), we obtain the Vlasov equation:

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0, \tag{2.7.1}$$

which lies at the heart of plasma physics.

We must now address the problem we created when we assumed that the particles were all noninteracting, when we know that interactions are the very essence of plasma behaviour. In fact this is not much of a problem. We can justify using the Vlasov equation in the following way.

First, we assume no collisions – no short range, local interactions. These are certainly not described by the Vlasov equation as presented. In collisionless plasmas, each particle moves in the average Coulomb field due to thousands of others, and this is in fact our second assumption – that the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in the equation are fields due to the rest of the plasma, and *they* describe the interaction of the particles. They are often called self-consistent fields. Any externally applied fields can also be included, of course. This separation of the collisional and long range interactions is valid only if the Debye sphere contains a large number of particles, and the plasma is therefore a truly cooperative system, but this is usually the case.

The field $\boldsymbol{E}$, and in general also the field $\boldsymbol{B}$, which are determined by the distribution in the rest of the plasma, both depend on the distribution function f. Thus the Vlasov equation is nonlinear, and analytic solutions are in general not possible.

We should also add that a Vlasov equation is needed for each separate species of particle in the plasma, with different mass, charge, and distribution function f.

In a few specialized applications, where temperatures are very high, relativistic effects may need to be taken into account, and the relativistic Vlasov equation is then needed. To obtain this, only a small modification to Eq. (2.7.1) is needed: we put

$$\boldsymbol{p} = m\boldsymbol{v} = \gamma m_0 \boldsymbol{v},$$

where m_0 is the rest mass. The Vlasov equation then becomes

$$\frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{m}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \boldsymbol{F}\cdot\frac{\partial f}{\partial \boldsymbol{p}} = 0, \tag{2.7.2}$$

where

$$\boldsymbol{F} = \boldsymbol{F}_{\text{ext}} + q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

2.8 The effect of collisions

We must now include the effect of collisions of the type which we have so far excluded from the description: short range interactions not included in the effect of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$.

If the plasma is only partially ionized, collisions with neutrals will occur. In a fully ionized plasma, as well as the cooperative effects described by the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ ($\boldsymbol{B}$ will often contain an externally applied component), particles do in fact undergo microcollisions, i.e. they are gradually deflected by large numbers of small deflections due to local Coulomb interactions. This is the cause of the ultimate thermalization. These collisions are not described by the Vlasov equation, and are treated by the inclusion of a collision term $(\partial f/\partial t)_c$:

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})\cdot\frac{\partial f}{\partial \boldsymbol{v}} = \left(\frac{\partial f}{\partial t}\right)_c. \tag{2.8.1}$$

One has to choose a suitable representation for the collision term. The simplest is the Krook collision term (Bhatnagar, Gross and Krook, 1954):

$$\left(\frac{\partial f}{\partial t}\right)_c = -\left(\frac{f - f_m}{\tau}\right), \tag{2.8.2}$$

where f_m is the Maxwellian equilibrium distribution function, to which the system is tending, and τ is the mean (constant) collision time. Eq. (2.8.2) integrates to yield

$$f(t) = f_m + (f(0) - f_m)\exp(-t/\tau). \tag{2.8.3}$$

The model is not very good if the masses of the colliding species are very different, such as in electron-neutral collisions. Under such circumstances, the collision time τ should be replaced by $(m_n/2m_e)\tau$. Discussions of the Krook collision approximation are given by Boyd and Sanderson (1969) and Clemmow and Dougherty (1969).

2.9 The Fokker–Planck equation

In fully ionized plasmas, the collisionality is the effect of many small Coulomb collisions, and the usual procedure is to use the Fokker–Planck equation for the collision term. It has its origins in the study of Brownian motion, a similar phenomenon in that there also the deflections of the large suspended particles are due to the accumulation of many microdeflections. We may derive it in the following way.

We define the probability $\psi(\boldsymbol{v}, \Delta\boldsymbol{v})$ that a particle initially with velocity $\boldsymbol{v}$, undergoing many microcollisions, acquires an increment of velocity $\Delta\boldsymbol{v}$ in a time Δt. We remember that $f(\boldsymbol{x}, \boldsymbol{v}, t)$ is itself a probability distribution, and its form at time t can be written as a product of the distribution at a time Δt earlier, multiplied by the probability of change in the time Δt, and integrated over the possible $\Delta\boldsymbol{v}$:

$$f(\boldsymbol{x}, \boldsymbol{v}, t) = \int f(\boldsymbol{x}, \boldsymbol{v} - \Delta\boldsymbol{v}, t - \Delta t)\psi(\boldsymbol{v} - \Delta\boldsymbol{v}, \Delta\boldsymbol{v})\, d^3(\Delta\boldsymbol{v}). \tag{2.9.1}$$

Using Taylor's theorem, we can expand the product $f\psi$ inside the integral to second order:

$$f(\boldsymbol{x},\boldsymbol{v},t) = \int \mathrm{d}^3(\Delta\boldsymbol{v})\left[f(\boldsymbol{x},\boldsymbol{v},t-\Delta t)\psi(\boldsymbol{v},\Delta\boldsymbol{v}) - \Delta\boldsymbol{v}\cdot\left(\frac{\partial}{\partial\boldsymbol{v}}(f\psi)\right)\right.$$
$$\left. + \tfrac{1}{2}\Delta\boldsymbol{v}\Delta\boldsymbol{v}:\left(\frac{\partial^2}{\partial\boldsymbol{v}\,\partial\boldsymbol{v}}(f\psi)\right)\right]. \tag{2.9.2}$$

(We are using standard tensor notation here. A brief summary will be found in Appendix 2.17.)

We now work through the integrals in Eq. (2.9.2), remembering that $\int \psi \mathrm{d}^3(\Delta\boldsymbol{v}) = 1$, by definition:

$$f(\boldsymbol{x},\boldsymbol{v},t) = f(\boldsymbol{x},\boldsymbol{v},t-\Delta t) - \frac{\partial}{\partial\boldsymbol{v}}\cdot(f\langle\Delta\boldsymbol{v}\rangle) + \frac{1}{2}\frac{\partial^2}{\partial\boldsymbol{v}\,\partial\boldsymbol{v}}:(f\langle\Delta\boldsymbol{v}\Delta\boldsymbol{v}\rangle) \tag{2.9.3}$$

where

$$\langle\Delta\boldsymbol{v}\rangle = \int \psi\Delta\boldsymbol{v}\,\mathrm{d}^3(\Delta\boldsymbol{v})$$

and

$$\langle\Delta\boldsymbol{v}\Delta\boldsymbol{v}\rangle = \int \psi\Delta\boldsymbol{v}\Delta\boldsymbol{v}\,\mathrm{d}^3(\Delta\boldsymbol{v}).$$

Now again by definition,

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} = \frac{f(\boldsymbol{x},\boldsymbol{v},t) - f(\boldsymbol{x},\boldsymbol{v},t-\Delta t)}{\Delta t},$$

and hence

$$\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}}\Delta t = -\frac{\partial}{\partial\boldsymbol{v}}\cdot(f\langle\Delta\boldsymbol{v}\rangle) + \frac{1}{2}\frac{\partial^2}{\partial\boldsymbol{v}\,\partial\boldsymbol{v}}:(f\langle\Delta\boldsymbol{v}\Delta\boldsymbol{v}\rangle), \tag{2.9.4}$$

which is the Fokker–Planck equation.

The physical meaning of the first term on the right of Eq. (2.9.4) can be understood by noting that $\langle\Delta\boldsymbol{v}\rangle/\Delta t$ is an acceleration, or force/unit mass. This term therefore describes the frictional force slowing down fast particles and accelerating slow ones. The negative divergence in velocity space describes a narrowing of the distribution.

