Order of Magnitude Astrophysics - *a.k.a. Astronomy 111*

Stellar Structure and Virial Theorem

The simplest description of stellar structure is that the stars are *spherical* and *static* (no rotation, magnetic field, no pulsation, oscillation, ...), and we will deal with such objects here.

Equations of Stellar Structure

The basic idea is to write down quantities that describe the stellar interior as a function of radius r, then to write down relations between them, either algebraic, or else differential equations, until we get a closed set of equations (as many equations as unknowns). Then we can think about solving them.

Equation of Hydrostatic Equilibrium

Consider an element of gas in equilibrium in the star (Figure 1). The pressure $\wp(r)$ is larger than the pressure $\wp(r + dr)$ by just the weight per unit area of the material between r and $r + dr$ in

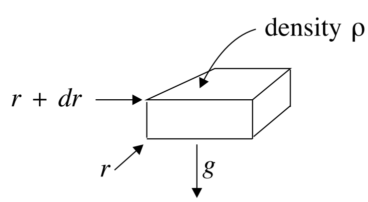

Fig. 1.— Element of gas in equilibirum.

the local gravitational acceleration g. If the element is of area dA, we have

$$[\text{mass of element}] = \rho\, dA\, dr \tag{1}$$

$$[\text{weight of element}] = g\rho\, dA\, dr = \left[\frac{GM(r)}{r^2}\right]\rho(r)\, dA\, dr \tag{2}$$

$$[\text{weight of element per unit area}] = \frac{G\rho(r)M(r)}{r^2}\, dr \tag{3}$$

So,

$$\frac{d\wp}{dr} = \frac{\wp(r+dr) - \wp(r)}{dr} = -\frac{G\rho(r)M(r)}{r^2}. \tag{4}$$

The minus sign is because pressure increases as r gets smaller (downward direction).

Mass Equation

Mass (interior to radius r) is just the integral of the density in spherical coordinates:

$$M(r) = \int_0^r \rho(r')4\pi r'^2 dr'. \tag{5}$$

We usually prefer to write this as a differential equation. Taking the derivative with respect to the upper limit of integration gives trivially,

$$\frac{dM}{dr} = 4\pi r^2 \rho. \tag{6}$$

Equation of State

If we could just find an algebraic relation between pressure and density

$$\wp = \wp(\rho) \tag{7}$$

we would be done: 3 equations for 3 unknowns $\wp$, ρ, M as a function of the independent variable r. In real life, however, pressure depends not only on density but also on *temperature* and *composition*. For a mixture of a prefect gas and radiation, we have

$$\wp = \wp_{\rm gas} + \wp_{\rm radiation} = \left(\frac{\rho}{\mu}\right) kT + \frac{1}{3} a T^4. \tag{8}$$

Luckily, it is often true that $\wp_{\rm radiation} \ll \wp_{\rm gas}$, so that the second term can be neglected, and that we can either (i) derive from other physics, or (ii) make a good guess, about how T in the first term varies with ρ. Further, in many cases of interest, μ is constant. Then we will have arrived at the so called *barytropic* equation of state $\wp = \wp(\rho)$. Let us know make that assumption.

Polytropes

A polytropic equation of state is a special case of a barytropic equation of state where the relation between P and ρ is a pure power law

$$\wp = K\rho^{1+\frac{1}{n}} = K\rho^{\Gamma} \tag{9}$$

Here n which need not be an integer is the so called polytropic index. The weird notation $1 + \frac{1}{n}$ can be thought of as arising from the perfect gas law

$$\wp \propto \rho T \tag{10}$$

along with an assumed power law relating T and ρ,

$$T \propto \rho^{\frac{1}{n}} \quad \text{or} \quad \rho \propto T^n \tag{11}$$

But that is just notational history.

It turns out that main sequence stars are pretty well modeled by $n = 3$ polytropes. That is, the run of temperature and density in the star roughly follows

$$T \propto \rho^{\frac{1}{3}} \tag{12}$$

So the value $n = 3$ is a good one to keep in mind as we proceed although we will meet other values later.

