Advanced Fluids Report

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Abstract

I have constructed a code to simulate the evolution of fluid under the Boussinesq approximation contained within a box. [Discuss boundary conditions here] Vertical resolution is represented using a grid for which derivatives are computed by the finite difference method while horizontal resolution is represented as a fourier sequence. In order to test that this code worked, I ran it under the standard Rayleigh-Benard convection problem. Second, I investigated allowing the bottom boundary $n = 2$ temperature mode to fluctuate periodically in time, holding the top boundary at constant temperature.

1 Introduction

Often we can represent fairly complicated astrophysical or terrestrial phenomenon with simple simulations in the goal of gaining understanding or intuition about the system. In this work, I have constructed a simulation of Boussinesq fluid inside a box, which perhaps is a model for convection in the ocean, accretion disks or a thin convective zone on the sun.

This simulation makes the following important assumptions: (1) the Boussinesq approximation, (2) the fluid is 2D and (3) the boundaries are given by a box with the specified boundary conditions. These are quite unrealistic for any system of research interest so should only be used to build intuition. Certainly quantitative predictions in no way are accurate.

2 Methods

In this section, I present the non dimensional Boussinesq approximation as well as discuss details of implementing this in the code.

2.1 Boussinesq Approximation

Define the Rayleigh number and Prantdl number as follows

$$\mathrm{Ra} \equiv \frac{g\alpha\Delta T D^3}{\nu\kappa} \tag{1}$$

$$\mathrm{Pr} \equiv \frac{\nu}{\kappa} \tag{2}$$

where the Rayleigh number measures the ratio of buoyancy force to the viscous force. The Prantdl number is the ratio of viscous to thermal diffusion coefficients. g is the magnitude of the gravitational force (assumed to be in the vertical direction), ΔT is the magnitude of the difference in temperature between top and bottom, D is the distance from top to bottom, ν is the kinematic viscosity, and κ is the thermal diffusivity.

The Boussinesq approximation is given by the following equations

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v}\cdot\nabla)\mathbf{v} - \nabla p + \mathrm{RaPrT}\hat{z} + \mathrm{Pr}\nabla^2\mathbf{v} \tag{3}$$

$$\frac{\partial \mathrm{T}}{\partial t} = -(\mathbf{v}\cdot\nabla)\mathrm{T} + \nabla^2\mathrm{T} \tag{4}$$

$$\nabla\cdot\mathbf{v} = 0 \tag{5}$$

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The main assumption is that the thermal expansion of the fluid is small so that the fluid is approximately incompressible, yet thermal expansion still creates a buoyancy force affecting the momentum equation. Temperature is transported by advection and diffusion.

2.2 Rayleigh-Benard Convection

The Rayleigh-Benard convection problem is defined to have fluid confined between two plates each held at constant temperature. Convection occurs when the bottom plate is held at a sufficiently higher temperature than the top plate. Gravity waves occur when the top plate is held at a sufficiently higher temperature than the bottom plate.

This physical situation is modified often by adding (1) magnetic fields, (2) rotation or (3) inserting a salinity gradient.

The boundary conditions are as follows. The temperature is held fixed on the top and bottom and the sides are set to have $\partial T/\partial x = 0$. The stress free boundary conditions were employed for the momentum equation.

2.3 Code & Algorithm

From the original Boussinesq equations, we take the curl of the momentum equation to get an evolution equation for the vorticity. The governing equations are then

$$\frac{\partial \omega}{\partial t} = -(\mathbf{v}\cdot\nabla)\omega - \mathrm{RaPr}\frac{\partial \mathrm{T}}{\partial x} + \mathrm{Pr}\nabla^2\omega \tag{6}$$

$$\frac{\partial \mathrm{T}}{\partial t} = -(\mathbf{v}\cdot\nabla)\mathrm{T} + \nabla^2\mathrm{T} \tag{7}$$

$$\mathbf{v} = \nabla \times \psi\hat{y} \tag{8}$$

$$\omega = -\nabla^2\psi \tag{9}$$

We horizontally decompose ω, T and ψ into their nonzero Fourier terms

$$\omega(x,z,t) = \sum_{n\geq 1}\omega_n(z,t)\sin\left(\frac{n\pi x}{a}\right) \tag{10}$$

$$\mathrm{T}(x,z,t) = \sum_{n\geq 0}\mathrm{T}_n(z,t)\cos\left(\frac{n\pi x}{a}\right) \tag{11}$$

$$\psi(x,z,t) = \sum_{n\geq 1}\psi_n(z,t)\sin\left(\frac{n\pi x}{a}\right) \tag{12}$$

ω and T are evolved forward at each time step and ψ is determined from ω of the previous time step using a tri-diagonal matrix solver. The time step is determined to be the minimum of $(\delta z)^2/(4\,\mathrm{Pr})$ or the Courant-Fridrichs-Lewy condition.