In the second term, $\langle\Delta\boldsymbol{v}\,\Delta\boldsymbol{v}\rangle/\Delta t$ is a coefficient of diffusion in velocity space. This term then describes the fact that a narrow velocity distribution (e.g. a beam) will broaden as a result of collisions. The two terms thus operate in opposite senses, and are in balance for an equilibrium (Maxwellian) distribution.

The actual physics of the collisional process is contained in the function $\psi(\boldsymbol{v},\Delta\boldsymbol{v})$, and this function is commonly derived using a Rutherford scattering model (see, for example, Krall and Trivelpiece, 1973). We shall not delve further into the details of collisional processes in this chapter, but we will include the effects of collisions in the later development of the fluid equations.

2.10 The equivalence of kinetic theory and orbit theory

In this section, we follow the useful approach given by Boyd and Sanderson (1969) and Clemmow and Dougherty (1969).

The collisionless Boltzmann equation is the root of kinetic theory:

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + \frac{\boldsymbol{F}}{m}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = 0. \tag{2.10.1}$$

The base equation of orbit theory is Newton's law: $m\,\mathrm{d}^2\boldsymbol{x}/\mathrm{d}t^2 = \boldsymbol{F}$. Being a second order differential equation in three dimensions, the general solution of Newton's equation must contain six constants of integration, $\alpha_1, \ldots, \alpha_6$. We write the solution

$$\left.\begin{aligned} \boldsymbol{x} &= \boldsymbol{x}(\alpha_1, \ldots, \alpha_6, t) \\ \boldsymbol{v} &= \boldsymbol{v}(\alpha_1, \ldots, \alpha_6, t). \end{aligned}\right\} \tag{2.10.2}$$

In principle, we can formally solve these six scalar equations for the α_i:

$$\alpha_i = \alpha_i(\boldsymbol{x}, \boldsymbol{v}, t), \quad i = 1\text{–}6. \tag{2.10.3}$$

Now any arbitrary function of the α_i, $f = f(\alpha_i, \ldots, \alpha_6)$, is a solution of the Boltzmann equation above; we can show this by substituting the function f directly into Eq. (2.10.1):

$$\sum_i \left(\frac{\partial \alpha_i}{\partial t} + \boldsymbol{v}\cdot\frac{\partial \alpha_i}{\partial \boldsymbol{x}} + \frac{\boldsymbol{F}}{m}\cdot\frac{\partial \alpha_i}{\partial \boldsymbol{v}}\right) = \sum_i \frac{\partial f}{\partial \alpha_i}\frac{\mathrm{d}\alpha_i}{\mathrm{d}t} \equiv 0. \tag{2.10.4}$$

The result is identically zero because the α_i's are constants.

Thus the general solution of the Vlasov equation is an arbitrary function of the integrals of Newton's law, and the two approaches are equivalent. This was first shown by Jeans in connection with stellar dynamics, and is sometimes called Jeans's theorem (Chandrasekhar, 1960).

2.11 The fluid equations

We have seen in Section 2.5 that real physical quantities are derived as moments of the distribution function. We can therefore expect to obtain physical equations relating macroscopic variables such as density, fluid velocity and pressure by taking moments of the Vlasov equation, Eq. (2.7.1). This is how we proceed.

We will first derive the equation of continuity, describing conservation of particles for each species. This equation follows from taking the zeroth order moment. The first order moment will give us the force balance equation, describing conservation of momentum. The second order moment gives us an equation describing conservation of energy; however, the derivation is quite lengthy and the reader will be referred to specialist texts for a full treatment. We shall also discover that this sequence of equations obtained from moments of the Vlasov equation is in principle not closed: each equation contains quantities which have to be derived from the next higher order equation. An approximation has to be found at some stage, which allows the set of equations to be closed in practice.

2.11.1 *The zeroth order moment*

We derive first, then, the continuity equation describing particle conservation. We could write this down as self-evident, but its derivation is straightforward and a very good illustration of the method.

Taking the zeroth order moment, equivalent to a straight integration of the equation, we obtain:

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\boldsymbol{v} + \int \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} \mathrm{d}^3\boldsymbol{v} + \frac{q}{m}\int (\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial \boldsymbol{v}} \mathrm{d}^3\boldsymbol{v} = \int \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \mathrm{d}^3\boldsymbol{v}. \tag{2.11.1}$$

The first term gives

$$\int \frac{\partial f}{\partial t} \mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial t}\int f\, \mathrm{d}^3\boldsymbol{v} = \frac{\partial n}{\partial t}. \tag{2.11.2}$$

The second term integrates as follows:

$$\int \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} \mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial \boldsymbol{x}}\cdot\int \boldsymbol{v} f\, \mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial \boldsymbol{x}}\cdot(n\bar{\boldsymbol{v}}) \equiv \frac{\partial}{\partial \boldsymbol{x}}\cdot(n\boldsymbol{u}). \tag{2.11.3}$$

We have used the fact that $\boldsymbol{x}$ and $\boldsymbol{v}$ are independent variables. In Eq. (2.11.3), $\boldsymbol{u}$ is the 'fluid' velocity.

The term in $\boldsymbol{E}$ in fact vanishes. We show this by rewriting it using Gauss's theorem in velocity space:

$$\int \boldsymbol{E}\cdot\frac{\partial f}{\partial \boldsymbol{v}} \mathrm{d}^3\boldsymbol{v} = \int \frac{\partial}{\partial \boldsymbol{v}}\cdot(f\boldsymbol{E})\, \mathrm{d}^3\boldsymbol{v} = \int_{\mathrm{s}} f\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S}_{\mathrm{v}} = 0. \tag{2.11.4}$$

The surface integral vanishes if we take the surface S to infinity: the surface area increases as v^2, but a real distribution $f(\boldsymbol{v})$ (for example, a Maxwellian) will in practice always go to zero much faster, typically like $\exp - (v^2/v_{\mathrm{T}}^2)$. The B term also vanishes:

$$\int (\boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial \boldsymbol{v}} \mathrm{d}^3\boldsymbol{v} = \int \frac{\partial}{\partial \boldsymbol{v}}\cdot[f(\boldsymbol{v}\times\boldsymbol{B})]\, \mathrm{d}^3\boldsymbol{v} - \int f\frac{\partial}{\partial \boldsymbol{v}}\cdot(\boldsymbol{v}\times\boldsymbol{B})\, \mathrm{d}^3\boldsymbol{v} = 0. \tag{2.11.5}$$

The first term again reduces to a vanishing surface integral, whereas the second is zero because $\boldsymbol{v}\times\boldsymbol{B}$ is perpendicular to $\partial/\partial\boldsymbol{v}$ (this point was discussed in Section 2.6).

The collision term also vanishes, because

$$\int \left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \mathrm{d}^3\boldsymbol{v} = \left[\frac{\partial}{\partial t}\int f\, \mathrm{d}^3\boldsymbol{v}\right]_{\mathrm{c}} = 0, \tag{2.11.6}$$

since the total number of particles of the species considered must remain constant as collisions proceed. (We do not consider recombination events in this context. Such events would of course remove particles from the distribution, but they are extremely rare, being essentially a three-particle collision.)

All that remains from Eq. (2.11.1) is

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \boldsymbol{x}}\cdot(n\boldsymbol{u}) = 0, \tag{2.11.7}$$

which expresses conservation of particles. It is only a matter of multiplying by m or q to convert this to an equation describing conservation of mass, or charge.

2.11.2 *The closure problem*

This result immediately illustrates the closure problem associated with this procedure: taking the zeroth moment of Eq. (2.7.1) has introduced a higher order moment, the first velocity moment of f, which is $\boldsymbol{u}$. This will have to be obtained by taking the first moment of the equation, but this in turn will introduce a second moment of f, etc. The sequence will have to be terminated by some justifiable procedure. For most purposes, the closure is effected by setting the third velocity moment of f, describing thermal conductivity, to zero. We shall not in fact pursue the details in this treatment, but shall quote results.

