Basic Nuclear Physics – 2

Nuclear Reactions

Glatzmaier and Krumholz 7,8
Prialnik 4
Pols 6
Clayton 4

A nuclear reaction turns one nucleus to another.
We have already discussed several kinds:

Beta decay, electron capture, positron emission

Alpha decay and fission (technically everything heavier iron is unstable)

Neutron or proton drip

http://www.nndc.bnl.gov/wallet/wc8.html

Here we expand the discussion to include fusion reactions where two (or rarely more) nuclei come together to produce a third, often with the emission of some lighter particles and energy.

The generic binary fusion reaction is:

$$\boxed{I + j \rightarrow L + k \qquad \text{or} \qquad \mathrm{I(j,k)L}}$$

I(j,k)L

I = Target nucleus   j = incident particle
L = Product nucleus   k = outgoing particle or particles

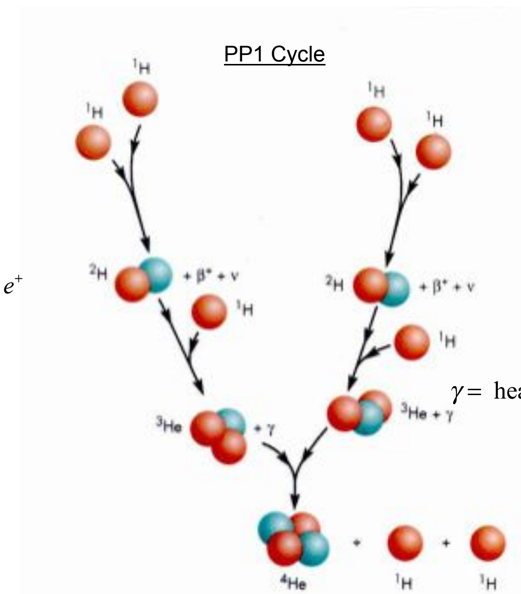
If there is no incident particle put the outgoing particles together without a comma

E.g., pp1; the main reaction sequence powering the sun

$$\mathrm{p(p,e^+\nu)^2H(p,\gamma)^3He(^3He,\ 2p)^4He}$$

The symbol γ in the second reaction is is sort of a catch-all for energy that comes out in forms other than neutrinos, electrons, and positrons. Energy comes out as photons as well as kinetic energy (heat).

Here $\beta^+ = e^+$



$\gamma =$ heat and light

The cross section for the reaction I(j,k)L is

$$\sigma_{jk}(I) = \frac{\text{number of reactions/nucleus I/sec}}{\text{number of incident particles j/cm}^2\text{/sec}}$$

σ thus has units of cm^2 though other units of area are sometimes used (typically "barns" are the unit in nuclear physics; 1 barn = 10^{-24} cm^2 as in "big as a barn")

The reaction rate for the reaction I (j,k) L is given by the product of the number densities of the reactants times their relevant speed and cross section. n_jv can loosely be thought of as the flux of particles j on a nucleus of species I

$$n_I n_j \sigma_{Ij} \mathrm{v}$$

This has units of reactions cm^{-3} s^{-1}

It is more convenient to write things in terms of the Y's previously defined

$$Y_I = \frac{X_I}{A_I} \qquad n_I = \rho N_A Y_I \qquad \left(\frac{gm}{cm^3}\right)\left(\frac{atoms}{Mole}\right)\left(\frac{Mole}{gm}\right)$$

so that the rate becomes

$$(\rho N_A)^2 Y_I Y_j \sigma_{Ij} \mathrm{v}$$

and a term in a rate equation decribing the destruction of I might be

$$\boxed{\begin{aligned}\frac{dY_I}{dt} &= -\rho Y_I Y_j N_A \langle \sigma_{jk}(I)\mathrm{v}\rangle + \ldots \\ &= -\rho Y_I Y_j \lambda_{jk}(I) + \ldots\end{aligned}}$$

Equivalent to

$$\frac{dn_I}{dt} = -n_I n_j \langle \sigma_{Ij} v\rangle + \ldots$$

Here $\langle \;\; \rangle$ denotes a suitable average over energies and angles and the reactants are usually assumed to be in thermal equilibrium.

For a Maxwell-Boltzmann distribution of reactant energies the average of the cross section times velocity is

$$\langle \sigma_{jk}(I)\mathrm{v}\rangle = 4\pi \left(\frac{\mu}{2\pi kT}\right)^{3/2} \int_0^\infty \sigma_{jk}(\mathrm{v})\, \mathrm{v}^3 e^{-\mu \mathrm{v}^2/2kT}\, d\mathrm{v}$$

$$\boxed{\langle \sigma_{jk}(I)\mathrm{v}\rangle = \left(\frac{8}{\pi\mu}\right)^{1/2} \left(\frac{1}{kT}\right)^{3/2} \int_0^\infty \sigma_{jk}(E)\, E\, e^{-E/kT}\, dE}$$

where μ is the "reduced mass"

$$\mu = \frac{M_I m_j}{M_I + m_j}$$

for the reaction I (j, k) L and E = $\frac{1}{2}\mu v^2$ is the kinetic energy in the center of mass frame.

For T in 10^9 K = 1 GK, σ in barns (1 barn = 10^{-24} cm^2), E_6 in MeV, and k = 1/11.6045 MeV/GK, the thermally averaged rate factor in cm^3 s^{-1} is

$$\langle \sigma_{jk} \mathrm{v}\rangle = \frac{6.197 \times 10^{-14}}{\hat{A}^{1/2} T_9^{3/2}} \int_0^\infty \sigma_{jk}(E_6)\, E_6\, e^{-11.6045 E_6/T_9}\, dE_6$$

$$\hat{A} = \frac{A_I A_j}{A_I + A_j} \qquad \text{for the reaction I(j,k)L}$$

If you know σ_{jk} from the lab, or a calculation, just put it in and integrate. But often we don't know the cross section at all, or know it in a limited energy range or only at a few points. How to estimate, interpolate, extrapolate?

The Cross Section

$$ƛ = \frac{\hbar}{p} = \frac{1}{k}$$

$$\sigma(E) = \overbrace{\pi ƛ^2}^{\text{Area subtended by a de Broglie wavelength in the c/m system}} \quad \underbrace{P(E)}_{\text{probability a flux of particles with energy E at infinity will reach the nuclear surface. Must account for charges and QM reflection.}} \quad \overbrace{X(E, A)}^{\text{How much the nucleus I+j looks like the target nucleus I with j sitting at its surface. Liklihood of staying inside R once you get there.}}$$

geometry term (Cla 4-180) — penetration factor — nuclear structure

see Clayton Chapter 4

$ƛ$ is the de Broglie wavelenth in the c / m system

$$\pi ƛ^2 = \frac{\pi\hbar^2}{\mu^2 v^2} = \frac{\pi\hbar^2}{2\mu E} = \frac{0.656\text{ barns}}{\hat{A}\ \text{E(MeV)}}$$

where 1 barn = 10^{-24} cm^2 is large for a nuclear cross section.

$$\mu = \frac{M_1 M_2}{M_1 + M_2}$$

$$\hat{A} = \frac{A_1 A_2}{A_1 + A_2} \sim 1 \text{ for neutrons and protons on heavy nuclei}$$

$$\sim 4 \text{ for } \alpha\text{-particles if } A_I \text{ is large, } >> 1$$

Classical turning radius

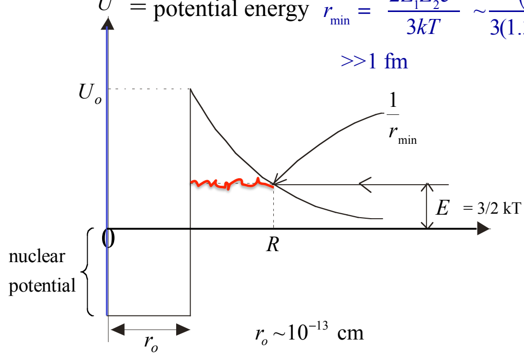
$$\frac{1}{2}mv^2 = KE = \frac{Z_1 Z_2 e^2}{r}$$

$$\frac{3}{2}kT = \frac{Z_1 Z_2 e^2}{r}$$

U = potential energy

$$r_{\min} = \frac{2Z_1 Z_2 e^2}{3kT} \sim \frac{2\left(4.80\times10^{-10}\right)^2}{3(1.38\times10^{-16})(10^7)} = 1.1\times10^{-10}\text{ cm}$$

