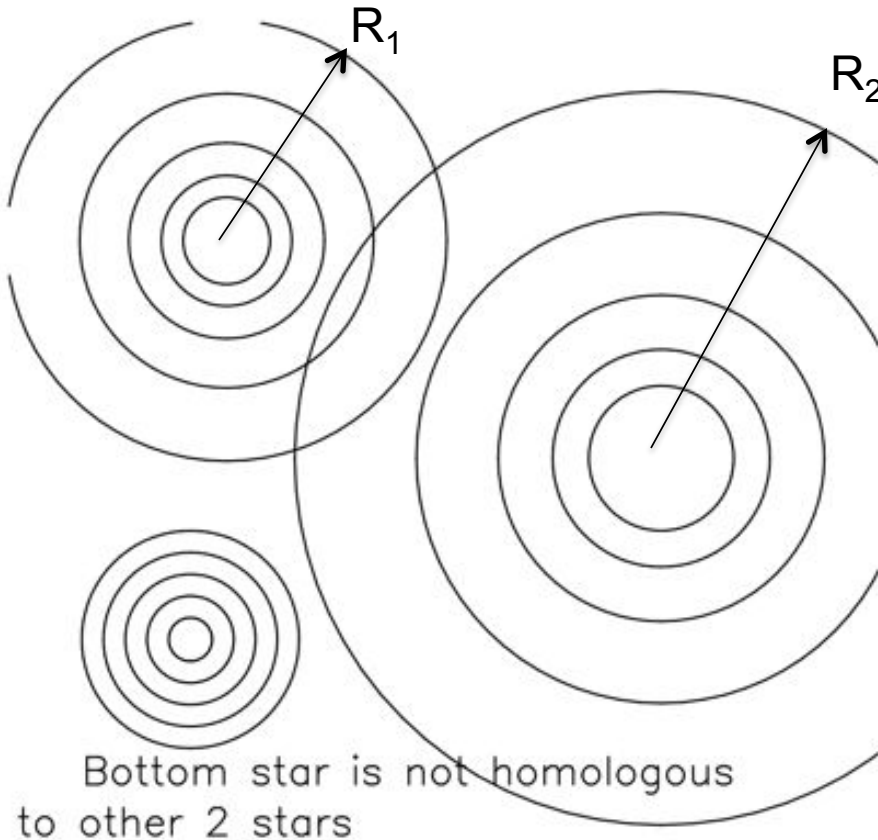


Hydrogen Burning on the Main Sequence

GK 14
Pols 7.7.4
Prialnik 7

Homology relations



Consider 2 stellar models with mass M_1 and M_2 and radius R_1 and R_2

$$\text{Let } x = \frac{m_1}{M_1} = \frac{m_2}{M_2} \quad 0 \leq x \leq 1$$

be a mass coordinate such that $x = 1$ at the surface. The two models are said to be homologous if

$$\frac{r_1(x)}{R_1} = \frac{r_2(x)}{R_2} \quad \text{or} \quad \frac{r_1(x)}{r_2(x)} = \frac{R_1}{R_2}$$

This slide from JS Pineda shows circles indicating the radius that encloses 20% mass increments of two stars that are homologous and one that is not

Then for example the mass conservation equation can be written

for anywhere inside star number 1: $dm_1 = 4\pi r_1^2 \rho_1 dr_1$

$$\frac{dr_1}{dm_1} = \frac{1}{4\pi r_1^2 \rho_1} \quad x = \frac{m_1}{M_1} \Rightarrow \frac{dr_1}{dx} = \frac{M_1}{4\pi r_1^2 \rho_1}$$

and since $r_1 = r_2 \left(\frac{R_1}{R_2} \right) \quad \left(\frac{R_1}{R_2} \right) \frac{dr_2}{dx} = \frac{dr_1}{dx}$