2.11.3 *The first order moment*

To take the first order moment of the Vlasov equation, we multiply Eq. (2.7.1) by $m\boldsymbol{v}$ and integrate. This will yield a macroscopic equation describing conservation of momentum. We have

$$m\int \boldsymbol{v}\frac{\partial f}{\partial t}\,\mathrm{d}^3\boldsymbol{v} + m\int \boldsymbol{v}\left(\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)f\,\mathrm{d}^3\boldsymbol{v} + q\int \boldsymbol{v}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial \boldsymbol{v}}\,\mathrm{d}^3\boldsymbol{v} = \int m\boldsymbol{v}\left(\frac{\partial f}{\partial t}\right)\mathrm{d}^3\boldsymbol{v}. \tag{2.11.8}$$

The first term of Eq. (2.11.8) gives

$$m\int \boldsymbol{v}\frac{\partial f}{\partial t}\,\mathrm{d}^3\boldsymbol{v} = m\frac{\partial}{\partial t}\int \boldsymbol{v}f\,\mathrm{d}^3\boldsymbol{v} = m\frac{\partial}{\partial t}(n\boldsymbol{u}). \tag{2.11.9}$$

Taking the third term next, it may be expanded to become:

$$\int \boldsymbol{v}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial f}{\partial \boldsymbol{v}}\,\mathrm{d}^3\boldsymbol{v} = \int \frac{\partial}{\partial \boldsymbol{v}}\cdot[f\boldsymbol{v}(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})]\,\mathrm{d}^3\boldsymbol{v} - \int f\boldsymbol{v}\frac{\partial}{\partial \boldsymbol{v}}\cdot(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\,\mathrm{d}^3\boldsymbol{v}$$
$$- \int f(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})\cdot\frac{\partial \boldsymbol{v}}{\partial \boldsymbol{v}}\,\mathrm{d}^3\boldsymbol{v}. \tag{2.11.10}$$

(Note: products like $\boldsymbol{ab}$ are called tensor products, or dyads: see the appendix, Section 2.17.)

The first integral in Eq. (2.11.10) vanishes, as can be seen by applying Gauss's theorem as we did in Section 2.11.1 earlier; the second integral is also zero, because $\boldsymbol{E}$ is not a function of $\boldsymbol{v}$, and $(\boldsymbol{v}\times\boldsymbol{B})$ is perpendicular to $\partial/\partial\boldsymbol{v}$; and $\partial\boldsymbol{v}/\partial\boldsymbol{v}$ is the identity tensor $\boldsymbol{I}$. This third term therefore reduces to

$$-q\int(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B})f\,\mathrm{d}^3\boldsymbol{v} = -qn(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}). \tag{2.11.11}$$

Let us now consider the second term in Eq. (2.11.8). Remembering that $\boldsymbol{x}$ and $\boldsymbol{v}$ are independent variables, we write

$$\int \boldsymbol{v}\left(\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}\right)f\,\mathrm{d}^3\boldsymbol{v} = \int \frac{\partial}{\partial \boldsymbol{x}}\cdot(f\boldsymbol{vv})\,\mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial \boldsymbol{x}}\cdot\int f\boldsymbol{vv}\,\mathrm{d}^3\boldsymbol{v} = \frac{\partial}{\partial \boldsymbol{x}}\cdot(n\overline{\boldsymbol{vv}}). \tag{2.11.12}$$

At this point, it is helpful to separate the velocity $\boldsymbol{v}$ into a mean (fluid) velocity $\boldsymbol{u}$ and a random (thermal) velocity $\boldsymbol{w}$: $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{w}$. Then

$$\frac{\partial}{\partial \boldsymbol{x}} \cdot (n\overline{\boldsymbol{vv}}) = \nabla \cdot (n\overline{\boldsymbol{vv}}) = \nabla \cdot (n\boldsymbol{uu}) + \nabla \cdot (n\overline{\boldsymbol{ww}}) + \nabla \cdot n(\boldsymbol{u}\bar{\boldsymbol{w}} + \bar{\boldsymbol{w}}\boldsymbol{u}). \tag{2.11.13}$$

The final term here is zero, since $\bar{\boldsymbol{w}} \equiv 0$ by definition. The first term can be written

$$\nabla \cdot (n\boldsymbol{uu}) = \boldsymbol{u}\nabla \cdot (n\boldsymbol{u}) + n(\boldsymbol{u} \cdot \nabla)\boldsymbol{u}. \tag{2.11.14}$$

In the second term, the quantity $mn\overline{\boldsymbol{ww}}$ appears. This clearly has the dimensions of energy density, and is called the stress tensor or pressure tensor $\boldsymbol{P}$. We will discuss the interpretation of this quantity later in this section.

There remains the collision term, which we shall represent as

$$\int m\boldsymbol{v}\left(\frac{\partial f}{\partial t}\right)_{\mathrm{c}} \mathrm{d}^3\boldsymbol{v} = \left[\frac{\partial}{\partial t}\int m\boldsymbol{v}f\,\mathrm{d}^3\boldsymbol{v}\right]_{\mathrm{c}} = \boldsymbol{P}_{ij}. \tag{2.11.15}$$

It represents the rate of change of momentum density due to collisions between different species i and j. Collisions between like particles cannot produce a net change of momentum of that species.

Collecting all the terms together, we have for Eq. (2.11.8), the first moment equation,

$$m\frac{\partial}{\partial t}(n\boldsymbol{u}) + m\boldsymbol{u}\nabla \cdot (n\boldsymbol{u}) + mn(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \cdot \boldsymbol{P} - qn(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{P}_{ij}. \tag{2.11.16}$$

Finally we combine the first two terms using the continuity equation above, Eq. (2.11.7), giving

$$mn\left[\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u}\right] = qn(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) - \nabla \cdot \boldsymbol{P} + \boldsymbol{P}_{ij}. \tag{2.11.17}$$

2.11.4 *The pressure tensor*

Let us now examine the meaning of the pressure tensor $\boldsymbol{P}$, defined in Section 2.11.3 above as $\boldsymbol{P} = nm\overline{\boldsymbol{ww}}$. It can be written

$$\boldsymbol{P} = \begin{pmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{pmatrix}. \tag{2.11.18}$$

$\boldsymbol{P}$ can be interpreted in the following way. Consider a closed surface S in the plasma, and consider a small element $\mathrm{d}S$ of this, with normal vector $\hat{\boldsymbol{n}}$. The force/unit area on $\hat{\boldsymbol{n}}\,\mathrm{d}S$ is

$$-\boldsymbol{P} \cdot \hat{\boldsymbol{n}} = -nm\langle \boldsymbol{w}(\boldsymbol{w} \cdot \hat{\boldsymbol{n}})\rangle, \tag{2.11.19}$$

where $\langle\ \rangle$ signifies an average. This result follows from the following observations: (1) $n(\boldsymbol{w} \cdot \hat{\boldsymbol{n}})$ is the *outward* flux/unit area of particles from the volume enclosed by S; (2) $nm\boldsymbol{w}(\boldsymbol{w} \cdot \hat{\boldsymbol{n}})$ is then the outward flux density in the $\hat{\boldsymbol{n}}$-direction of momentum in the $\boldsymbol{w}$-direction. The minus sign appears because an outward flux gives rise to a *lower* pressure.

Now consider a special case, where $\hat{\boldsymbol{n}}$ is in the x-direction: $\hat{\boldsymbol{n}} = (1, 0, 0)$. Then $-\boldsymbol{P}\cdot\hat{\boldsymbol{n}} = -(p_{xx}, p_{yx}, p_{zx})$. Fig. 2.2 shows that p_{xx} is a *pressure* force, whereas p_{yx} and p_{zx} are *shear* forces. Clearly, $p_{xy} = p_{yx}$, etc. These off-diagonal elements describe viscosity in the plasma: transfer of momentum in directions perpendicular to the particle motion. This may often be ignored in plasmas. In conventional fluids viscous effects are most prominent in the interaction of the fluid with boundaries, such as walls of pipes or confining vessels. In the situations in which plasmas are normally considered, there are no material confining vessels, and no walls. Viscosity therefore has very little importance for plasmas, but when needed it can be included in this term.