Adiabatic Indices for a Perfect Gas

If you compress a gas element whose thermal conductivity is small so that there is no heat conducted into or out of the element then its pressure is said to increase adiabatically In this case we can use the 1st Law of Thermodynamics to get a relation between $\wp$ and ρ for the gas element.

Consider a volume V containing N particles. Then the total energy E is given by the rule $\frac{1}{2}kT$ per degree of freedom, namely

$$E = \frac{\beta}{2}NkT \tag{13}$$

where β is the number of degrees of freedom. For a monatomic, fully ionized, gas (the usual case in stars) we have $\beta = 3$ corresponding to x, y, and z translational motions.

The perfect gas law involves only the number density of particles N/V not β, and is

$$\wp = \left(\frac{N}{V}\right)kT = nkT \tag{14}$$

which gives a relation between $\wp$ and E for perfect gasses

$$E = \frac{\beta}{2}\wp V. \tag{15}$$

Now the 1st Law of Thermodynamics, which is really just conservation of energy, says that when you squeeze a gas from volume V to volume $V - dV$ (smaller volume) the increase in its internal energy is just the work you have done squeezing it:

$$dE = -\wp dV \tag{16}$$

(The minus sign is because the internal energy increases with decreasing volume). Combining the last two equations (taking a differential of the first):

$$\frac{\beta}{2}(\wp dV + V d\wp) = -\wp dV \tag{17}$$

$$\frac{\beta}{2} V d\wp = -\left(1 + \frac{\beta}{2}\right) \wp dV \tag{18}$$

$$\frac{d\wp}{\wp} = -\left(\frac{2+\beta}{\beta}\right) \frac{dV}{V} \tag{19}$$

Integrating, we get

$$\wp = \text{constant} \times V^{-(1+\frac{2}{\beta})}. \tag{20}$$

But since the density of a fixed quantity of gas (N particles) varies inversely with its volume, this is just

$$\wp \propto \rho^{(1+\frac{2}{\beta})} \tag{21}$$

which is polytropic with index $n = \beta/2$. The most common case $\beta = 3$ gives $\wp \propto \rho^{5/3}$ ($n = 3/2$). Notice that as the number of degrees of freedom β increases, the polytropic index increases. In fact, the limiting case of an isothermal gas ($\wp \propto \rho T$, T constant, so $\wp \propto \rho$) corresponds to $\beta \to \infty$. This is because one can view the work compressing the gas as being spread over an infinite number β of internal degrees of freedom resulting in no increase in temperature.

Fully Convective Stars

Convection is the buoyancy-driven process of dynamical circulation that carries heat upward in gas or liquid in a gravitational field. You see it when you heat a pot of water on the stove or when the Sun heats the ground and hence the nearby air on the Earth. In essence convection is nothing more than hot air rises and cool air sinks.

In general, convection transports heat much faster than conduction does. Thus, a fluid element in the convective flow is very nearly adiabatically compressed as it sinks or decompressed as it rises. Since convection is also generally turbulent the mixing of different fluid elements is efficient. Thus, a gas in turbulent convection is quite accurately all on the same adiabat. That is, if it is monatomic and fully ionized ($\beta = 3$ above), it satisfies: $\wp \propto \rho^{5/3}$, where all fluid elements have the same constant. This is just what is needed for the validity of a polytropic model with $n = 3/2$. The Sun is not fully convective; most of it is stably stratified with the deeper, denser material being on a lower adiabat. That is why the run of density and pressure in the Sun is better described by $\wp \propto \rho^{4/3}$ (polytropic index $n = 3$) than by $\wp \propto \rho^{5/3}$, even though the material in the Sun is monatomic and fully ionized. The Sun has an outer convective envelope only in the last $1/6$ or so of its radius.