3 Tests and Results

The code was run first in the standard Rayleigh Benard problem as shown in Figure (1) with Rayleigh number 100000. Notice that there is a well defined steady convective cell that develops.

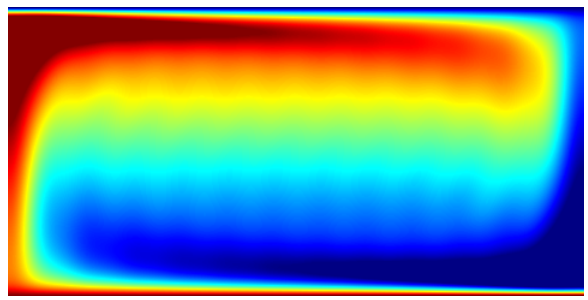

Figure 1: Snapshot of temperature profile in the standard Rayleigh-Benard problem. Ra = 100000, Pr = 2, Number of horizontal spectral modes = 25, number of vertical zones = 201, aspect ratio = 2

The code was also run at various Rayleigh number to test the Rayleigh-Nusselt relationship that has been historically studied. The Nusselt number is defined to be ratio of the total energy flux to the conductive energy flux. It turns out that this is also equal to the conductive flux at the boundary for this nondimensionalized problem. The trend is shown in Figure (2). Quantitative results from this simulation are unlikely to match behavior of 3D simulations, however it is nice to see that the Rayleigh number for which convection begins is close to the value 10^3.

3.1 A modification

I modified this code by setting the bottom to have a modulating $n = 2$ temperature profile. The result of this simulation is shown in Figure (3)

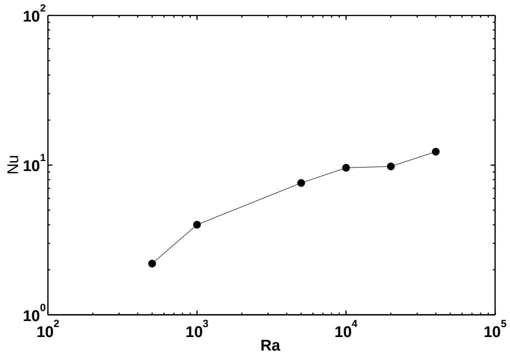

Figure 2: Rayleigh Nusselt relationship. Each fluid was run for 1 unit time. The Nusselt number was determined by measuring the average temperature gradient at the bottom boundary. In these runs, there are 51 vertical zones, 25 horizontal spectral modes, the Prandtl number is 2, the aspect ratio is 3. It is promising that the relationship is monotonically increasing.

4 Conclusions

In this work I have presented a 2D Boussinesq code applicable to the Rayleigh-Benard convection problem. The code is useful because it reveals intuition on how these simple flows work. Quantitative measurements should not be taken seriously, but it is nice to see that the "order of magnitude" behavior is quantitatively matched in the Rayleigh-Nusselt relationship. Interesting future modifications include adding Rotation, shear, magnetic fields or salinity.

5 Acknowledgment

This code was developed in the guidance of the "Advanced Fluids" class at UC Santa Cruz taught by Gary Glatzmaier.

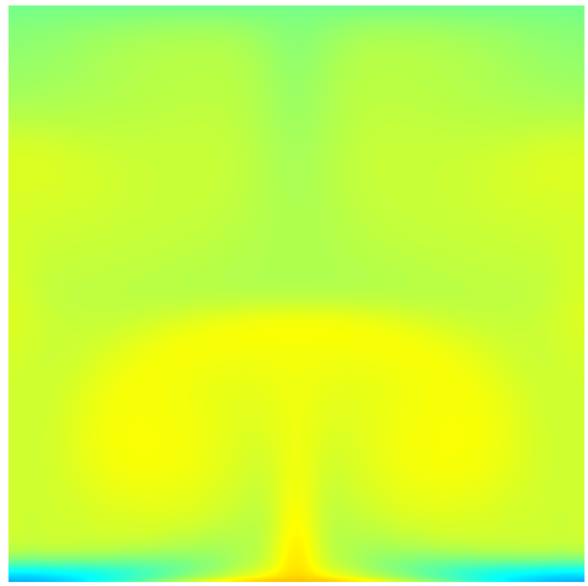

Figure 3: Snapshot of temperature profile where bottom boundary condition has temperature profile $T_2 = A\cos(\Omega t)$ where T_2 denotes the amplitude of the second spectral mode. $\Omega = 200$. Also, the number of vertical zones = 101, the number of horizontal spectral modes = 25, the Rayleigh number is set to 10000, the Prandtl number = 2 and the aspect ratio is 1.