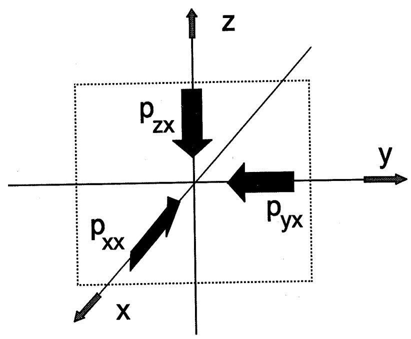

Fig. 2.2. The presssure tensor $\boldsymbol{P}$: three components are shown; p_{yx} and p_{zx} represent shear forces, and p_{xx} represents a pressure force.

The diagonal elements represent normal hydrostatic pressure. In an isotropic plasma, $p_{xx} = p_{yy} = p_{zz} = p$, the scalar pressure. Thus for most purposes, the pressure tensor can be written $\boldsymbol{P} = p\boldsymbol{I}$, where $\boldsymbol{I}$ is the unit tensor, and $\nabla\cdot\boldsymbol{P}$ becomes ∇p, the pressure gradient.

The square bracket in Eq. (2.11.17) is the convective derivative $\mathrm{d}\boldsymbol{u}/\mathrm{d}t$. The equation is the *fluid equation of motion*, and describes the force balance in this component of the plasma. There will be such an equation for each species, but it is often useful to combine them into single-fluid equations describing the neutral plasma. We will develop this in Section 2.12.

2.11.5 *The second order moment*

The second moment is obtained by multiplying Eq. (2.7.1) by $m\boldsymbol{v}\boldsymbol{v}/2$ and integrating. We shall not pursue it here; the procedure is well presented in several texts (for example Bittencourt, 1986). The second order equations, one for each particle species, describe energy conservation within and between the species: when reduced to a single equation for the neutral fluid (a procedure we shall follow in the next section), it becomes

$$p/\rho^{\gamma} = \text{constant}. \qquad (2.11.20)$$

the adiabatic equation of state.

2.12 The single-fluid equations

We first define the appropriate parameters for a neutral plasma: mass density

$$\rho = n_{\mathrm{i}}M + n_{\mathrm{e}}m \approx nM,$$

mass velocity

$$\boldsymbol{v} = \frac{1}{\rho}(n_i M \boldsymbol{u}_i + n_e m \boldsymbol{u}_e) \approx \frac{M\boldsymbol{u}_i + m\boldsymbol{u}_e}{M + m} \approx \boldsymbol{u}_i,$$

current density

$$\boldsymbol{J} = e(n_i \boldsymbol{u}_i - n_e \boldsymbol{u}_e) \approx ne(\boldsymbol{u}_i - \boldsymbol{u}_e). \tag{2.12.1}$$

These approximations indicate that the mass and velocity of the single fluid are essentially provided by the ions. We can therefore write the continuity equation, describing conservation of mass in the neutral plasma, using Eq. (2.11.7) for the ions:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho \boldsymbol{v}) = 0. \tag{2.12.2}$$

We will also need a useful form for the electron–ion collision term $\boldsymbol{P}_{ei}$ in Eq. (2.11.17). Since it describes momentum exchange between species, it is this term which describes resistive effects in the plasma. We present here a simple argument, starting with the resistive Ohm's law $\boldsymbol{E} = \eta \boldsymbol{J}$. Though simple, the procedure allows us to develop an approximate form for the resistive term which is both valid and useful. The accurate treatment of this problem in plasma physics is a difficult challenge.

We write:

$$\text{force/unit volume on the electrons} = ne\boldsymbol{E} = ne\eta\boldsymbol{J} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t, \tag{2.12.3}$$

where $\boldsymbol{p}$ is the momentum density. We equate this rate of change of momentum density to the collision term $\boldsymbol{P}_{ei}$. Thus, using the expression from Eq. (2.12.1) for $\boldsymbol{J}$, we have

$$\boldsymbol{P}_{ei} = n^2 e^2 \eta(\boldsymbol{u}_i - \boldsymbol{u}_e). \tag{2.12.4}$$

The ion–electron collision term $\boldsymbol{P}_{ie}$ is clearly given by $-\boldsymbol{P}_{ei}$: total momentum must be conserved in the collisions between the two fluids. This simple model gives the result that momentum exchange is proportional to the difference in the fluid velocities, and is approximately correct if this difference is fairly small and the distributions are Maxwellian. Assuming the validity of this velocity dependence, we can also write for the rate of change of momentum density

$$\boldsymbol{P}_{ei} = nm\nu_{ei}(\boldsymbol{u}_i - \boldsymbol{u}_e), \tag{2.12.5}$$

where ν_{ei} is the electron–ion collision frequency. Comparing with the previous expression for $\boldsymbol{P}_{ei}$, we obtain the relation between resistivity and collision frequency:

$$\eta = \frac{m\nu_{ei}}{ne^2}. \tag{2.12.6}$$

Finally we find that, since $\boldsymbol{P}_{ei} = -\boldsymbol{P}_{ie}$,

$$m\nu_{ei} = M\nu_{ie}. \tag{2.12.7}$$

It may be a surprise to find that ν_{ei} and ν_{ie} are not equal. This relationship makes clear that it is the mass difference which causes that inequality: if we consider $1/\nu_{ei}$

to be the mean time for an electron to be deviated by $\pi/2$ rad, due to a process involving many small interactions, it is clear that it will take a much larger number of such interactions, and hence a longer time, to deviate an ion by the same amount.

The two fluid equations, from Eq. (2.11.17), are:

$$Mn\left(\frac{\partial \boldsymbol{u}_{\mathrm{i}}}{\partial t} + (\boldsymbol{u}_{\mathrm{i}} \cdot \nabla)\boldsymbol{u}_{\mathrm{i}}\right) = en(\boldsymbol{E} + \boldsymbol{u}_{\mathrm{i}} \times \boldsymbol{B}) - \nabla p_i + \boldsymbol{P}_{\mathrm{ie}}, \tag{2.12.8}$$

$$mn\left(\frac{\partial \boldsymbol{u}_{\mathrm{e}}}{\partial t} + (\boldsymbol{u}_{\mathrm{e}} \cdot \nabla)\boldsymbol{u}_{\mathrm{e}}\right) = -en(\boldsymbol{E} + \boldsymbol{u}_{\mathrm{e}} \times \boldsymbol{B}) - \nabla p_{\mathrm{e}} + \boldsymbol{P}_{\mathrm{ei}}. \tag{2.12.9}$$

We can obtain two powerful equations by taking different linear combinations of these two fluid equations. The first is obtained by adding, and yields the force balance equation for the neutral plasma, and the second is obtained by taking a different linear combination to yield the effective Ohm's law for the system.

Adding the equations, we obtain

$$n\left(\frac{\partial}{\partial t}(M\boldsymbol{u}_{\mathrm{i}} + m\boldsymbol{u}_{\mathrm{e}}) + M(\boldsymbol{u}_{\mathrm{i}} \cdot \nabla)\boldsymbol{u}_{\mathrm{i}} + m(\boldsymbol{u}_{\mathrm{e}} \cdot \nabla)\boldsymbol{u}_{\mathrm{e}}\right)$$
$$= en(\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) \times \boldsymbol{B} - \nabla(p_i + p_{\mathrm{e}}) \tag{2.12.10}$$

(note $\boldsymbol{P}_{\mathrm{ei}} = -\boldsymbol{P}_{\mathrm{ie}}$). Since $p = p_i + p_{\mathrm{e}}$, and using the above definitions for $\boldsymbol{v}, \boldsymbol{J}$, and the approximation $m \ll M$, we obtain

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{v}\right) = \boldsymbol{J} \times \boldsymbol{B} - \nabla p + \boldsymbol{F}, \tag{2.12.11}$$

where we have generalized the equation by including a term $\boldsymbol{F}$ representing any external force, such as gravity. This important equation describes momentum conservation in the neutral plasma, or force balance.