$$\left(\frac{R_1}{R_2} \right) \frac{dr_2}{dx} = \frac{M_1}{4\pi r_1^2 \rho_1} = \frac{M_1}{4\pi r_2^2 \rho_1 \left(\frac{R_2}{R_1} \right)^2} = \frac{M_2}{4\pi r_2^2 \rho_2} \cdot \left[\frac{\rho_2}{\rho_1} \frac{M_1}{M_2} \left(\frac{R_2}{R_1} \right)^2 \right]$$

$$\frac{dr_2}{dx} = \frac{M_2}{4\pi r_2^2 \rho_2} \cdot \left[\frac{\rho_2}{\rho_1} \frac{M_1}{M_2} \left(\frac{R_2}{R_1} \right)^3 \right]$$

$$\frac{dr_2}{dx} = \frac{M_2}{4\pi r_2^2 \rho_2} \cdot \left[\frac{\rho_2}{\rho_1} \frac{M_1}{M_2} \left(\frac{R_2}{R_1} \right)^3 \right]$$

but mass conservation for star 2 implies $\frac{dr_2}{dx} = \frac{M_2}{4\pi r_2^2 \rho_2}$, so

$$\left[\frac{\rho_2}{\rho_1} \frac{M_1}{M_2} \left(\frac{R_2}{R_1} \right)^3 \right] = 1 \Rightarrow \frac{\rho_2(x)}{\rho_1(x)} = \frac{M_2}{M_1} \left(\frac{R_2}{R_1} \right)^{-3} \quad \rho(x) \propto \frac{M}{R^3}$$

This must hold for any mass shell $0 \leq x \leq 1$ and for $x = 0$

$$\frac{\rho_{c2}}{\rho_{c1}} = \frac{M_2}{M_1} \left(\frac{R_2}{R_1} \right)^{-3} = \frac{\bar{\rho}_2}{\bar{\rho}_1}$$

Similarly using the HE equation $\frac{dP}{dm} = -\frac{Gm}{4\pi r^2}$,

(Pols p.104) shows

$$P(x) \propto P_c \propto \frac{M^2}{R^4}$$

This is the same result one gets by dimensional analysis or by just "canceling the d's" in the differential equation and capitalizing the letters (i.e., using full star values).

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

Putting this together with $\rho(x) \propto \frac{M}{R^3} \Rightarrow R \propto (\rho / M)^{1/3}$, one gets a "new" result

$$P(x) \propto M^{2/3} \rho(x)^{4/3} \quad (\text{i.e., } P_c = \text{const } M^{2/3} \rho_c^{4/3})$$

which we have actually seen several times before, e.g., when talking about polytropes. (polytropes of the same index n are homologous).

and the whole set for radiative stars supported by ideal gas pressure

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

$$\rho \propto \frac{M}{R^3} \quad 1)$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

$$P \propto \frac{M^2}{R^4} \propto \frac{M\rho}{R} \quad 2)$$

$$\frac{dT}{dm} = -\frac{3}{4ac} \frac{\kappa}{T^3} \frac{L(r)}{(4\pi r^2)^2}$$

$$L \propto \frac{R^4 T^4}{\kappa M} \quad 3)$$

$$\frac{dL(m)}{dm} = \varepsilon$$

$$L \propto M\varepsilon \quad 4)$$

$$P = P_0 \rho T / \mu$$

$$P \propto \frac{\rho T}{\mu} \quad 5)$$

$$\varepsilon = \varepsilon_0 \rho T^\nu$$

$$\varepsilon \propto \rho T^\nu \quad 6)$$

$$\kappa = \kappa_0 \rho^a T^b$$

$$\kappa \propto \rho^a T^b \quad 7)$$

These are 7 equations in 9 unknowns.