$>>1$ fm



QM Barrier Penetration

The *classical turning radius* is given by energy conservation

$$\frac{1}{2}mv^2 = KE = \frac{Z_1 Z_2 e^2}{r} \Rightarrow \boxed{r_{classical} = \frac{2Z_1 Z_2 e^2}{mv^2}}$$

The De Broglie wavelength of the particle with mass m is,

$$\lambda = \frac{\hbar}{p} = \frac{\hbar}{mv}$$

So the ratio

$$\frac{r_{classical}}{\lambda} = \frac{2Z_1 Z_2 e^2}{mv^2}\frac{mv}{\hbar} = \frac{Z_1 Z_2 e^2}{\hbar v} \equiv \eta$$

The Sommerfeld parameter

Which is close to the factor in exponential in the penetration function

$$\boxed{\text{P} \sim \exp(-\,r_{classical}/\lambda) \propto \exp\left(-\frac{2\pi Z_1 Z_2 e^2}{\hbar v}\right)} \qquad note: r_{classical} >> \lambda$$

Note that as the charges become big or E gets small, P gets very small.

For particles with charge, providing X(A,E) does not vary rapidly.
with energy (exception to come), i.e., the nucleus is "structureless"

$$\sigma(E) = \pi\lambda\!\!\!^{-2} P_l\, X(A,E) \propto \frac{e^{-2\pi\eta}}{E}$$

This motivates the definition of an "S-factor" which should be ~ constant

$$\boxed{S(E) = \sigma(E)\, E \exp(2\pi\eta)}$$

$$\eta = \frac{Z_I Z_j e^2}{\hbar v} = 0.1575\, Z_I Z_j \sqrt{\hat{A}/E}$$

$$\hat{A} = \frac{A_I A_j}{A_I + A_j} \qquad \text{E in MeV}$$

This S-factor should vary slowly with energy. The first order effects of the Coulomb barrier and Compton wavelength have been factored out.

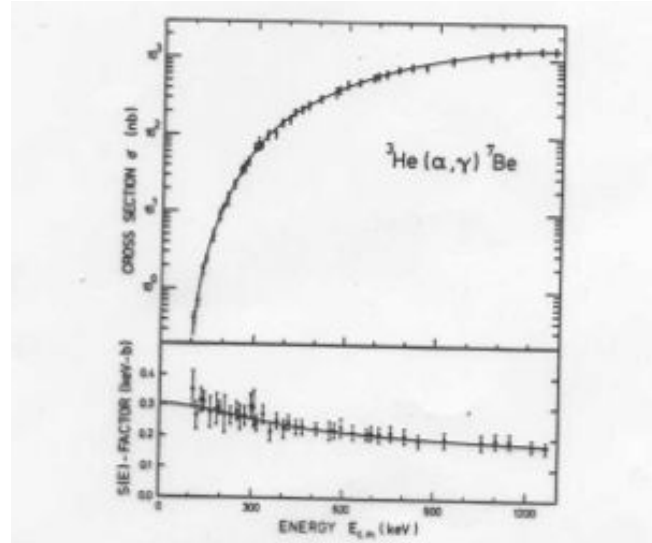


FIGURE 4.4. Energy dependence of the cross section $\sigma(E)$ and the factor $S(E)$ for the $^3He(\alpha,\gamma)^7Be$ reaction (Krä82). The line through the data points represents a theoretical description of the cross section in terms of the direct-capture model. This theory is used to extrapolate the data to zero energy.

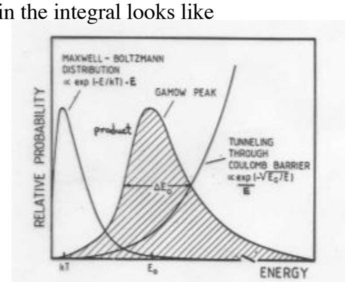
For those reactions in which S(E) is a slowly varying function of energy in the range of interest and can be approximated by its value at the energy where the integrand in the reaction rate formula is a maximum, E_0,

$$\sigma(\mathrm{E}) \simeq \frac{S(E_0)}{E} \exp(-2\pi\eta)$$

$$N_A \langle \sigma v \rangle \approx N_A \left(\frac{8}{\pi\mu}\right)^{1/2} \left(\frac{1}{kT}\right)^{3/2} S(E_o) \int_0^\infty \exp(-E/kT - 2\pi\eta(E))\, dE$$

where $\eta(E) = \dfrac{Z_I Z_j e^2}{\hbar v} = 0.1575 \sqrt{\hat{A}/E(MeV)}\; Z_I Z_j$

The quantity in the integral looks like



For accurate calculations we would just enter the energy variation of S(E) and do the integral numerically. However, Pols 6.2.2 shows that

$\exp\left(\dfrac{-E}{kT} - 2\pi\eta\right)$ can be replaced to good accuracy by

$\mathrm{C}\exp\left(\dfrac{-(E-E_0)^2}{(\Delta/2)^2}\right)$, i.e. a Gaussian

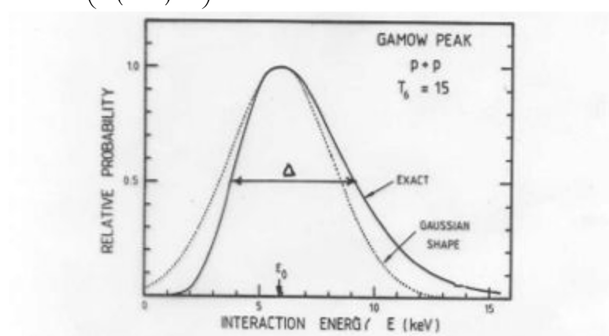


FIGURE 4.7. Curves for the Gamow peak for the $p + p$ reaction at $T_6 = 15$, as obtained from the exact expression and from the approximation using the Gaussian function.

Detail: $\exp\left(\frac{-E}{kT}-2\pi\eta\right) \approx C\exp\left(\frac{-(E-E_0)^2}{(\Delta/2)^2}\right)$

1) Get E_0 from $\frac{d}{dE}\exp\left(\frac{-E}{kT}-2\pi\eta\right) = 0$

with $\eta = 0.1575\sqrt{\hat{A}/E(MeV)}\, Z_I Z_j = \eta_0/\sqrt{E(MeV)}$

gives $E_0 = (\pi\eta_0 kT)^{2/3}$

$= (\pi(0.1575\hat{A}^{1/2} Z_I Z_j)kT)^{2/3}$

$= 0.122(Z_I^2 Z_j^2 \hat{A} T_9^2)^{1/3}$ MeV

2) Evaluate C at E_0 $\quad C\exp\left(\frac{-(E-E_0)^2}{(\Delta/2)^2}\right) = \exp\left(\frac{-E}{kT}-2\pi\eta\right)_{E_0}$

gives $C = \exp\left(\frac{-E_0}{kT} - 2\pi\eta(E_0)\right) = \exp\left(\frac{-3E_0}{kT}\right)$ [Pols 6.2.2]

3) Δ is determined by matching the 2^{nd} derivatives at E_0

[Pols 6.2.2] $\quad \Delta = 4\left(\frac{E_0 kT}{3}\right)^{1/2}$

Detail:

Then if $\tau = \frac{3E_0}{kT}$

$$\lambda \approx N_A\left(\frac{8}{\mu\pi}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2} e^{-\tau}\int_{-\infty}^{\infty} S(E)\exp\left[-\left(\frac{E-E_0}{\Delta/2}\right)^2\right]dE$$

