Ay 112 Problem Set 2
Solutions

1) 5 points How does Kelvin-Helmholtz time scale vary with mass?

on the main sequence, approximately $L \propto M^3$

$$\tau_{KH} = \frac{\alpha G M^2}{2LR}$$

$$R \propto M^{2/3} \Rightarrow \tau_{KH} \propto \frac{M^2}{M^3 M^{2/3}} = \boxed{M^{-5/3}}$$

considerably shorter for more massive stars.

2) 15 points Virial theorem for a star with a mixture of gas and radiation pressure

$$U = \int u\, dm = \int (u_{gas} + u_{rad})\, dm$$

$$u_{gas} = 3/2\ P_{gas}/\rho = 3/2 \frac{\beta P}{\rho}$$

$$u_{rad} = 3\ P_{rad}/\rho = 3\ (1-\beta)\frac{P}{\rho}$$

$$U = 3/2\,\beta + 3(1-\beta) \int_0^M P/\rho\, dm$$

but the Virial theorem implies

$$\int_0^M P/\rho\, dm = -\Omega/3$$

$$U = -[3/2\,\beta + 3(1-\beta)]\ \Omega/3$$

$$= (3/2\,\beta - 3)\,\Omega/3 = (\beta/2 - 1)\,\Omega$$

$$U = \frac{\beta - 2}{2}\,\Omega$$

The total energy

$$E = U + \Omega = \left(\frac{\beta-2}{2} + 1\right)\Omega$$

$$= \frac{\beta}{2}\,\Omega = \frac{\beta}{2}\left(\frac{2}{\beta-2}\right) U$$

2) continued

$$E = \frac{\beta}{2}\left(\frac{2}{\beta-2}\right)u = \frac{\beta}{\beta-2}\,u$$

$$\boxed{E = -\frac{\beta}{2-\beta}\,U}$$

If $\beta = 1$, ie ideal gas pressure

$$E = -u = \frac{\Omega}{2}$$

the star is bound with a total energy equal to 1/2 its binding energy. The Virial Theorem we have discussed before. The star is stable and contraction leads to heating.

If $\beta = 0$ $E = 0$, the star has zero total energy. It can expand or contract without spending energy. It is neutrally (un)stable.

3) 15 points equation of state

35% H 65% ^{4}He by mass. 150 g cm^{-3} 1.5×10^{7} K

ideal gas $P = \frac{\rho N_A kT}{\mu}$. What is μ?

a) $\mu = \left(\sum_i Y_i + Y_e\right)^{-1}$

$$\frac{1}{\mu} = \sum \frac{(Z_i+1)}{A_i}\,X_i = \frac{2}{1}(0.35) + \frac{3}{4}(0.65)$$

$$= 0.7 + 0.4875 = 1.1875 \Rightarrow \boxed{\mu = 0.842}$$

b) $Y_e = \sum Y_i Z_i = \frac{0.35}{1}(1) + \frac{0.65}{4}(2)$

$$= 0.35 + 0.325 = \boxed{0.675}$$

c) $n_e = \rho N_A Y_e = (0.675)(6.02\times10^{23})\cancel{(1.50\times10^{-16})}\,(150)$

$$= \cancel{9.07\times10^{12}}\ \boxed{6.10\times10^{25}}$$

3) Continued (typo repaired)

$$P_{ideal} = \frac{\rho N_A k T}{\mu} = 1.1875\,(150)(1.5\times10^{7})(6.02\times10^{23})(1.38\times10^{-16})$$

$$= \boxed{2.21\times10^{17}\ \text{dyne cm}^{-2}}$$

e) $$u = 3/2\, P/\rho = 3/2\ \frac{2.21\times10^{17}}{150}$$

$$= \boxed{2.21\times10^{15}\ \text{erg gm}^{-1}}$$

f) $$P_{rad} = 1/3\, aT^4 = 1/3\,(7.56\times10^{-15})(1.5\times10^{7})^4$$

$$= 1.28\times10^{14}\ \text{dyne cm}^{-2}$$

$$\beta = \frac{P_{gas}}{P_{rad}+P_{gas}} \qquad 1-\beta = \frac{P_{rad}}{P_{gas}+P_{rad}}$$