$$\rho, T, \mu, P, L, R, M, \varepsilon, \kappa$$

Once can solve for any one of them in terms of at most two others. e.g. L as $f(\mu, M)$

$$\begin{aligned}
\rho &\propto \frac{M}{R^3} + P \propto \frac{M^2}{R^4} & \Rightarrow P &\propto \frac{M^2 \rho^{4/3}}{M^{4/3}} = M^{2/3} \rho^{4/3} \\
+ P &\propto \frac{\rho T}{\mu} & \Rightarrow T &\propto \frac{\mu P}{\rho} \propto \frac{\mu}{\rho} M^{2/3} \rho^{4/3} \propto \frac{\mu M}{R} \\
+ L &\propto \frac{R^4 T^4}{\kappa M} & \Rightarrow L &\propto \frac{\mu^4 M^4}{\kappa M} = \frac{\mu^4 M^3}{\kappa} \\
+ L &\propto M \varepsilon + \varepsilon = \varepsilon_0 \rho T^\nu & \Rightarrow \frac{\mu^4 M^3}{\kappa} &\propto M \frac{M}{R^3} \left(\frac{\mu M}{R} \right)^\nu \\
& & R^{3+\nu} &\propto M^{\nu+2-3} \mu^{\nu-4} \kappa \\
& & R &\propto M^{\left(\frac{\nu-1}{\nu+3}\right)} \mu^{\left(\frac{\nu-4}{\nu+3}\right)} \kappa^{\left(\frac{1}{\nu+3}\right)}
\end{aligned}$$

These have been evaluated for constant κ , e.g., electron scattering, but the generalization to $\kappa = \kappa_0 \rho^a T^b$ is straightforward.

$$R \propto M^{\left(\frac{\nu-1}{\nu+3}\right)} \mu^{\left(\frac{\nu-4}{\nu+3}\right)} \kappa^{\left(\frac{1}{\nu+3}\right)}$$

e.g. *pp* cycle ($\nu = 4$) and electron scattering $\kappa = \text{constant}$

$$R \propto M^{3/7}$$

while for the *CNO* cycle ($\nu = 18$) and electron scattering $\kappa = \text{constant}$

$$R \propto \mu^{2/3} M^{17/21}$$

If one further includes the density and temperature variation of κ other relations result. E.g. if $\kappa = \kappa_0 \rho T^{-7/2}$

$$L \propto \mu^{7.5} \frac{M^{5.5}}{R^{1/2}} \quad (HW \text{ problem})$$

and

$$R \propto \mu^{\frac{\nu-7.5}{\nu+2.5}} M^{\frac{\nu-3.5}{\nu+2.5}}$$

Note that the relevant values of e.g., κ and μ , are averages for the whole star, not just the photosphere

e.g. $\nu = 4$ $R \propto \mu^{-0.54} M^{0.0769}$ and $L \propto \mu^{7.77} M^{5.46}$

In general, the radius is weakly dependent on the mass

Given these relations one can also estimate how the central temperature and density will vary on the main sequence.

For illustration, just the electron scattering case

$$T_c \propto \frac{\mu M}{R} \propto \mu M^{0.57} \text{ (pp)} \quad \text{or} \quad \mu^{1/3} M^{0.19} \text{ (CNO)}$$

$$\rho_c \propto \frac{M}{R^3} \propto M^{-0.29} \text{ (pp)} \quad \text{or} \quad \mu^{-2} M^{-1.43} \text{ (CNO)}$$

Summary Table for Mass Luminosity and Mass-Temperature

Assume:

$$P = \frac{N_A \rho T}{\mu}$$

$$\kappa = \kappa_0 \rho^a T^{-s} \quad a = b = 0 \text{ electron scattering}$$

$$a = 1, b = 3.5 \text{ Kramers}$$

$$\varepsilon = \varepsilon_0 \rho T^\nu \quad \nu = 4 \text{ } pp \text{ cycle}$$

$$\nu = 17 \text{ CNO cycle (Cox and Guilli's choice)}$$

$$R = \text{const} \left(\varepsilon_0 \kappa_0 \right)^{\frac{1}{3+\nu-b+3a}} \mu^{\frac{\nu-b-4}{3+\nu-b+3a}} M^{\frac{1+\nu-b+a-2}{3+\nu-b+3a}}$$

$$L = \text{const } \varepsilon_0^{-\alpha} \kappa_0^{-\beta} \mu^\gamma M^\delta$$