Then if S(E) is constant in the vicinity of E_0

$$\lambda = N_A\left(\frac{8}{\mu\pi}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2} e^{-\tau} S(E_0)\int_{-\infty}^{\infty} \exp\left[-\left(\frac{E-E_0}{\Delta/2}\right)^2\right]dE$$

$$= N_A\left(\frac{8}{\mu\pi}\right)^{1/2}\left(\frac{1}{kT}\right)^{3/2} e^{-\tau} S(E_0)\ \sqrt{\pi}\,\frac{\Delta_{FW}}{2}$$ (Pols 6.2.7 with $\Delta_{1/2} \to \frac{\Delta_{FW}}{2}$)

but $\Delta/(2(kT)^{3/2}) = \left(\frac{4}{9\sqrt{3}}\right)\left(\frac{1}{2\pi\eta_0}\right)\tau^2$

So $\lambda = N_A\langle\sigma v\rangle = N_A\left(\frac{2}{3\mu}\right)^{1/2}\left(\frac{8}{9}\right)\left(\frac{1}{2\pi\eta_0}\right) S(E_o)\tau^2 e^{-\tau}$

$= \frac{4.34\times10^8}{\hat{A} Z_I Z_j} S(E_0)\,\tau^2 e^{-\tau}$ [Pols 6.28]

Summary

We replaced the exponential term with a Gaussian that was analytically integrable. Matching the first and second derivatives of the two functions, the maximum and full width of the Gaussian were determined

The peak is E_o, the *Gamow Energy*. This is the center-o-mass energy where most reactions happen. It is >>kT

$$\boxed{E_o = 0.122\left(Z_I^2 Z_j^2 \hat{A} T_9^2\right)^{1/3}\ \text{MeV}}$$

$\hat{A} = \frac{A_I A_j}{A_I + A_j}$

$T_9 = T/10^9$ K

The Gaussian has a full width at $1/e$ times the maximum. This gives the range of interesting energies, e.g., in which the cross section needs to be measured.

$$\boxed{\Delta = \frac{4}{\sqrt{3}}(E_o kT)^{1/2} = 0.237\left(Z_I^2 Z_j^2 \hat{A} T_9^5\right)^{1/6}\ \text{MeV}}$$

Example

$^3He + {}^3He$ at 1.5×10^7 K

$E_o = 0.122\left(Z_I^2 Z_j^2 \hat{A} T_9^2\right)^{1/3}$ MeV

$= 0.122\,(2^2 2^2 \frac{9}{6}(0.015)^2)^{1/3} = 0.0214$ MeV $= 21.4$ keV

$\Delta = 0.237\left(Z_I^2 Z_j^2 \hat{A} T_9^5\right)^{1/6}$ MeV

$= 0.237\left(2^2 2^2 \frac{9}{6}(0.015)^5\right)^{1/6} = 0.0122$ MeV $= 12.2$ keV

The relevant energy range in which we need to know S(E) is 21 ± 6 keV

The value of kT for comparison is 1.3 keV

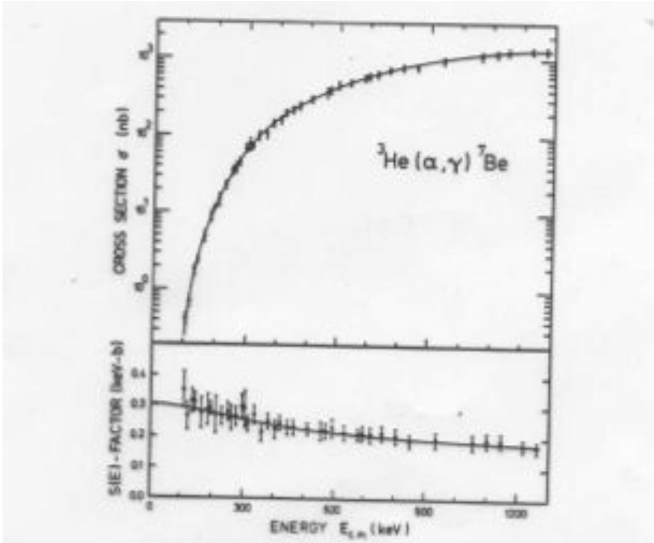


FIGURE 4.4. Energy dependence of the cross section $\sigma(E)$ and the factor $S(E)$ for the ^{3}He(α,γ)^{7}Be reaction (Kr82). The line through the data points represents a theoretical description of the cross section in terms of the direct-capture model. This theory is used to extrapolate the data to zero energy.

A Gaussian integral is analytic $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$ and so

$$N_A\langle\sigma v\rangle = \frac{4.34\times10^8}{\hat{A}Z_IZ_j}\,S(E_0)\,\tau^2 e^{-\tau}\ \mathrm{cm}^3/(\mathrm{Mole\ s})$$

where S(E_0) is measured in MeV barns. If we define

$$\lambda_{jk} = N_A\langle\sigma_{jk}v\rangle$$

then a term in the rate equation for species I such as $Y_j\rho\lambda_{jk}$ has units

$$\left(\frac{Mole}{gm}\right)\left(\frac{gm}{cm^3}\right)\left(\frac{cm^3}{Mole\ s}\right) = s^{-1}$$

Different people use different conventions for λ which sometimes do or do not include ρ or N_A. This defines mine.

Note that τ here is

$$\tau = \frac{3E_0}{kT} = 4.248\left(\frac{Z_I^2Z_j^2\,\hat{A}}{T_9}\right)^{1/3}$$

The lifetime of the species I against the reaction I(j,k)L is given by

$$\tau_{jk}(I) = \left[\frac{1}{Y_I}\frac{dY_i}{dt}\right]_{jk}^{-1} = \left(\rho Y_j\lambda_{jk}(I)\right)^{-1}$$

E.g., the lifetime of ^{14}N against the reaction ^{14}N (p,γ)^{15}O is

$$\tau_{p\gamma}(^{14}N) = \left(\rho Y_p\lambda_{p\gamma}(^{14}N)\right)^{-1}$$

Adelberger et al, RMP, (1998(

TABLE I. Best-estimate low-energy nuclear reaction cross-section factors and their estimated 1σ errors.

Reaction	$S(0)$ (keV b)	$S'(0)$ (b)
^{1}H($p,e^+\nu_e$)^{2}H	$4.00(1\pm0.007^{+0.020}_{-0.011})\times10^{-22}$	4.48×10^{-24}
^{1}H(pe^-,ν_e)^{2}H	Eq. (19)	
^{3}He(^{3}He,2p)^{4}He	$(5.4\pm0.4)^a\times10^{-3}$	
^{3}He(α,γ)^{7}Be	0.53 ± 0.05	-3.0×10^{-4}
^{3}He($p,e^+\nu_e$)^{4}He	2.3×10^{-20}	
^{7}Be(e^-,ν_e)^{7}Li	Eq. (26)	
^{7}Be(p,γ)^{8}B	$0.019^{+0.004}_{-0.002}$	See Sec. VIII.A
^{14}N(p,γ)^{15}O	$3.5^{+0.4}_{-1.6}$	See Sec. IX.A.5

$$\lambda \propto f(T) = \tau^2 e^{-\tau} \qquad \tau = \frac{A}{T^{1/3}} \qquad \frac{d\tau}{dT} = -\frac{A}{3T^{4/3}} = -\frac{\tau}{3T}$$

$$\frac{df}{dT} = 2\tau e^{-\tau}\frac{d\tau}{dT} - \tau^2 e^{-\tau}\frac{d\tau}{dT}$$

$$\frac{T}{f}\left(\frac{df}{dT}\right) = \frac{T}{\tau^2 e^{-\tau}}(2\tau e^{-\tau})(-\frac{\tau}{3T}) - \frac{T}{\tau^2 e^{-\tau}}(\tau^2 e^{-\tau})(-\frac{\tau}{3T})$$

$$= \left(\frac{d\ln f}{d\ln T}\right) = \frac{\tau-2}{3}$$

$$\therefore\ f \propto T^n \qquad \boxed{n = \frac{\tau-2}{3}}$$

For example, $^{12}C + {}^{12}C$ at 8×10^8 K

$$\tau = 4.248\left(\frac{6^2\, 6^2\, \frac{12 \cdot 12}{12+12}}{0.8}\right)^{1/3}$$

$$= 90.66$$

$$n = \frac{90.66-2}{3} = 29.5$$

p + p at 1.5×10^7 K

$$\tau = 4.248\left(\frac{1 \cdot 1 \cdot \frac{1 \cdot 1}{1+1}}{0.015}\right)^{1/3}$$

$$= 13.67$$

$$n = \frac{13.67 - 2}{3} = 3.89$$

Resonant reactions

The previous discussion was predicated upon the assumption that the nuclear structure factor X was slowly varying with energy. This led to an S-factor that was also nearly constant.