Low mass stars $\leq 0.3 M_\odot$ are almost completely convective, so they are good $n = 1.5$ polytropes. Also, stars of all masses go through an initial convective phase (called the Hayashi phase) before they settle down to the main sequence. This phase can typically last several million years. It, too is well described by an $n = 1.5$ polytrope.

Stellar Structure Virial Theorem

We can use thermodynamic results and the pressure equation to deduce a virial theorem relating to stellar structure. Start from the equation of hydrostatic equilibrium

$$\frac{d\wp}{dr} = -\frac{G\rho(r)M(r)}{r^2}. \tag{22}$$

and define $V(r)$ = volume occupied by gas inside radius r, so that dV = volume of dr−shell, containing mass $\rho(r) 4\pi r^2 dr$.

Multiply the pressure equation by $V(r)dr$:

$$V(r)\frac{d\wp}{dr}dr = -\frac{GM(r)\rho(r)}{r^2}V(r)dr, \tag{23}$$

i.e.,

$$V(r)d\wp = -GM(r)\rho(r)\frac{4}{3}\pi r^3 dr\frac{1}{r^2} = -\frac{1}{3}GM(r)dM(r)\frac{1}{r}. \tag{24}$$

Integrating over the star

$$\int_{r=0}^{r=R} V(r)d\wp = -\frac{1}{3}\int_{r=0}^{r=R}\frac{GM(r)dM(r)}{r} = \frac{1}{3}U, \tag{25}$$

where U is the total gravitational potential energy. Hence, integrating by parts,

$$\frac{1}{3}U = [\wp V]_{r=0}^{r=R} - \int_{r=0}^{r=R}\wp(r)dV. \tag{26}$$

At $r = 0$, $V = 0$. At $r = R$, $\wp \approx 0$. T(he surface of a star is approximately a vacuum.) Therefore,

$$[\wp V]_{r=0}^{r=R} = 0 \tag{27}$$

Hence,

$$U + 3\int_{r=0}^{r=R}\wp dV = 0. \tag{28}$$

This is the most general form of the stellar structure virial theorem.

For an ideal gas, the energy per volume u is

$$u = \frac{\wp}{\gamma - 1}. \tag{29}$$

Thus,

$$\int \wp dV = (\gamma - 1)\int u dV = (\gamma - 1)E, \tag{30}$$

where E is the total internal energy of gas. Therefore,

$$U + 3(\gamma - 1)E = 0. \tag{31}$$

Note that for a perfect, monoatomic gas, $\gamma = 5/3$, and the total internal energy/unit mass is just the total kinetic energy of the gas particles, i.e., $E \equiv T$. So $U + 3(\gamma - 1)E = 0$ is equivalent to $U + 2T = 0$, the same as self-gravitating particle virial theorem.

The total energy of the star is

$$E_{\rm tot} = E + U \tag{32}$$

so

$$E_{\rm tot} = E - 3(\gamma - 1)E = -(3\gamma - 4)E \tag{33}$$

$$= \frac{(3\gamma - 4)}{3(\gamma - 1)}U. \tag{34}$$

We now see that

$$\text{if} \;\; \gamma > \frac{4}{3}, \quad E_{\rm tot} < 0 \quad (\text{therefore, bound star}) \tag{35}$$

$$\text{if} \;\; \gamma < \frac{4}{3}, \quad E_{\rm tot} > 0 \quad (\text{therefore, unbound star}) \tag{36}$$

We see that stars are stable only if their adiabatic index γ exceeds $\frac{4}{3}$. Otherwise, they are unstable to converting their internal energy into expansion velocity and they blow themselves apart!

For example, a molecule a like NH_4 has not only the obvious degrees of freedom (3 translational and 3 rotational) but also 9 vibrational modes. So a star made of convecting ammonia would be a polytrope with

$$\gamma = 1 + \frac{2}{15} < \frac{4}{3}. \tag{37}$$

This is of course fanciful, since stars are always so hot that molecules are destroyed. But there is a point of principle to understand here.