The second equation, known as the *generalized Ohm's law*, is obtained by taking a particular linear combination of the fluid equations. We multiply the ion equation, Eq. (2.12.8), by m, and the electron equation, Eq. (2.12.9), by M, and subtract:

$$Mmn\left(\frac{\partial}{\partial t}(\boldsymbol{u}_{\mathrm{i}} - \boldsymbol{u}_{\mathrm{e}}) + (\boldsymbol{u}_{\mathrm{i}} \cdot \nabla)\boldsymbol{u}_{\mathrm{i}} - (\boldsymbol{u}_{\mathrm{e}} \cdot \nabla)\boldsymbol{u}_{\mathrm{e}}\right)$$
$$= en(M + m)\boldsymbol{E} + en(m\boldsymbol{u}_{\mathrm{i}} + M\boldsymbol{u}_{\mathrm{e}}) \times \boldsymbol{B} - m\nabla p_{\mathrm{i}} + M\nabla p_{\mathrm{e}} - (M + m)\boldsymbol{P}_{\mathrm{ei}}. \tag{2.12.12}$$

Let us first deal with the convective terms on the left-hand side. For many purposes, they may be neglected, and this is the common case. The simplest justification for this is to restrict ourselves to small velocities, in which case these terms, being essentially quadratic in $\boldsymbol{u}_{\mathrm{i}}$ and $\boldsymbol{u}_{\mathrm{e}}$, may be neglected. This, however, is not very satisfactory. A better insight can be gained by roughly assessing the relative magnitude of the terms by a dimensional argument, and we shall do this later in this section. For the moment, we shall neglect the convective terms.

Using our earlier definitions for $\boldsymbol{J}$, Eq. (2.12.1) and $\boldsymbol{P}_{\mathrm{ei}}$, Eq. (2.12.4), the above equation reduces, in the limit $m \ll M$, to

$$\frac{Mmn}{e}\frac{\partial}{\partial t}\left(\frac{\boldsymbol{J}}{n}\right) = e\rho\boldsymbol{E} - Mne\eta\boldsymbol{J} + M\nabla p_e + en(m\boldsymbol{u}_i + M\boldsymbol{u}_e) \times \boldsymbol{B}. \tag{2.12.13}$$

The last term can be simplified by the ingenious device of writing

$$\begin{aligned} m\boldsymbol{u}_i + M\boldsymbol{u}_e &= M\boldsymbol{u}_i + m\boldsymbol{u}_e - (M - m)(\boldsymbol{u}_i - \boldsymbol{u}_e) \\ &\simeq \frac{\rho}{n}\boldsymbol{v} - \frac{M}{ne}\boldsymbol{J}. \end{aligned} \tag{2.12.14}$$

Dividing by $e\rho$ and rearranging terms we finally obtain

$$\frac{m}{ne^2}\frac{\partial \boldsymbol{J}}{\partial t} = \boldsymbol{E} + (\boldsymbol{v} + \boldsymbol{B}) - \frac{1}{ne}(\boldsymbol{J} \times \boldsymbol{B}) + \frac{1}{ne}\nabla p_e - \eta\boldsymbol{J}, \tag{2.12.15}$$

which is indeed a generalized Ohm's law. Each term is an emf, measured in volts/metre: the left-hand side describes the effect of electron inertia, and only matters for very high frequency phenomena. In the special case of uniform, collisionless plasma without magnetic field, Eq. (2.12.15) reduces to

$$\frac{m}{ne^2}\frac{\partial \boldsymbol{J}}{\partial t} = \boldsymbol{E},$$

and putting $\boldsymbol{J} = ne\boldsymbol{v}$ this in turn reduces to Newton's law:

$$m\frac{\partial \boldsymbol{v}}{\partial t} = e\boldsymbol{E}.$$

If the electron mass is ignored, and collisions included, Eq. (2.12.15) becomes $\boldsymbol{E} = \eta$, the simple Ohm's law. The $(\boldsymbol{J} \times \boldsymbol{B})$ term describes the Hall effect, and the ∇p_e term describes the fact that electron pressure gradients will drive currents.

It is often the case that some of the terms in the generalized Ohm's law Eq. (2.12.15) can be neglected, and this can be illustrated by the dimensional analysis mentioned earlier. It is a rough-and-ready argument, but useful. It should be noted that the argument follows a slightly different course depending on whether we are considering electrostatic phenomena (where the field $\boldsymbol{B}$ is essentially a vacuum field, or zero) or true magnetohydrodynamic (MHD) phenomena, where $\boldsymbol{B}$ is determined by the current density $\boldsymbol{J}$ within the plasma.

We use the rough dimensional relationhip $v = \omega L$, where $1/\omega$ and L are characteristic time and length scales. For MHD, we use Maxwell's equation reduced to a scalar form: $\nabla \times \boldsymbol{B} \approx B/L \approx \mu_0 \boldsymbol{J}$. We also use the relationships $\omega_c = eB/m$, $\omega_p = ne^2/m\varepsilon_0$, $c^2 = 1/\varepsilon_0\mu_0$, $V_s^2 \approx p/\rho$.

Take the first two terms on the right of Eq. (2.12.15). It is generally the case that the velocity v is of the order E/B, hence these terms are the same order of magnitude. Now (for electrostatic cases) compare the left-hand side term with the $\boldsymbol{E}$ term; it helps to take the divergence of both. The ratio of the terms is:

$$\frac{m}{ne^2}\frac{\partial}{\partial t}(\nabla\cdot\boldsymbol{J}):\nabla\cdot\boldsymbol{E} = \frac{m}{ne^2}\frac{\partial^2\rho}{\partial t^2}:\frac{\rho}{\varepsilon_0}.$$

Putting $\partial/\partial t = \mathrm{i}\omega$, the ratio may be written

$$\frac{\varepsilon_0 m\omega^2}{ne^2} = \frac{\omega^2}{\omega_p^2},$$

demonstrating that the left-hand side term matters only at frequencies approaching the plasma frequency. For the MHD case we compare with the $\boldsymbol{v} \times \boldsymbol{B}$ term, and use $B/L \approx \mu_0 J$, to obtain the ratio

$$\left(\frac{\omega}{\omega_{\mathrm{p}}} \frac{c}{v}\right)^2.$$

Putting $J = nev_{\mathrm{c}}$, where v_{c} represents the velocity of the charge carriers producing the current, the ratio of the Hall term to the $(\boldsymbol{v} \times \boldsymbol{B})$ term is of the order v_{c}/v, and the Hall term may be neglected for small currents and high densities. Alternatively, using the relation $B/L \approx \mu_0 J$ one can show that the ratio is of the order

$$\left(\frac{\omega\omega_{\mathrm{ce}}}{\omega_p^2}\right)\left(\frac{c^2}{v^2}\right),$$

again showing that the term may be neglected for sufficiently low frequencies.

The ratio of the ∇p_{e} term to the $(\boldsymbol{v} \times \boldsymbol{B})$ term for MHD is of the order

$$\frac{\omega}{\omega_{\mathrm{ci}}}\left(\frac{V_{\mathrm{s}}}{v}\right)^2.$$

Finally, the ratio of the resistive term to the $\boldsymbol{E}$ term for electrostatic cases is $\omega\nu/\omega_{\mathrm{p}}^2$, where ν is the collision frequency, and is $(\omega\nu/\omega_{\mathrm{p}}^2)(c/v)^2$ for MHD.