This turns out to be the case when the excited state structure of the nucleus can be ignored. But suppose I and j come together at the Gamow energy to produce L in an excited state that has just the right spin, angular momentum, and parity to closely resemble I + j.

Such reactions are called "resonant" and the cross section in a narrow energy range can be many of orders of magnitude larger than if the resonant state were not there. See appendix to these notes.

Specific Reactions in Hydrogen and Helium Burning

- pp1 chain
- pp2 and pp3 chains
- CNO cycle
- 3-alpha and $^{12}C(\alpha,\gamma)^{16}O$

Proton-proton reaction (the basic first step for pp1, 2, 3):

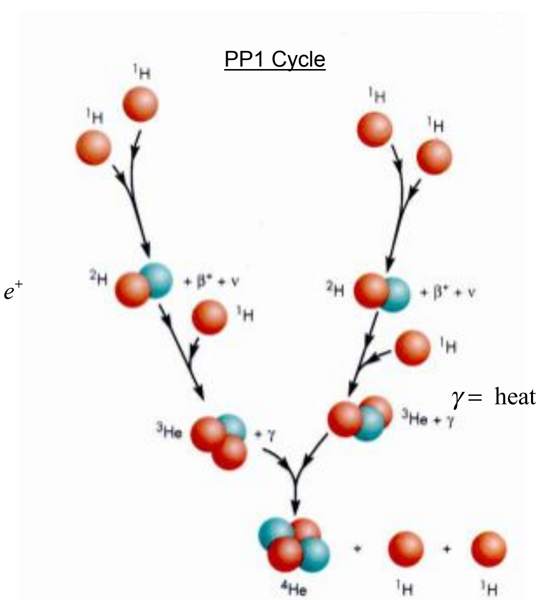
$$p(p,e^+\nu_e)^2\mathrm{H} \quad (+0.42\,\mathrm{MeV})$$

This cross section is far too small ($\sim 10^{-47}$ cm^2 at 1 MeV) to measure in the laboratory but it does have a nearly constant, calculable S-factor.

Two stages:

- Temporarily form diproton. The initial wave function is the same as for proton scattering. Initially the diproton must have its protons spins counter alligned because can't have protons in identical states. Unbound.
- Diproton experiences a weak interaction (with a spin flip) to make deuteron with the neutron and proton alligned. Higher this is more tightly bound than the counter alligned state

Here $\beta^+ = e^+$



$\gamma =$ heat and light

So now we have protons, ^{4}He, and ^{2}H. Next

$$^2H + p \rightarrow {}^3He + \gamma + 5.49\,\mathrm{MeV}$$

no weak interaction needed, very fast

or ^{2}H(p,γ)^{3}He. This may be followed by either

$$a)\ {}^3He + {}^3He \rightarrow {}^4He + 2p + 12.86\,\mathrm{MeV} \qquad \text{pp1}$$

or $\qquad$ ^{4}Li is unbound

$$b)\ {}^3He + {}^4He \rightarrow {}^7Be + \gamma + 1.59\,MeV \qquad \text{pp2,3}$$

H. A. Bethe (b 1906)
Nobel 1967

Lifetimes against various reactions

Reaction	Lifetime (years)
^{1}H(p,e$^+\nu$)^{2}H	7.9 x 10^9
^{2}H(p,γ)^{3}He	4.4 x 10^{-8}
^{3}He(^{3}He,2p)^{4}He	2.4 x 10^5
^{3}He(^{4}He,γ)^{7}B	9.7 x 10^5

For 50% H, 50% He at a density of 100 g cm^{-3} and a temperature of 15 million K

The time between proton collisions, for a given proton, is about a hundred millionth of a second.

Nuclear energy yield

How much energy is produced and how fast is it liberated. We could just add up the energies of each reaction and divide that number by the time scale for the slowest reaction. Instead let's solve the general problem (make things easier in the future).

We have a set of nuclei {Y_i } that is transformed into a new set {$Y_i+\delta Y_i$}. Each nucleus has binding energy BE_i in MeV.

$$q_{nuc} = 1.602\times10^{-6}\, N_A[\sum(\delta Y_i)(BE_i - q_{weak})] \text{ erg/gm}$$

Here 1.602 x 10^{-6} is the conversion factor from MeV (which are the units of BE) to erg. q_{weak} is a correction term that is only non zero for weak interactions. It accounts for neutrinos lost and mass changes when protons and electrons turn to neutrons or vice versa.

q_{weak}

Basically important *only in hydrogen burning*

q_{weak} accounts for the neutron proton mass difference, the creation and annihilation of positrons, and energy lost to neutrinos

$$\begin{aligned} q_{weak} &= \Delta Z * [(n-p \text{ mass difference}) + m_e c^2] - q_\nu \\ &= \Delta Z * [1.294 \text{ MeV-}0.511 \text{ MeV}] - q_\nu \\ &= \Delta Z * [0.783 \text{ MeV}] - q_\nu \end{aligned}$$

For example in the pp1 process 4 protons urn into 2 neutons and 2 protons so $\Delta Z = -2$. In addition two neutrinos are emitted with an average of 0.26 MeV each, so $q_\nu = 0.52$ MeV and $q_{weak} = -2.09$MeV

Example: Hydogen burning

$a)$ 100% ^{1}H $\to$ ^{4}He $\quad \delta Y(^1H) = -1 \quad BE(^1H) = 0$

$$\delta Y(^4He) = \frac{1}{4} \quad BE(^4He) = 28.296\,\text{MeV}$$

From previous page $q_{weak} = -2.09$

$$q = 9.65\times10^{17}\left(\frac{28.296-2.09}{4}\right) = 6.32\times10^{18} \text{ erg g}^{-1}$$

$b)$ 70% ^{1}H; 30% ^{4}He $\to$ ^{4}He $\quad \delta Y(^1H) = -0.7$

$$\delta Y(^4He) = \frac{1}{4} - \frac{0.3}{4}$$

$$q = 9.65\times10^{17}\left(\frac{1}{4} - \frac{0.3}{4}\right)(28.296-2.09) = \boxed{4.42\times10^{18} \text{ erg g}^{-1}}$$

$$\left.\begin{array}{l} BE(^{12}C) = 92.162 \text{ MeV} \\ BE(^{16}O) = 127.619 \\ BE(\alpha) = 28.296 \text{ MeV} \end{array}\right\} \text{ values for helium burning}$$

$q_{weak} = 0$

A related quantity, the energy generation rate is given by

$$\boxed{\varepsilon_{nuc} = 9.65\times10^{17}\sum \frac{dY_i}{dt}(BE_i - q_{weak}) \text{ erg g}^{-1}\text{ sec}^{-1}}$$

Additional nuclear physics needed for post-helium burning stages will be covered when we treat the advanced burning stages of massive stars later in the course.