$$1-\beta = \frac{1.28\times10^{14}}{(1.28\times10^{14}+2.21\times10^{17})} = \boxed{5.79\times10^{-4}}$$

$$= 0.058\%$$

4) 15 points neutron star EOS

Neutrons have spin 1/2 and obey Fermi-Dirac statistics

Neutron degeneracy is given by

$$P = 1/3 \int_0^{p_0} \frac{dn(p)}{dp}\, p v\, dp \qquad p_0 = \left(\frac{3h^3 n_n}{8\pi}\right)^{1/3}$$

just like electrons but a) $v = p/m_n$ has the mass of the neutron, not electron and b) n_n is the neutron number density $n_n = \rho N_A Y_n$

Assume $Y_n = 1$ $\quad n_n = \rho N_A$

So

$$P_n = 1/3 \int_0^{p_0} \frac{2}{h^3} 4\pi p^2\, p \left(\frac{p}{m_n}\right) dp$$

$$= \frac{8\pi}{3h^3 m_n}\ \frac{p_0^5}{5}$$

4 continued

$$P_n^{\deg NR} = \frac{8\pi}{3h^3 m_n}\left(\frac{p_o^5}{5}\right) = \frac{8\pi}{15h^3 m_n}\left(\frac{3h^3 n_n}{8\pi}\right)^{5/3}$$

$$= \frac{h^2}{5m_n}\left(\frac{3}{8\pi}\right)^{2/3}\frac{1}{m_n}\, n_n^{5/3} \qquad \text{(typo)}$$

$$= \boxed{(3/\pi)^{2/3}\frac{h^2}{20m_n}(\rho N_A)^{5/3}} = \boxed{5.45\times10^{9}\,\rho^{5/3}\ \text{dyne cm}^{-2}}$$

$$5.45\times10^{9}\rho^{5/3}$$

$$P_n/P_e\Big)_{NR-deg} = \frac{1}{m_n}\Big/\frac{Y_e^{5/3}}{m_e}$$

$$= \boxed{\frac{m_e}{m_n Y_e^{5/3}}} = \frac{1}{1838\,Y_e^{5/3}}$$

The ratio of neutron degeneracy pressure to electron degeneracy pressure is reduced by the ratio of the electron mass to neutron mass, plus a factor that gives the number of electrons per baryon to the 5/3 power. Typically $Y_e = 0.5$

You might want to explore the implications of this for the neutron star mass-radius relation!

b) The neutrons would become relativistically degenerate when

$$\frac{p_F^2}{2m} = p_F c \quad \text{or} \quad \frac{h}{2}\left(\frac{3n_n}{\pi}\right)^{1/3} = 2m_n c$$

$$n_n = \frac{64\pi}{3}\left(\frac{m_n c}{h}\right)^3 = \rho N_A$$

$$\Rightarrow \rho = \frac{64\pi}{3N_A}\left(\frac{m_n c}{h}\right)^3$$

$$= (1838)^3\,\rho_{crit}(e)$$

$$= (1838)^3\,(7.80\times10^{6}\ \text{g cm}^{-3})$$

$$= \boxed{4.8\times10^{16}\ \text{g cm}^{-3}}$$

4) continued

One could also use $\frac{p_F}{m} = v = c$ and get a similar answer though smaller by 8

Such densities are probaly not achieved because a 1.4 $M_\odot$ neutron star would have a radius smaller than the Schwartzchild radius, but we won't count off if you didn't realize that.