Cox and Guilli Chap 22

$$\kappa = \kappa_0 \rho^a T^{-b}; \quad \varepsilon = \varepsilon_0 \rho T^\nu$$

			cno	pp
Exponent	a	b	$\nu=17$	$\nu=4$
α	1 0	3.5 0	0.026 0	0.077 0
β	1 0	3.5 0	1.026 1	1.077 1
γ	1 0	3.5 0	7.256 4	7.769 4
δ	1 0	3.5 0	5.154 3	5.462 3

$$L = \text{const } \varepsilon_0^{-\alpha} \kappa_0^{\beta} \mu^{-\gamma} T_{\text{eff}}^{\delta}$$

Cox and Guilli Chap 22

$$\kappa = \kappa_0 \rho^a T^{-b}; \quad \varepsilon = \varepsilon_0 \rho T^{\nu}$$

			cno	pp
Exponent	a	b	$\nu=17$	$\nu=4$
α	1	3.5	0.15	0.319
	0	0	0.214	0.400
β	1	3.5	0.517	0.348
	0	0	1.357	0.800
γ	1	3.5	1.333	1.333
	0	0	1.786	1.600
δ	1	3.5	5.469	4.116
	0	0	8.571	5.600

Implications of homology - summary

These results from homology have many interesting implications.

- The mass luminosity relation varies with mass. For lighter stars on the pp cycle with Kramers opacity L is predicted to be proportional to $M^{5.46}$. For stars where electron scattering dominates it is M^3 . For very high masses where radiation dominates (not included in the examples), L becomes proportional to M (this could be shown by repeating the derivations assuming $P = 1/3 aT^4$). The observed mass-luminosity relation for stars lighter than about 0.5 solar masses is not consistent with homology because the convective structure of the star, neglected here, changes things. But overall the agreement with observations is good.

Homology works well for massive main sequence stars
but does not give the mass luminosity relation correctly below 1 Msun