Now we must return to consider the convective terms in Eq. (2.12.12), which we so glibly neglected earlier in this section. The ratio of the $\boldsymbol{u}_i$ term to the $(\boldsymbol{v} \times \boldsymbol{B})$ term is $\omega/\omega_{\mathrm{ce}}$, and is thus negligible for frequencies small compared to the electron cyclotron frequency. For the $\boldsymbol{u}_{\mathrm{e}}$ term, the ratio is

$$\frac{\dfrac{m}{e}(\boldsymbol{u}_{\mathrm{e}} \cdot \nabla)\boldsymbol{u}_{\mathrm{e}}}{vB} \approx \frac{vu_{\mathrm{e}}^2}{Lv^2\omega_{\mathrm{ce}}} \approx \frac{\omega}{\omega_{\mathrm{ce}}}\left(\frac{u_{\mathrm{e}}}{v}\right)^2.$$

giving the additional condition that the electron fluid velocity should be not large compared with the global velocity.

These arguments are not totally reliable: we have ignored all vector relationships, which may not always be admissible. We do see that it may well be necessary to consider the convective terms in some circumstances. An interesting example where their effect is important has been highlighted by Jones and Hugrass (1981) and Hugrass (1982), who point out that for frequencies ω intermediate between ω_{ci} and ω_{ce}, the magnetic field is tied to the electrons, but not to the ions. In effect, the field is frozen into the species with the smallest Larmor radius; the field is even tied preferentially to the lower energy electrons in the distribution. A practical application of the effect is seen in the rotamak (Hugrass *et al.*, 1980), where a rotating magnetic field is used to drive a continuous unidirectional electron current.

In conclusion to this section, the two single-fluid equations we have just derived are the very powerful starting point for the wide subject of magnetohydrodynamics (MHD).

2.13 The MHD equations

For low frequency phenomena, the MHD equations are now found, from the results obtained in Sections 2.11 and 2.12 as Eqs (2.11.20), (2.12.2), (2.12.11), and (2.12.15), to be:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0. \tag{2.13.1}$$

$$\rho \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = (\boldsymbol{J} \times \boldsymbol{B}) - \nabla p. \tag{2.13.2}$$

$$\frac{p}{\rho^{\gamma}} = \text{const.}, \tag{2.13.3}$$

$$\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B}) = \eta \boldsymbol{J}, \tag{2.13.4}$$

and we include from Maxwell's equations

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{2.13.5}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}. \tag{2.13.6}$$

There are 14 scalar variables $(\rho, p, \boldsymbol{v}, \boldsymbol{J}, \boldsymbol{E}, \boldsymbol{B})$ and 14 equations, which is an encouraging outcome from our analysis, showing that the set is complete and in principle soluble.

If the resistivity η is neglected, we find that $\boldsymbol{E}$ can be eliminated, giving

$$\nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t}. \tag{2.13.7}$$

This situation, in which the effects of resistivity and electron inertia are neglected, is commonly known as *ideal MHD*.

2.14 An application of kinetic theory: electron plasma waves

These are waves at very high frequency, close to the plasma frequency, and are in essence compressional waves in the electron fluid. The ions remain virtually fixed at these frequencies, and may be treated merely as a constant background. Electron plasma waves (Langmuir waves) are an excellent example of the power of the kinetic theory approach. The dispersion relation can be derived from the fluid equations (see Chapter 3), which yield

$$\omega^2 = \omega_{\mathrm{p}}^2 + \gamma v_{\mathrm{T}}^2 k^2, \tag{2.14.1}$$

where v_{T} is the electron thermal speed. However, some very important and significant physics has been lost in this derivation. The fluid equations, as we have seen, are obtained by averaging over the distribution functions, and thus any effects due to nonequilibrium (non-Maxwellian) distribution functions have been irretrievably lost. When the same phenomenon – electron plasma waves – is approached using the kinetic equations, we find a new, interesting and important phenomenon, known as Landau damping after the Soviet physicist Lev D. Landau who first predicted it mathematically (Landau, 1946).

The essence of Landau damping is that the electron plasma waves are strongly damped, even in the complete absence of resistivity, and when there is therefore absolutely no dissipative mechanism in the conventional sense. The phenomenon is not restricted to electron plasma waves, and similar damping is widely observed in other circumsances. We shall present a simplified derivation of the effect, and then discuss the physical interpretation of this strange behaviour.

2.14.1 *The dispersion relation*

We start by going right back to the Vlasov equation for the electron gas. We are dealing with a high frequency phenomenon, and so the ions can be considered to be a stationary background system taking no part in the oscillations. We assume no magnetic field, and we shall work in a one-dimensional system for simplicity. We have

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + \frac{qE}{m}\frac{\partial f}{\partial v} = 0. \tag{2.14.2}$$

When a wave is propagating through the electron gas, the local distribution function, which in the absence of the wave is a Maxwellian $f_0(x, v, t)$, will be perturbed by an amount $\tilde{f}(x, v, t)$. We write the new distribution function f as $f = f_0 + \tilde{f}$. Let $E = \tilde{E}$, a small field due to the wave in the plasma. Making these substitutions, the Vlasov equation may be linearized by retaining only first order terms:

$$\frac{\partial \tilde{f}}{\partial t} + v\frac{\partial \tilde{f}}{\partial x} + \frac{q\tilde{E}}{m}\frac{\partial f_0}{\partial v} = 0. \tag{2.14.3}$$

We now assume that the perturbations are due to plane waves of the form $\exp(\mathrm{i}kx - \mathrm{i}\omega t)$. We can put $\partial/\partial t \to -\mathrm{i}\omega$, $\partial/\partial x \to \mathrm{i}k$. We obtain

$$-\mathrm{i}\omega\tilde{f} + \mathrm{i}kv\tilde{f} + \frac{q\tilde{E}}{m}\frac{\partial f_0}{\partial v} = 0; \tag{2.14.4}$$

therefore

$$\tilde{f} = -\frac{q\tilde{E}}{m}\frac{\partial f_0/\partial v}{\mathrm{i}(kv - \omega)}. \tag{2.14.5}$$

The density perturbation can be found as $\tilde{n} = \int_{-\infty}^{+\infty} \tilde{f}\,\mathrm{d}v$; therefore

$$\tilde{n} = -\frac{q\tilde{E}}{\mathrm{i}m}\int_{-\infty}^{+\infty} \frac{\partial f_0/\partial v}{kv - \omega}\,\mathrm{d}v. \tag{2.14.6}$$

Let $\tilde{E} = -\partial\tilde{\phi}/\partial x = -\mathrm{i}k\tilde{\phi}$. Poisson's equation gives $\partial\tilde{E}/\partial x = \tilde{\rho}/\varepsilon_0$, from which $k^2\tilde{\phi} = \tilde{n}q/\varepsilon_0$, and we have

$$k^2 = \frac{q^2}{m\varepsilon_0}\int_{-\infty}^{+\infty} \frac{\partial f_0/\partial v}{v - \omega/k}\,\mathrm{d}v. \tag{2.14.7}$$

This is, in fact, a dispersion relation, although not yet in a comprehensible form. We observe that the integrand contains a singularity at $v = \omega/k$. The proper way to handle this is to use contour integration in the complex plane: the solution can be found in standard texts (for example Bittencourt, 1986). However, we can

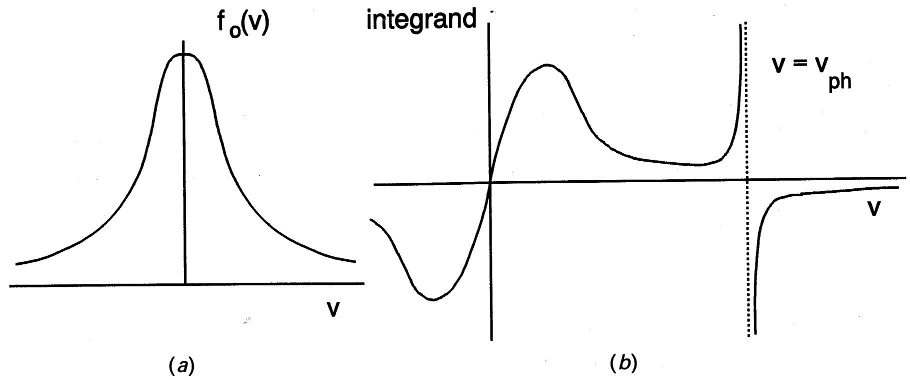

Fig. 2.3. The integrand appearing in Eq. (2.14.7). (*a*) The Maxwellian distribution $f_0(v)$; (*b*) The form of the integrand, showing a central region dominated by $\partial f_0/\partial v$, and a region in the wings dominated by the singularity.

obtain the correct result very simply by using a less rigorous approach.