For hydrogen burning by the pp1 process

We could include all the reactions and evaluate $\frac{dY_i}{dt}$ for each species but it is much easier to just realize that the net result is that every time 4 protons are burned by the reaction p(p,e$^+\nu$), one ^{4}He appears. So

$$\frac{dY(^4He)}{dt} = -\frac{1}{4}\frac{dY(^1H)}{dt} = \frac{1}{4}2\rho Y^2(^1H)\lambda_{pp} = \frac{1}{2}\rho Y^2(^1H)\lambda_{pp}$$

$$\varepsilon_{nuc} = 9.65\times10^{17}\sum \frac{dY_i}{dt}(BE_i - q_{weak}) \text{ erg g}^{-1}\text{ s}^{-1}$$

$$= \frac{9.65\times10^{17}}{2}\rho Y^2(^1H)(26.2)\lambda_{pp}$$

$$= 1.26\times10^{19}\ \rho Y^2(^1H)\ \lambda_{pp} \quad \text{erg g}^{-1}\ \text{s}^{-1}$$

Previously we showed

$$\lambda = \frac{4.34\times10^8}{\hat{A}Z_IZ_j}S(E_0)\,\tau^2 e^{-\tau}\ \text{cm}^3/(\text{Mole s})$$

for the pp reaction $\hat{A} = 1*1/(1+1) = 1/2$

$$\tau = 4.248\left(\frac{Z_I^2Z_j^2\,\hat{A}}{T_9}\right)^{1/3} = 4.248\left(\frac{1\bullet1\bullet1/2}{T_9}\right)^{1/3}$$

$$= 3.37\ T_9^{-1/3} = 33.7\ T_6^{-1/3}$$

$$S_0 = 4.0\times10^{-22}\text{ keV b} = 4.0\times10^{-25}\text{ MeV b}$$

$$\lambda_{pp} = \frac{1}{2}\frac{4.34\times10^8}{1/2}4.0\times10^{-25}\left(\frac{33.7}{T_6^{1/3}}\right)^2 e^{-33.7/T_6^{1/3}}$$

$$= 1.97\times10^{-13}T_6^{-2/3}e^{-33.7/T_6^{1/3}}$$

Compare with Clayton 5.11

Putting it together

$$\varepsilon = 1.26\times10^{19}\cdot1.97\times10^{-13}\ \rho Y_H^2T_6^{-2/3}e^{-33.7/T_6^{1/3}} \quad \text{erg g}^{-1}\text{ s}^{-1}$$

e.g. $\rho = 150$ g cm^{-3} $T_6 = 15$ $Y_H = 0.35$

$$\varepsilon = (2.48\times10^6)(150)(0.35)^2\ (15)^{-2/3}\ e^{-13.66} \quad \text{erg g}^{-1}\text{ s}^{-1}$$

$$= 8.7 \text{ erg g}^{-1}\text{ s}^{-1}$$

Given our previous discussion this can also be written as a power law of the temperature with n = $\frac{\tau-2}{3}$

$$\tau = 33.7/(15)^{1/3} = 13.66 \Rightarrow n = 3.89$$

So $$\boxed{\varepsilon_{pp} \approx 8.7\left(\frac{\rho}{150\text{ g cm}^{-3}}\right)\left(\frac{Y_H}{0.35}\right)^2\left(\frac{T}{15\times10^6\text{ K}}\right)^{3.89} \text{ erg g}^{-1}\text{ s}^{-1}}$$

Reality checks:

1) Energy budget for the sun

$$Q_\odot = L_\odot \tau_{MS}(\odot) \approx (4\times10^{33}\ \text{erg s}^{-1})(10^{10}\ \text{yr})(3.16\times10^{7}\ \text{s/yr})$$
$$\sim 1\ \times 10^{51}\ \text{erg}$$
$$Q_{nuc} \approx (4.4\times10^{18}\ \text{erg g}^{-})(0.1)(M_\odot)$$
$$= (4.4\times10^{18}\ \text{erg g}^{-1})(2\times10^{32}\ \text{g}) = 9\ \times10^{50}\ \text{erg}$$

2) Luminosity of the sun

$$L_\odot = 3.84\times10^{33}\ \text{erg s}^{-1}$$
$$\varepsilon_{pp} \bullet (0.1M_\odot) \sim 9\,\text{erg g}^{-1}\ \text{s}^{-1}(2\times10^{32}\ \text{g}) \sim 2\times10^{33}\text{erg s}^{-1}$$

Could adjust Y_H, T, ρ, or fraction burning to get better agreement

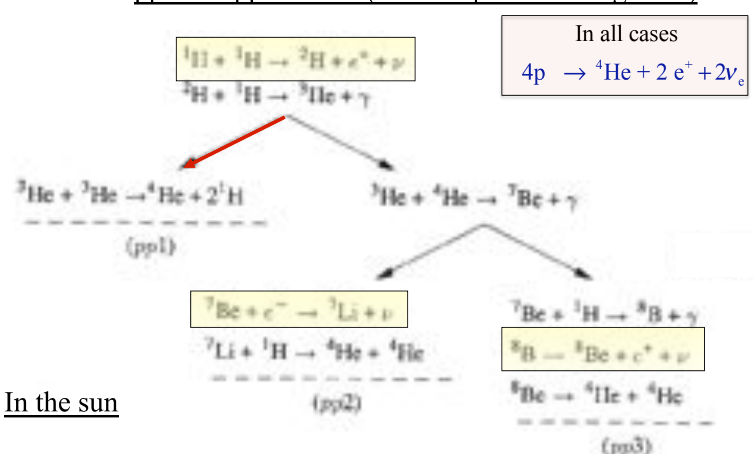
3) Lifetime

$$\tau_{MS} \sim \tau_{pp} = \left(Y_H \,/\, \frac{dY_H}{dt}\right)^{-1}$$
$$= \left(\frac{Y_H}{\rho Y_H^2 \lambda_{pp}}\right)^{-1}$$
$$= (1.97\times10^{-13}\rho Y_H T_6^{-2/3} e^{-33.7/T_6^{1/3}})^{-1}$$

At $\rho = 150\,\text{g cm}^{-3};\ T_6 = 15;\ Y_H = 0.7$

$$\tau_{MS} \sim \left((2x10^{-13})(150)(0.7)(0.164)(1.1x10^{-6})\right)^{-1}$$
$$= \left(3.8\times10^{-18}\right)^{-1}\ \text{s}\ = 8\ \text{billion years}$$

pp2 and pp3 chains (more important at higher T)



In the sun

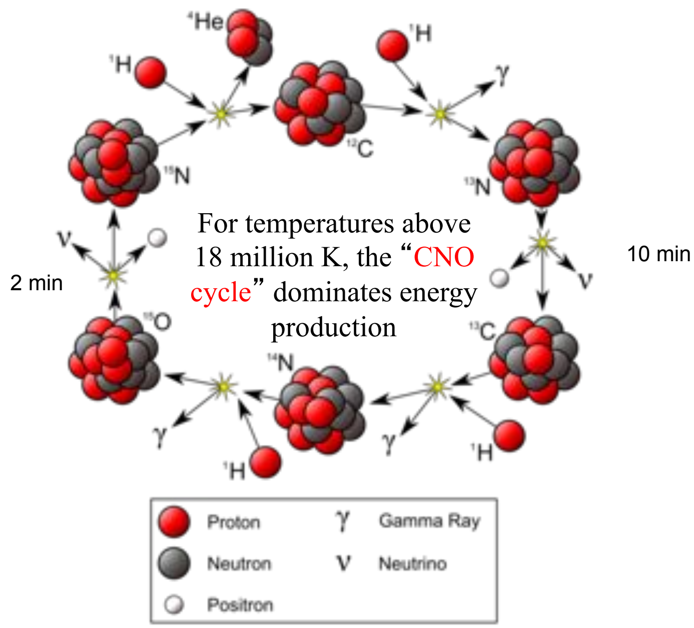
- pp1 85%
- pp2 15%
- pp3 0.02%

$T_{central} = 15.7$ Million K

Neutrino Energies

Species	Average energy	Maximum energy	
p+p	0.267 MeV	0.420 MeV	
^{7}Be	0.383 MeV 0.861	0.383 MeV 0.861	10% 90%
^{8}B	6.735 MeV	15 MeV	

In the case of ^{8}B and p+p, the energy is shared with a positron hence there is a spread. For ^{7}Be the electron capture goes to two particular states in ^{7}Li and the neutrino has only two energies