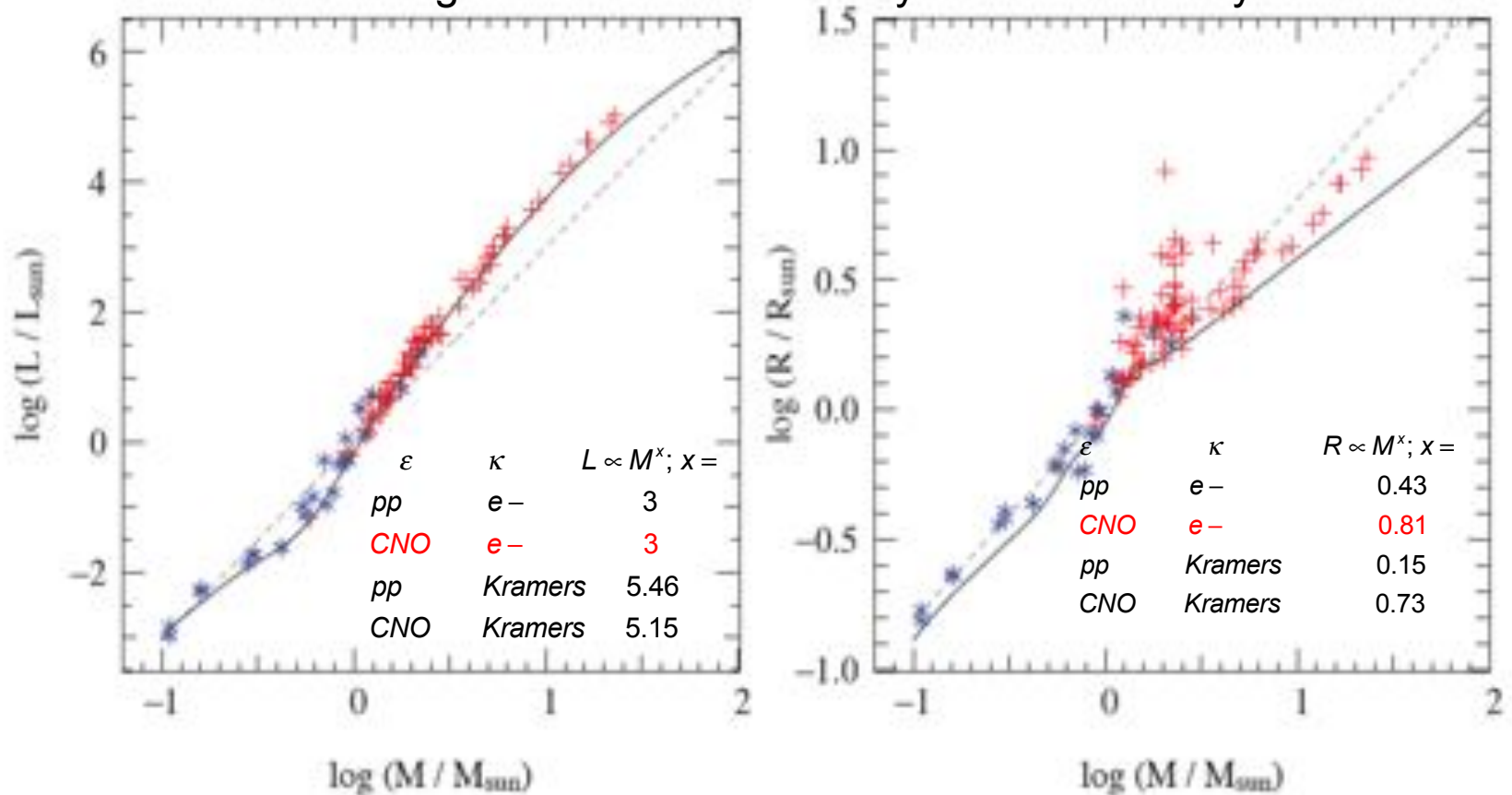
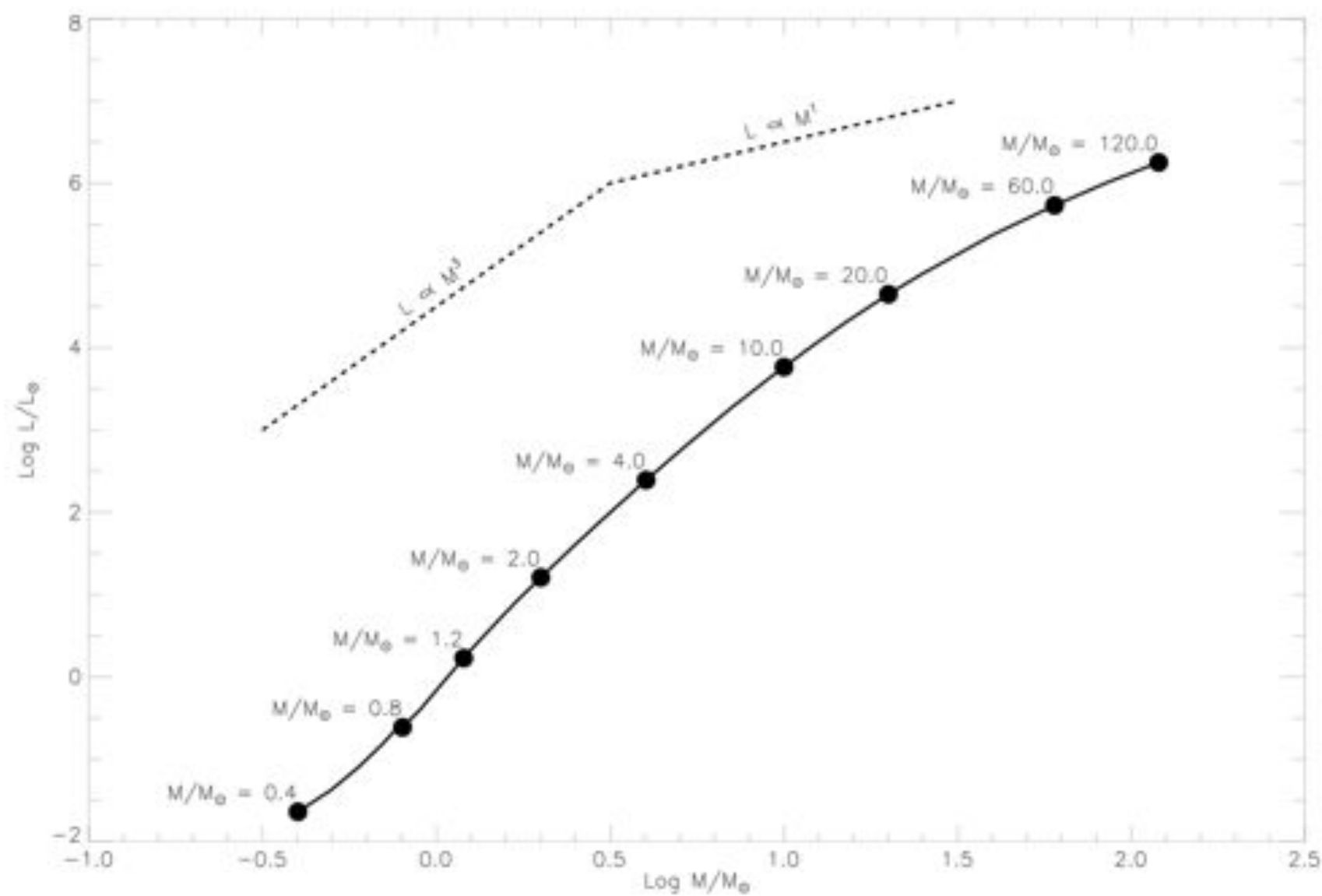