We note that $\omega/k = v_{\mathrm{ph}}$, the phase velocity of the wave. Fig. 2.3 shows the form of $f_0(v)$ and the full integrand. The integral can be simplified by assuming $v_{\mathrm{ph}} \gg \overline{v^2}$, for it can then be divided into two parts:

(*a*) a central region, dominated by $\partial f_0/\partial v$;
(*b*) a region near $v = v_{\mathrm{ph}}$, dominated by the singularity.

Do (*a*) first, remembering that $v \ll v_{\mathrm{ph}}$: the integral is

$$-\frac{1}{v_{\mathrm{ph}}}\int \frac{\partial f_0/\partial v}{1 - v/v_{\mathrm{ph}}}\mathrm{d}v \approx -\frac{1}{v_{\mathrm{ph}}}\int \frac{\partial f_0}{\partial v}\left\{1 + \frac{v}{v_{\mathrm{ph}}} + \left(\frac{v}{v_{\mathrm{ph}}}\right)^2 + \left(\frac{v}{v_{\mathrm{ph}}}\right)^3 + \cdots\right\}\mathrm{d}v$$

$$= -\frac{1}{v_{\mathrm{ph}}^2}\underbrace{\int v\frac{\partial f_0}{\partial v}\mathrm{d}v}_{[1]} - \frac{1}{v_{\mathrm{ph}}^4}\underbrace{\int v^3\frac{\partial f_0}{\partial v}\mathrm{d}v}_{[2]} \qquad (2.14.8)$$

(The terms in even powers of v are zero because of the antisymmetry of $\partial f_0/\partial v$.)

The term labelled [1] becomes

$$-\frac{1}{v_{\mathrm{ph}}^2}\left(\Big[vf_0\Big]_{-\infty}^{\infty} - \int f_0\,\mathrm{d}v\right) = \frac{n}{v_{\mathrm{ph}}^2}.$$

Term [2] becomes

$$-\frac{1}{v_{\mathrm{ph}}^4}\left(\Big[v^3 f_0\Big]_{-\infty}^{\infty} - 3\int f_0 v^2\,\mathrm{d}v\right) = \frac{3}{v_{\mathrm{ph}}^4}n\overline{v^2}.$$

Combining these results, we obtain for (*a*)

$$k^2 = \frac{q^2 n}{m\varepsilon_0}\left(\frac{k^2}{\omega^2} + \frac{3k^4}{\omega^4}\overline{v^2}\right).$$

Now $q^2 n/m\varepsilon_0 = \omega_{\mathrm{p}}^2$; therefore

$$\omega^2 = \omega_{\mathrm{p}}^2 + 3\frac{\omega_{\mathrm{p}}^2}{\omega^2}k^2\overline{v^2},$$

or, for $\omega \approx \omega_{\mathrm{p}}$,

$$\omega^2 = \omega_{\mathrm{p}}^2 + 3k^2\overline{v^2}. \qquad (2.14.9)$$

This result, known as the Bohm–Gross dispersion relation, is very similar to that obtained from the fluid theory. The difference in the numerical factor arises merely because we have worked in one dimension here, giving $\gamma = (N+2)/N = 3$, since N, the number of degrees of freedom, is 1.

We might think that the contribution of the singularity, part (b) of the integral, will not matter, as it might be expected by symmetry to integrate to zero. It does not, however. It is this feature which, correctly included, describes Landau damping.

2.14.2 *The contribution of the singularity*

We are now concerned only with an arbitrarily narrow range of velocity near $v - \omega/k = 0$. We can therefore regard $\partial f_0/\partial v$ as constant over this narrow range, and remove it outside the integral. Denoting $v - \omega/k$ by u, Eq. (2.14.7) becomes

$$k^2 = \frac{q^2}{m\varepsilon_0}\left[\frac{\partial f_0}{\partial v}\right]_{\omega/k}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{u}. \tag{2.14.10}$$

Now

$$\begin{aligned}\int_{-a}^{a}\frac{\mathrm{d}u}{u} = \ln a - \ln(-a) &= \ln a - \ln(a\exp(\mathrm{i}\pi + \mathrm{i}2\pi n))\\ &= \ln a - \ln a - (\mathrm{i}\pi + \mathrm{i}2\pi n)\\ &= -\mathrm{i}\pi,\end{aligned}$$

where we have taken $n = 0$, without justifying that it is in fact the correct value. Hence

$$k^2 = -\frac{\mathrm{i}q^2\pi}{m\varepsilon_0}\left[\frac{\partial f_0}{\partial v}\right]_{\omega/k}. \tag{2.14.11}$$

Including both contributions, we obtain the full dispersion relation for electron plasma waves:

$$k^2 = k^2\left(\frac{\omega_\mathrm{p}}{\omega}\right)^2\left(1 + 3\frac{k^2}{\omega^2}\overline{v^2}\right) - \mathrm{i}\pi\frac{\omega_\mathrm{p}^2}{n}\left[\frac{\partial f_0}{\partial v}\right]_{\omega/k}. \tag{2.14.12}$$

The imaginary term which has now appeared in the dispersion relation implies damping of the wave: for any real value of k the corresponding ω must be complex, and the imaginary part of ω will describe the damping rate. To see this quickly, neglect the thermal correction ($\overline{v^2}$) term:

$$k^2 = k^2\left(\frac{\omega_\mathrm{p}}{\omega}\right)^2 - \mathrm{i}\pi\frac{\omega_\mathrm{p}^2}{n}\left[\frac{\partial f_0}{\partial v}\right]_{\omega/k}$$

or

$$\omega = \omega_\mathrm{p}\left(1 - \frac{\mathrm{i}\pi\omega_\mathrm{p}^2}{2k^2 n}\left[\frac{\partial f_0}{\partial v}\right]_{\omega/k}\right),$$

where the binomial expansion has been used to first order.

The $\exp(-\mathrm{i}\omega t)$ oscillation becomes

$$\exp(-i\omega_p t)\exp\left(\frac{\pi\omega_p^2}{2k^2 n}\frac{\partial f_0}{\partial v}t\right).$$

Since $\partial f_0/\partial v$ is negative at $v = \omega/k$, this describes damping, even in the absence of any collisions or dissipative mechanism. This is really quite a remarkable phenomenon in physics, and we shall now try to understand it in physical terms.

2.15 The physics of Landau damping

We observe that the damping is associated with the singularity in the region $v \approx v_{ph}$, where the electrons are travelling at almost the same speed as the wave. Such electrons will not see a rapidly oscillating field; if v is the *same* as v_{ph}, the electron will see a *constant* field. It will then be subject, for example, to constant acceleration, but that obviously cannot continue indefinitely. The situation is often likened to that of a surfer riding a wave – the surfer is always going downhill, because he is able to 'lock on' to the wave velocity. In the theory we have just derived, however, no such locking, or 'trapping', is described, for such effects are in essence nonlinear, and we have only derived a linearized theory.