Detail

Rate equations:

$$\frac{dY(^{12}C)}{dt} = -Y(^{12}C)Y_p\rho\lambda_{p\gamma}(^{12}C)+Y(^{15}N)Y_p\rho\lambda_{p\alpha}(^{15}N)$$

$$\frac{dY(^{13}N)}{dt} = Y(^{12}C)Y_p\rho\lambda_{p\gamma}(^{12}C)-Y(^{13}N)\lambda_{e^+}(^{13}N)$$

$$\frac{dY(^{13}C)}{dt} = Y(^{13}N)\lambda_{e^+}(^{13}N)-Y(^{13}C)Y_p\rho\lambda_{p\gamma}(^{13}C)$$

$$\frac{dY(^{14}N)}{dt} = Y(^{13}C)Y_p\rho\lambda_{p\gamma}(^{13}C)-Y(^{14}N)Y_p\rho\lambda_{p\gamma}(^{14}N)$$

$$\frac{dY(^{15}O)}{dt} = Y(^{14}N)Y_p\rho\lambda_{p\gamma}(^{14}C)-Y(^{15}O)\lambda_{e^+}(^{13}N)$$

$$\frac{dY(^{15}N)}{dt} = Y(^{15}O)\lambda_{e^+}(^{13}N)-Y(^{15}N)Y_p\rho\lambda_{p\alpha}(^{15}N)$$

$$\frac{dY_p}{dt} = -Y(^{12}C)Y_p\rho\lambda_{p\gamma}(^{12}C)-Y(^{13}C)Y_p\rho\lambda_{p\gamma}(^{13}C)$$
$$-Y(^{14}N)Y_p\rho\lambda_{p\gamma}(^{14}N)-Y(^{15}N)Y_p\rho\lambda_{p\alpha}(^{15}N)$$

$$\frac{dY_\alpha}{dt} = Y(^{15}N)Y_p\rho\lambda_{p\alpha}(^{15}N)$$

The slowest reaction is $^{14}N(p,\gamma)^{15}O$ so for purposes of energy generation the rate equations can be approximately be written:

$$\frac{dY_p}{dt} = -4Y(^{14}N)Y_p\rho\lambda_{p\gamma}(^{14}N)$$

$$\frac{dY_\alpha}{dt} = Y(^{14}N)Y_p\rho\lambda_{p\gamma}(^{14}N)$$

where $Y(^{14}N) \approx \frac{1}{14}(X_C + X_N + X_O) \approx \frac{Z}{14}$

One still needs to subtract off the energy carried away by neutrinos and adjust for n - p mass differences

$$q_{weak} = \Delta Z * [0.783 \text{ MeV}] + q_\nu = (2)(0.783) + 1.70$$
$$= 3.26 \text{ MeV}$$

What is $\lambda_{p\gamma}(^{14}N)$?

The S factor for $^{14}N(p,\gamma)^{15}O$ is 1.64×10^{-3} MeV barns (including a recent downward revision)

$$\lambda = \frac{4.34\times10^8}{\hat{A}Z_IZ_j}\, S(E_0)\,\tau^2 e^{-\tau} \text{ cm}^3/(\text{Mole s})$$

for the $^{14}N(p,\gamma)$ reaction $\hat{A} = 1*14/(14+1) = 14/15$

$$\tau = 4.248\left(\frac{Z_I^2 Z_j^2 \hat{A}}{T_9}\right)^{1/3} = 4.248\left(\frac{1\bullet 7^2 \bullet 14/15}{T_9}\right)^{1/3}$$
$$= 15.19\, T_9^{-1/3}$$

$$\lambda_{p\gamma}(^{14}N) = \frac{4.34\times10^8}{7(14/15)}\left(1.64\times10^{-3}\right)(15.19)^2\, T_9^{-2/3} e^{-15.19/T_9^{1/3}}$$
$$= \boxed{2.51\times10^7 T_9^{-2/3} e^{-15.19/T_9^{1/3}}}$$
$$= 1.69\times10^{-16} \quad \text{for example at } T_9 = 0.02 \ (20\ M\ K)$$

$$\frac{dY_\alpha}{dt} = Y(^{14}N)Y_p \rho \lambda_{p\gamma}(^{14}N)$$
$$\approx \frac{Y_p Z}{14} \rho \lambda_{p\gamma}(^{14}N)$$
$$\text{At } T_9 = 20,\ \tau = 56.0,\ n = \frac{\tau - 2}{3} = 18$$
$$\lambda_{p\gamma}(^{14}N) \approx 1.7 \times 10^{-16} \left(\frac{T}{20\,MK}\right)^{18}$$
$$\frac{dY_\alpha}{dt} \approx 1.2 \times 10^{-17} \rho Y_p Z \left(\frac{T}{20\,MK}\right)^{18}$$
$$\varepsilon_{nuc} = 9.65 \times 10^{17} \sum \frac{dY_i}{dt}(BE_i - q_{weak}) \ \text{erg g}^{-1}\text{sec}^{-1}$$
$$= 11.7 \rho Y_p Z \left(\frac{T}{20\,MK}\right)^{18} (28.296 - 3.26)$$
$$\boxed{\varepsilon_{CNO} = 293\ \rho Y_p Z \left(\frac{T}{20\,MK}\right)^{18} \ \text{erg g}^{-1}\text{sec}^{-1}}$$

Ratio of CNO energy generation to pp

$$\varepsilon_{CNO} = 293\ \rho Y_p Z \left(\frac{T}{20\,MK}\right)^{18} \ \text{erg g}^{-1}\text{sec}^{-1}$$
$$\varepsilon_{pp} = 1.26 \times 10^{19} \cdot 1.97 \times 10^{-13}\ \rho Y_p^2 T_6^{-2/3} e^{-33.7/T_6^{1/3}}$$
$$= 1.36\ \rho Y_p^2 \left(\frac{T_9}{0.02}\right)^{3.89} \ \text{erg g}^{-1}\text{sec}^{-1}$$

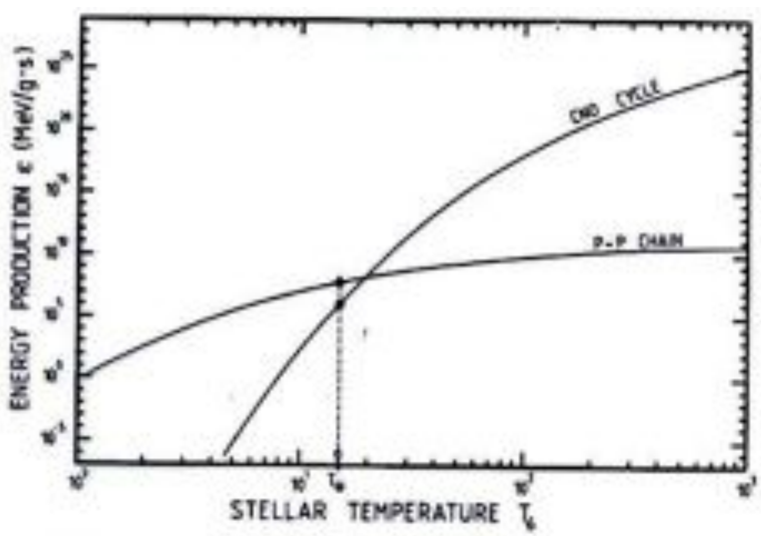
The ratio = 1 when

$$\frac{293}{1.36}\left(\frac{Z}{Y_p}\right)\left(\frac{T_9}{0.02}\right)^{14} = 1$$