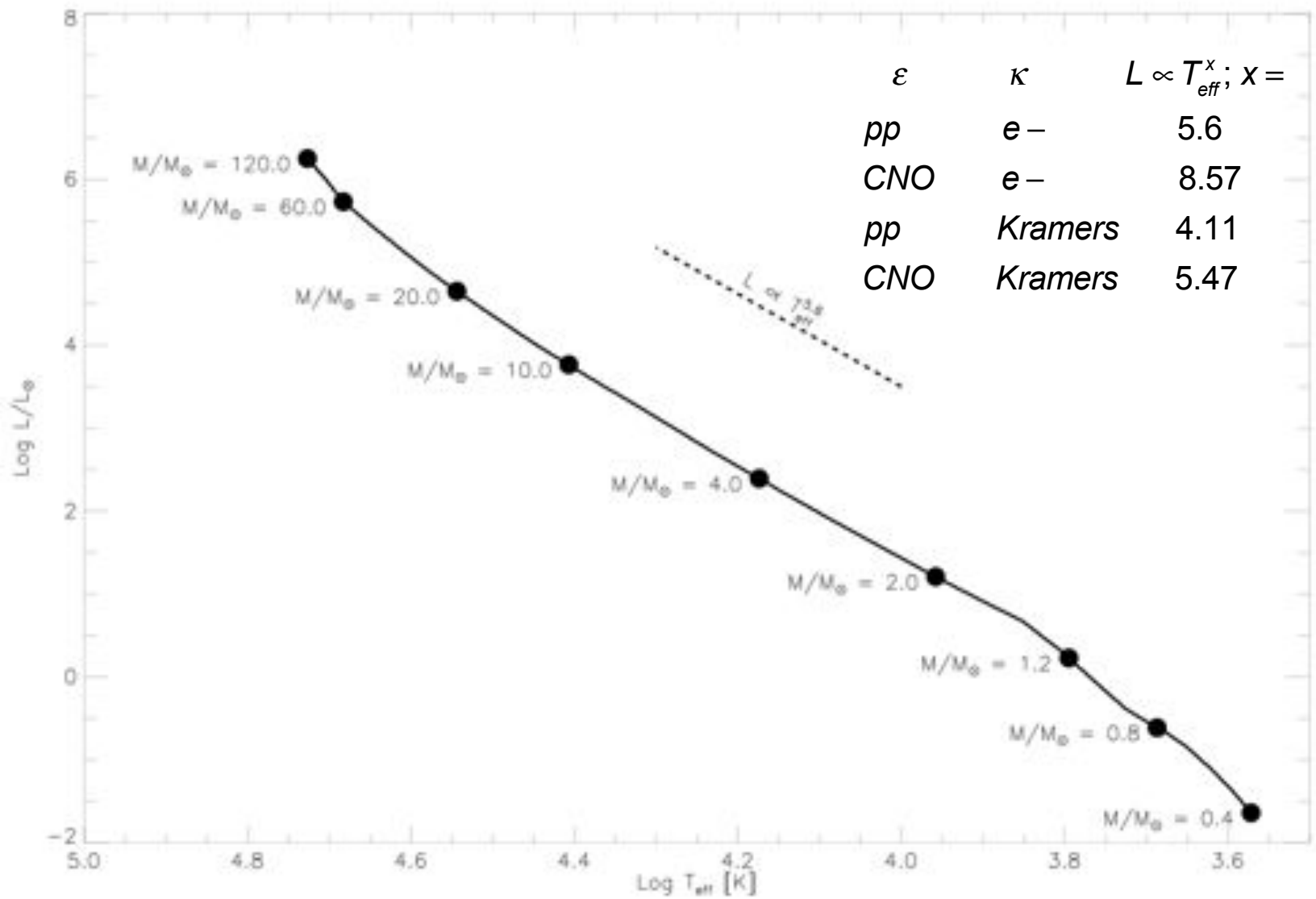