5) 15 points The luminosity of the sun and radiation transport.

a) $\frac{dT}{dr} \sim \frac{T_{\odot c}}{R_\odot} = \frac{1.57 \times 10^7 \text{ K}}{6.96 \times 10^{10} \text{ cm}} = \boxed{2.3 \times 10^{-4} \text{ K/cm}}$

across 0.1 cm $\Delta T = \underline{2.3 \times 10^{-5} \text{ K}}$

b) Each cube radiates $6 \sigma T^4$

$\boxed{\sigma T^4 = 3.44 \times 10^{+24} \text{ erg cm}^{-2} \text{ s}^{-1}}$

leaves each face

this equals $\frac{3.44 \times 10^{24}}{3.87 \times 10^{33}} = 8.97 \times 10^{-10} L_\odot$

c) 6 faces emit $5.38 \times 10^{-9} L_\odot$

Volume of cubes $\sim L_\odot / 5.38 \times 10^{-9} = \boxed{1.86 \times 10^8 \text{ cm}^3}$

This is $<< 1.4 \times 10^{33}$ cm^3 the volume of the sun or even the inner 10% (1.4×10^{30} cm^3)

In fact the radius of a sphere with volume 1.86×10^8 cm is only 350 cm !!

This is a nonsense model because in the solar interior, each cm^3 radiates into adjacent cm^3 that have almost exactly the same temperature (to 11 figures!!). The back reaction cannot

5) continued

d) each cm^3 actually radiates

$$\sigma (T+dT)^4 - \sigma T^4 \approx 4\sigma T^3 \, dT$$

pf:

$$\sigma T^4 \left(1 + \frac{dT}{T}\right)^4 - \sigma T^4$$

$$= \sigma T^4 \left(1 + 4\frac{dT}{T} - 1\right) = 4\sigma T^3 dT$$

$dT \approx \frac{dT}{dr} \ell_{mfp}$ $\qquad \ell_{mfp} = 0.1$ cm

$\approx (2.3 \times 10^{-4})(0.1) = 2.3 \times 10^{-5}$ as in a)

$L = 4\pi r^2 (4\sigma T^3)(2.3 \times 10^{-5})$ i.e. $4\pi r^2 4\sigma T^3 \frac{dT}{dr} \ell_{mfp}$

$= 16\pi\sigma r^2 T^3 (2.3 \times 10^{-5})$ $\qquad r = 6.96 \times 10^9 = R_\odot/10$

$= \boxed{3.2 \times 10^{33} \text{ erg/s}}$ $\qquad T = 10^7$ K

which is closer to $L_\odot$ than it deserves to be.

6) 10 points opacity

Lumping bound free and bound-bound together we discussed i) electron scattering ii) free-free iii) bound free and bound-bound

$$\kappa \propto \sigma \propto \rho^a T^b$$

For electron scattering $a = b = 0$

For the other two $a=1$, $b = -7/2$

ES is scattering, the other two are absorptive

ES dominates at high T

BF, BB, + FF will be less when metallicity is less.

7) ~~10~~ 5 points n=1 polytrope neutron star

$$\bar{\rho} = \frac{3M}{4\pi r^3} = 6.65 \times 10^{14}\ g\ cm^{-3}$$

$$\rho_c/\bar{\rho} = 3.29$$

$\boxed{\rho_c = 2.2 \times 10^{15}}$ about $10 \times \rho_{nuc}$

8) 20 points Calculate the radius of the helium core at the time of the helium flash and 2^{nd} ignition on the horizonal branch $M_{He} = 0.45\ M_\odot$

$T_{He} \approx 1.5 \times 10^8$ K (1.0 X10^8 might have been better for such low masses)

Low mass stars, like the sun, have a common mass, $\approx 0.45\ M_\odot$, when they ignite He. A smaller core can be supported by degeneracy pressure

a) On first ignition, the core is supported by non-relativistic electron degeneracy pressure. $P \propto (\rho Y_e)^{5/3}$. The effective polytropic index is given by $\gamma = (n+1)/n = 5/3 \Rightarrow$

$$\boxed{n = 1.5}$$

b) The radius would be given by the mass-radius relation for non-relativistic white dwarfs

$$R = 8800\ Km \left(\frac{Y_e}{0.5}\right)^{5/3} \left(\frac{M_\odot}{M}\right)^{1/3}$$