To begin to interpret the physical reason for linear damping, look at the graph (Fig. 2.4) of the potential energy $q\phi$ of an electron in the wave. Regions like 'a' are regions of acceleration, while regions like 'b' represent deceleration. An accelerated particle will thus move into a decelerating region, and will therefore be 'trapped' in the potential minimum. We have already seen that such trapping does occur, but is not linear Landau damping, as described by our theory.

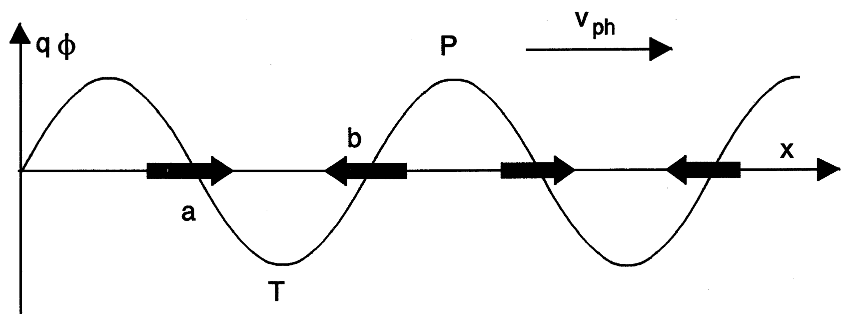

Fig. 2.4. Landau damping: the diagram shows schematically the potential energy $q\phi$ of an electron in an electron plasma wave. Regions like 'a' are regions of acceleration, and regions like 'b' give deceleration of the electron.

It is important to realize that our theory made the explicit assumption that at $t = 0$ the electrons are in thermal, Boltzmann equilibrium in the wave potential. The imposition of a wave at $t = 0$ modifies the distribution in x and t, but not in velocity v. Boltzmann equilibrium implies that the density n varies with potential energy $q\phi$ as $n = n_0 \exp(-q\phi/k_B T)$: this in turn means that at the peaks 'P' the electron density is lower than at the troughs 'T'.

Now consider particles at P going slightly faster than the wave. In the immediate future they will move into a region of acceleration. Those particles at the troughs T going faster than the wave will move into a region of deceleration. However, there are more particles at T than P, so the net effect is deceleration, i.e. electrons give energy to the wave.

We can apply the same argument to particles going slightly slower than the wave: the net effect is the opposite, a net transfer of energy from the wave to the particles.

These two effects do not cancel out, however. For we observe that from the shape of the distribution function $f_0(v)$, in particular the negative gradient $\partial f_0/\partial v$, there are more particles moving slower than the wave velocity v_{ph} than there are going faster. Thus the overall result is a net loss of energy by the wave, to the particles. The damping is explained, without any collisional process.

Since there is no dissipation, there can be no loss of information – the plasma must still 'know' that there was a wave there. This is indeed the case, for as the damping proceeds and wave energy is lost, there is a net transfer of particles in velocity space from just below v_{ph} to just above $\boldsymbol{v}_{ph}$ (Fig. 2.5). The information is being transferred from the waveform to the distribution function. This is no longer Maxwellian, but is distorted near $v = v_{ph}$. Only collisions can smooth away the bump.

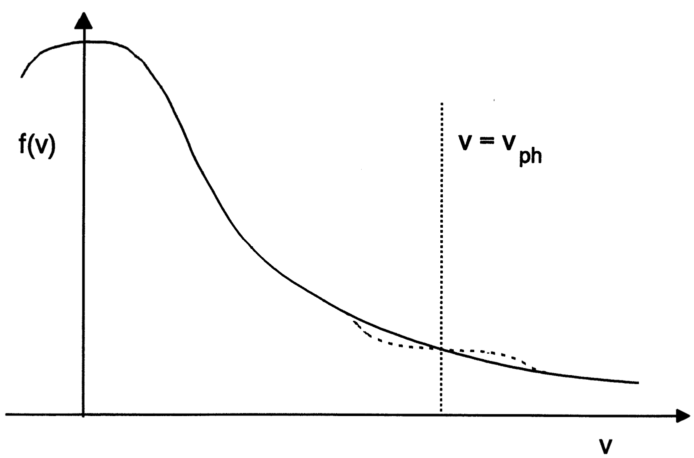

Fig. 2.5. The effect of Landau damping on the distribution function: energy is transferred from the wave to the electrons, thereby distorting the distribution function.

Negative Landau damping, or growth, is also observed. For this to occur, $\partial f/\partial v$ must be positive at $v = v_{ph}$. This situation can be created if the distribution is far from equilibrium and has a 'bump on the tail'. Such a distribution can be created, for example, by injecting a beam of fast particles into the plasma, and it is observed that the plama can then go into spontaneous oscillation. This is known as the *beam–plasma instability*.

2.16 Other examples of Landau damping

Landau damping can in principle occur in any wave in a plasma in which there is a component of electric field $\boldsymbol{E}$ parallel to the wave vector $\boldsymbol{k}$. A commonly observed occurrence is in ion acoustic waves, which are low frequency compressional waves in the ion fluid, but where the restoring force is electrostatic, not collisional as in a neutral gas. The velocity of ion acoustic waves is given by

$$\frac{\omega}{k} = V_s = \left(\frac{k_B T_e + \gamma_i k_B T_i}{M}\right)^{1/2}. \tag{2.16.1}$$

The thermal velocity of the ions is $\approx (k_B T_i/M)^{1/2}$; ion Landau damping thus occurs if $T_i \geqslant T_e$ and the particle velocity is comparable with the wave velocity. It is observed that ion acoustic waves only propagate in plasmas for which $T_e \gg T_i$. Such conditions are, however, quite commonly encountered in laboratory plasmas.

There are very interesting examples of Landau damping, and other aspects related to plasma kinetic theory, in other areas of physics. For example, an assembly of particles which are gravitationally coupled, (i.e. stars in galaxies), is subject to much the same physics as a plasma with Coulomb coupling. Some of the techniques and results described in this chapter will be encountered again in Chapter 11, when we consider gravitational plasmas.

2.17 Appendix: Tensor basics

We work in two dimensions, for simplicity of expression.

The *tensor product* or *dyad* $\boldsymbol{ab}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined by

$$\boldsymbol{T} = \boldsymbol{ab} = \begin{pmatrix} a_x b_x & a_x b_y \\ a_y b_x & a_y b_y \end{pmatrix}.$$

The *tensor dot product* is itself a vector, and is defined as

$$\boldsymbol{T}\cdot\boldsymbol{c} = \begin{pmatrix} a_x b_x & a_x b_y \\ a_y b_x & a_y b_y \end{pmatrix}\begin{pmatrix} c_x \\ c_y \end{pmatrix}; \quad \boldsymbol{c}\cdot\boldsymbol{T} = (c_x \quad c_y)\begin{pmatrix} a_x b_x & a_x b_y \\ a_y b_x & a_y b_y \end{pmatrix}.$$

From these definitions we can obtain the following relations:

$$(\boldsymbol{ab})\cdot\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c}) = (\boldsymbol{c}\cdot\boldsymbol{b})\boldsymbol{a},$$

$$\boldsymbol{c}\cdot(\boldsymbol{ab}) = (\boldsymbol{c}\cdot\boldsymbol{a})\boldsymbol{b}$$

and

$$\nabla\cdot(\boldsymbol{ab}) = \boldsymbol{b}(\nabla\cdot\boldsymbol{a}) + (\boldsymbol{a}\cdot\nabla)\boldsymbol{b}.$$

The product $\boldsymbol{S}:\boldsymbol{T}$ of two tensors $\boldsymbol{S}$ and $\boldsymbol{T}$ follows the rules of matrix multiplication, and is itself a tensor:

$$\boldsymbol{S}:\boldsymbol{T} = \begin{pmatrix} S_{xx} & S_{xy} \\ S_{yx} & S_{yy} \end{pmatrix}\begin{pmatrix} T_{xx} & T_{xy} \\ T_{yx} & T_{yy} \end{pmatrix}.$$

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