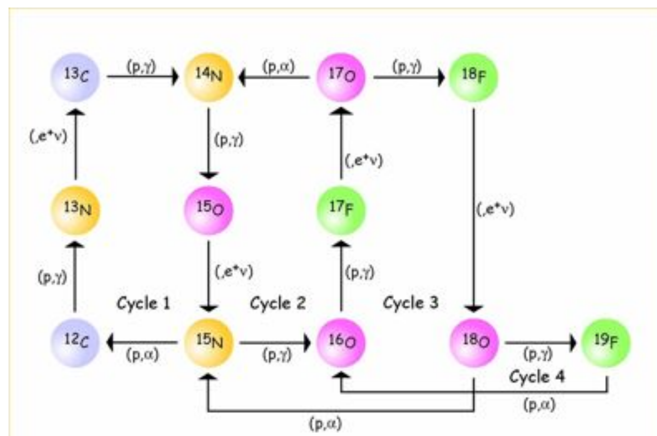
and if Z= 0.015, $Y_p = 0.7$

$$4.62\left(\frac{T_9}{0.02}\right)^{14} = 1 \qquad T_9 = 0.0179$$

CNO will dominate above about 18 MK



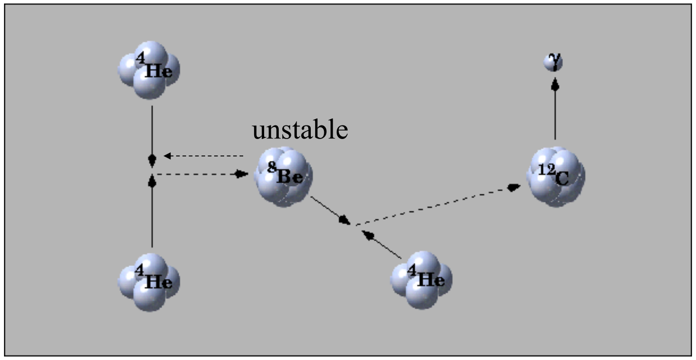
CNO Tri-cycle



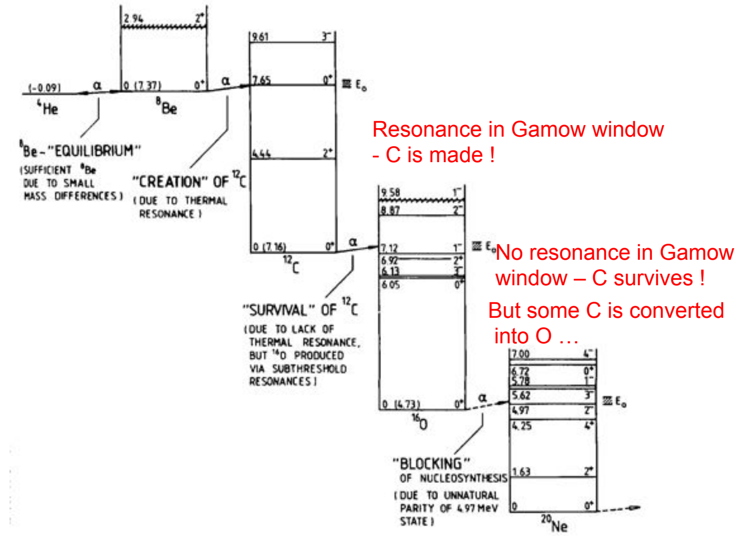
Note nucleosynthetic implications:
Synthesis of some ^{13}C. ^{14,15}N, ^{17,18}O, ^{19}F

Helium Burning

Helium burning is a two-stage nuclear process in which two alpha-particles temporarily form the ground state of unstable $^8Be^*$. Occasionally the $^8Be^*$ captures a third alpha-particle before it flies apart. No weak interactions are involved.



Helium burning 2 – the $^{12}C(\alpha,\gamma)$ rate



The current value is due to Caughlan and Fowler (1988) using mesurements from Sam Austin

$$\lambda_{3\alpha} = 2.79 \times 10^{-8}\ T_9^{-3}\ \exp(-4.403/T_9)$$

Slight revisions to Γ_γ here

T_9	$\frac{d\ln\lambda}{d\ln T}$
0.1	41
0.2	19
0.3	12

$$\frac{d\ln\lambda}{d\ln T} = \frac{4.403}{T_9} - 3$$

Unlike most reactions in astrophysics, the temperature dependence here is not determined by barrier penetration but by the Saha equation. In fact, at high temperature ($T_9 > 1.5$) the rate saturates and actually begins to decline slowly as the resonance slips out of the Gamow window.

Helium Burning

$$\frac{dY_\alpha}{dt} = -3\rho^2 Y_\alpha^3 \lambda_{3\alpha}/6 - Y_\alpha\, Y(^{12}C)\,\rho\,\lambda_{\alpha\gamma}(^{12}C)$$

$$\frac{dY(^{12}C)}{dt} = \rho^2 Y_\alpha^3 \lambda_{3\alpha}/6 - Y_\alpha\, Y(^{12}C)\,\rho\,\lambda_{\alpha\gamma}(^{12}C)$$

$$\frac{dY(^{16}O)}{dt} = Y_\alpha\, Y(^{12}C)\,\rho\lambda_{\alpha\gamma}(^{12}C)$$

For binary reactions, $\lambda \equiv N_A \langle \sigma v \rangle$

For Y_{12} small or ρ large

$$\alpha \rightarrow {}^{12}C$$

For Y_{12} large or ρ small

$$\alpha \rightarrow {}^{16}O$$

Energy Generation from Helium Burning

BE(^{4}He) = 28.296 MeV BE(^{12}C) = 92.162 MeV

BE(^{16}O) = 127.617 MeV

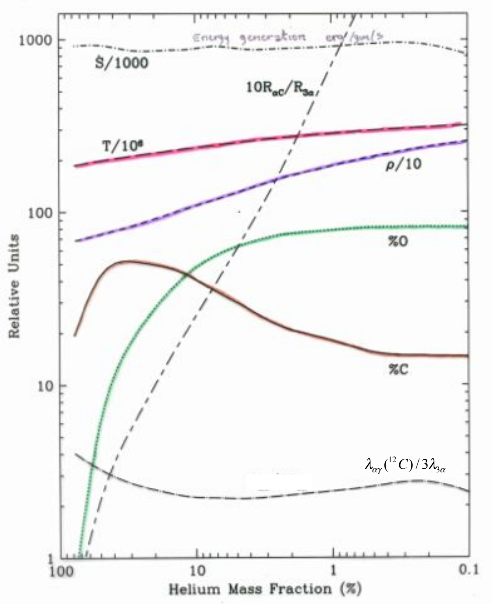
1) 3 ^{4}He $\rightarrow$ ^{12}C

$$q = 9.65\times 10^{17}\ \sum(\delta Y_i)(BE_i)\ \text{erg/gm} \qquad (q_{weak}=0)$$
$$= 9.65\times 10^{17}\left[Y_{final}(^{12}C)\ BE(^{12}C) - Y_{initial}(^4He)\ BE(^4He)\right]$$
$$= 9.65\times 10^{17}\left[\frac{1}{12}92.162 - \frac{1}{4}28.296\right]$$
$$= 5.85\times 10^{17}\ \text{erg g}^{-1}$$

2) 18 ^{4}He $\rightarrow$ ^{12}C + 4 ^{16}O (*i.e.* $\frac{12}{76} = 15.8\%\ C \quad 84.2\%\ O$)

$$= 9.65\times 10^{17}\ [Y_{final}(^{12}C)\ BE(^{12}C) + Y_{final}(^{16}O)\ BE(^{16}O) - Y_{initial}(^4He)\ BE(^4He)]$$
$$= 9.65\times 10^{17}\left[\frac{1}{12}\frac{12}{76}92.162 + \frac{1}{16}\frac{64}{76}127.617 - \frac{1}{4}28.296\right]$$
$$= 8.25\times 10^{17}\ \text{erg g}^{-1}$$

In a 15 solar mass star:



Energy Generation from Helium Burning

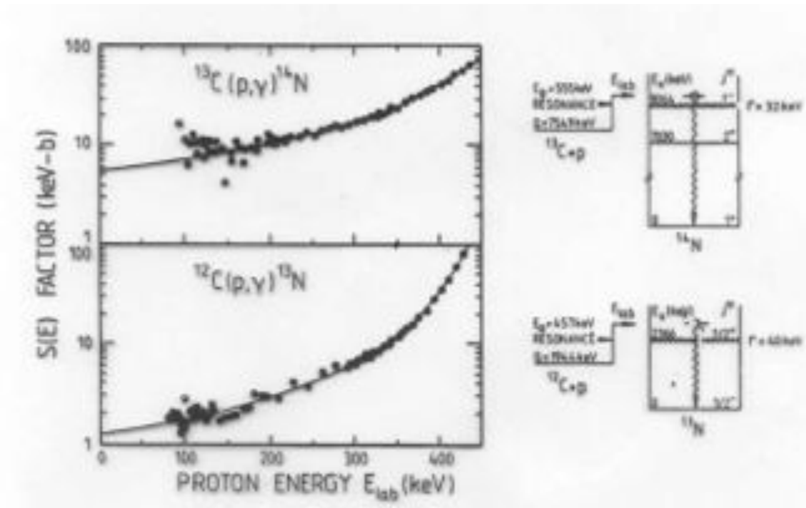
$$\varepsilon_{nuc} = 9.65\times 10^{17}\sum\frac{dY_i}{dt}(BE_i)\ \ \text{erg g}^{-1}\,\text{sec}^{-1}$$
$$\frac{dY_\alpha}{dt} = -3\rho^2 Y_\alpha^3\lambda_{3\alpha}/6 \qquad \text{ignore } ^{12}C(\alpha,\gamma)^{16}O$$
$$\frac{dY(^{12}C)}{dt} = \rho^2 Y_\alpha^3\lambda_{3\alpha}/6$$
$$\varepsilon_{nuc} = 9.65\times 10^{17}\sum\frac{dY_i}{dt}(BE_i)$$
$$= 9.65\times 10^{17}\rho^2 Y_\alpha^3\lambda_{3\alpha}/6\left[92.162 - 3\,(28.296)\right]$$
$$= 1.83\times 10^{16}\rho^2 X_\alpha^3\lambda_{3\alpha}$$
$$= 1.83\times 10^{16}\rho^2 X_\alpha^3\left(2.79\times 10^{-8}\ T_9^{-3}\exp(-4.403/T_9)\right)$$
$$\boxed{\varepsilon_{3\alpha} = 5.11\times 10^{8}\rho^2 X_\alpha^3\ T_9^{-3}\exp(-4.403/T_9)}$$

note typo in GK coeff off by 1000

Extra Material

Resonant Reactions

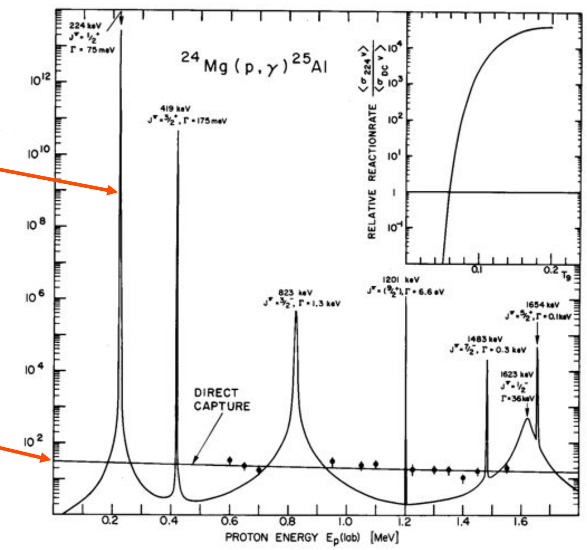
The resonances can be broad



or narrow

Not constant S-factor for resonances (log scale !!!!)

~ constant S-factor for direct capture



If the resonance is much broader than the Gamow energy, then treat with the formalism we have already developed by with a slowly varying S-factor.

For narrow resonances though, width << the Gamow energy, another formalism is employed. This would take some time to develop, so here we just give the result.

Rate of reaction through a narrow resonance

Narrow means: $\Gamma << \Delta E$

In this case, the resonance energy must be "near" the relevant energy range ΔE to contribute to the stellar reaction rate.

pull out front

Recall:

$$<\sigma v> = \sqrt{\frac{8}{\pi\mu}} \frac{1}{(kT)^{3/2}} \int_0^\infty \sigma(E) E \, e^{-\frac{E}{kT}} dE$$

and

$$\sigma(E) = \pi \bar{\lambda}^2 \, \omega \, \frac{\Gamma_1(E)\Gamma_2(E)}{(E-E_r)^2 + (\Gamma(E)/2)^2}$$

Here E_r is the energy of the resonance and the Γ's are the partial widths of the state to break up into I+j, L+k, etc.

Then one can carry out the integration analytically (Clayton 4-193) and finds:

For the contribution of a single narrow resonance to the stellar reaction rate:

$$N_A < \sigma v >= 1.54 \cdot 10^{11} (AT_9)^{-3/2}\, \omega\gamma\, [\mathrm{MeV}]\, \mathrm{e}^{\frac{-11.605\, \mathrm{E_r}\, [\mathrm{MeV}]}{\mathrm{T_9}}} \; \frac{\mathrm{cm}^3}{\mathrm{s\ mole}}$$

The rate is entirely determined by the "resonance strength" $\omega\gamma$

$$\omega\gamma = \frac{2J_r + 1}{(2J_j + 1)(2J_I + 1)} \frac{\Gamma_1 \Gamma_2}{\Gamma}$$

In general the reactions during the proton-proton cycles are non-resonant but the reactions for the CNO cycle and more advanced burning stages are resonant.

Light particle (n, p, α): $\lambda_{jk} \equiv \mathrm{N}_A \langle \sigma_{jk} v \rangle$

$$Y_I Y_p \,\rho\, \lambda_{p\gamma}(I); \quad Y_I Y_n \,\rho\, \lambda_{n\gamma}(I); \quad Y_I Y_p \,\rho\, \lambda_{p\alpha}(I)$$

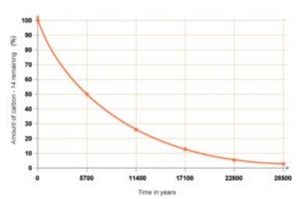
Heavy Ion

$$\frac{1}{6} Y_\alpha^3 \lambda_{3\alpha}; \quad \frac{1}{2} \rho Y^2 \left({}^{12}C\right) \lambda_{12,12}; \quad \frac{1}{2} \rho Y^2 \left({}^{16}O\right) \lambda_{16,16}$$

Weak interaction (beta decay, electron capture, positron emission)

$$Y_I \lambda_\beta(I); \quad Y_I \lambda_{ec}(I); \quad Y_I \lambda_\beta(I) \qquad \lambda = \frac{\ln 2}{\tau_{1/2}}$$

Example of solving a rate equation



E.g. the decay of ^{14}C

$$\frac{dY({}^{14}C)}{dt} = -\, Y({}^{14}C)\, \lambda_\beta({}^{14}C)$$

So

$$Y({}^{14}C) = Y_0({}^{14}C)\, \exp(-\lambda_\beta({}^{14}C)\, t)$$

$$\tau_{1/2} = 5730\ y = 1.81 \times 10^{11}\ \mathrm{sec}$$

$$\lambda = \frac{\ln 2}{1.81 \times 10^{11}} = 3.83 \times 10^{-12}\ \mathrm{sec}^{-1}$$

E.g., $pp1$ hydrogen burning:

$p(p,e^+\nu)^2H(p,\gamma)^3He(^3He,2p)^4He$

$$\frac{dY_p}{dt} = -2\left(\frac{1}{2}\right) Y_p^2 \,\rho\, \lambda_{pp} + 2\left(\frac{1}{2}\right) Y^2\left({}^3He\right) \rho\, \lambda_{3,3}$$

$$\frac{dY({}^2H)}{dt} = \left(\frac{1}{2}\right) Y_p^2 \,\rho\, \lambda_{pp} - Y_p Y\left({}^2H\right) \rho\, \lambda_{p\gamma}({}^2H)$$

$$\frac{dY({}^3He)}{dt} = Y_p Y\left({}^2H\right) \rho\, \lambda_{p\gamma}({}^2H) - 2\left(\frac{1}{2}\right) Y^2\left({}^3He\right) \rho\, \lambda_{3,3}$$

$$\frac{dY({}^4He)}{dt} = \left(\frac{1}{2}\right) Y^2\left({}^3He\right) \rho\, \lambda_{3,3}$$

The rate factors λ contain the temperature dependence of the rate equations. The density dependence has been separated out.