For ^4He $Y_e = 0.5$

8 continued

$$R = 8800\text{ Km}\ \frac{1}{(0.45)^{1/3}} = \boxed{11,500\text{ Km}}$$

the average density $\bar{\rho} = \dfrac{3M}{4\pi R^3}$

$$= \frac{(3)(0.45)(1.989 \times 10^{33})}{(4\pi)(1.15 \times 10^{9})^3}$$

$$= \boxed{1.40 \times 10^5\ g\ cm^{-3}}$$

the central density for an $n = 1.5$ polytrope is 5.99 times that

c)

$$\rho_c = (5.99)(1.40 \times 10^5) = \boxed{8.4 \times 10^5\ g\ cm^{-3}}$$

On a plot of pressures, this clearly lies in the non-relativistic degenerate domain. The core is thus unstable to runaway as we will discuss in class

d)

e) Now the star core heats up, expands, recontracts and ignites a second time at a lower density now, by assumption $n = 3$, β is close to, but not exactly $= 1$.

One can use equations in the notes, but they were given without derivation. You should, for completely full credit, derive them

For any polytrope

$$M = -4\pi\alpha^3 \rho_c\, \xi_1^2 \left.\frac{d\theta}{d\xi}\right)_{\xi_1}$$

$$= -4\pi \left(\frac{P_c(n+1)}{4\pi G \rho_c^2}\right)^{3/2} \rho_c\, \xi_1^2 \left.\frac{d\theta}{d\xi}\right)_{\xi_1}$$

For $\beta \simeq 1$

$$P_c \simeq \frac{\rho_c N_A k T_c}{\mu}$$

8e continued

$$M = 4\pi \overset{\xi_1^2 \frac{d\theta}{d\xi}\big)_{\xi_1}}{(2.01824)}\, 4^{3/2} \left(\frac{\rho_c N_A k T_c}{\mu 4\pi G \rho_c^{\,2}} \right)^{3/2} \rho_c$$

$n = 3$

$$= 4\pi (2.01824)\, 4^{3/2} \frac{(N_A k)^{3/2}}{(4\pi G)^{3/2}} \frac{T_c^{\,3/2}}{\mu^{3/2} \rho_c^{\,1/2}}$$

$$M = 2.00 \times 10^{23} \left(\frac{T_c^{\,3}}{\mu^3 \rho_c} \right)^{1/2}$$

$$T_c = 2.92 \times 10^{-16}\, M^{2/3} \mu\, \rho_c^{\,1/3}$$

$$= 4.62 \times 10^{6}\,\mathrm{K} \left(\frac{M}{M_\odot} \right)^{2/3} \mu\, \rho_c^{\,1/3}$$

but $\rho_c = 54.18\, \bar{\rho} = 54.18 \dfrac{3M}{4\pi R^3}$

$$\rho_c^{\,1/3} = 4.241 \left(M/M_\odot \right)^{1/3} \left(R_\odot / R \right)$$

so

$$T_c = 19.6 \times 10^{6}\,\mathrm{K}\ \mu \left(M/M_\odot \right) \left(R_\odot / R \right)$$

note $T_c \propto M/R$ as expected.

Now back to the problem For He $\mu = 4/3$

$$T_c = 1.5 \times 10^{8} = 1.96 \times 10^{7} (4/3)(0.45)(R_\odot/R)$$

$$R/R_\odot = \cancel{\frac{(1.5 \times 10^8)(0.75)}{(0.45)}}$$

$$= \frac{(1.96 \times 10^{7})(4/3)(0.45)}{1.5 \times 10^{8}} = \boxed{0.078}$$

$$\boxed{\bar{\rho} = \frac{7.23 \times 10^{4}}{54.18} = 1330\ \mathrm{g\ cm^{-3}}}$$

$$\rho_c = (4.24)^3 (0.45) \left(\frac{1}{0.078} \right)^3$$

$$= \boxed{7.23 \times 10^{4}\ \mathrm{g\ cm^{-3}}}$$

non-degenerate

If $T_{He} = 1.0 \times 10^{8}$ ρ_c would be $(1/1.5)^3 = 0.3$ times smaller $\approx 2 \times 10^{4}\ \mathrm{g\ cm^{-3}}$ as in Pols p 152