Figure 9.5. ZAMS mass-luminosity (left) and mass-radius (right) relations from detailed structure models with $X = 0.7, Z = 0.02$ (solid lines) and from homology relations scaled to solar values (dashed lines). For the radius homology relation, a value $\nu = 18$ appropriate for the CNO cycle was assumed (giving $R \propto M^{0.81}$); this does not apply to $M < 1 M_{\odot}$ so the lower part should be disregarded. Symbols indicate components of double-lined eclipsing binaries with accurately measured M, R and L , most of which are MS stars.





Implications of homology- continued

The radii of main sequence stars does not vary rapidly with mass ($M^{0.15 \text{ to } 0.81}$). This implies that for stars of higher M

The effective temperature $T_{\text{eff}} = \left(\frac{L}{4\pi\sigma R^2} \right)^{1/4}$ will be

higher. $L \propto M^{3 \text{ to } 5.5}$, $R^2 \propto M^{0.3 \text{ to } 1.62}$

The Kelvin helmholtz time scale $\tau_{KH} = \frac{\alpha GM^2}{RL}$ will be shorter

- More massive stars have higher central temperatures and will tend to be powered by the CNO cycle and have radiation as a larger component of their pressure
- At higher temperature and lower density (more massive stars) electron scattering opacity will dominate

Implications of homology

- Lower mass stars with Kramers opacity will have higher opacity (because of their lower T and larger ρ) especially near their surfaces and will tend to be convective there.
- Higher mass stars will shine by the CNO cycle and will therefore have more centrally concentrated energy generation. They will thus have convective cores.
- And to restate the obvious, massive stars with their higher luminosities will have shorter lifetimes.

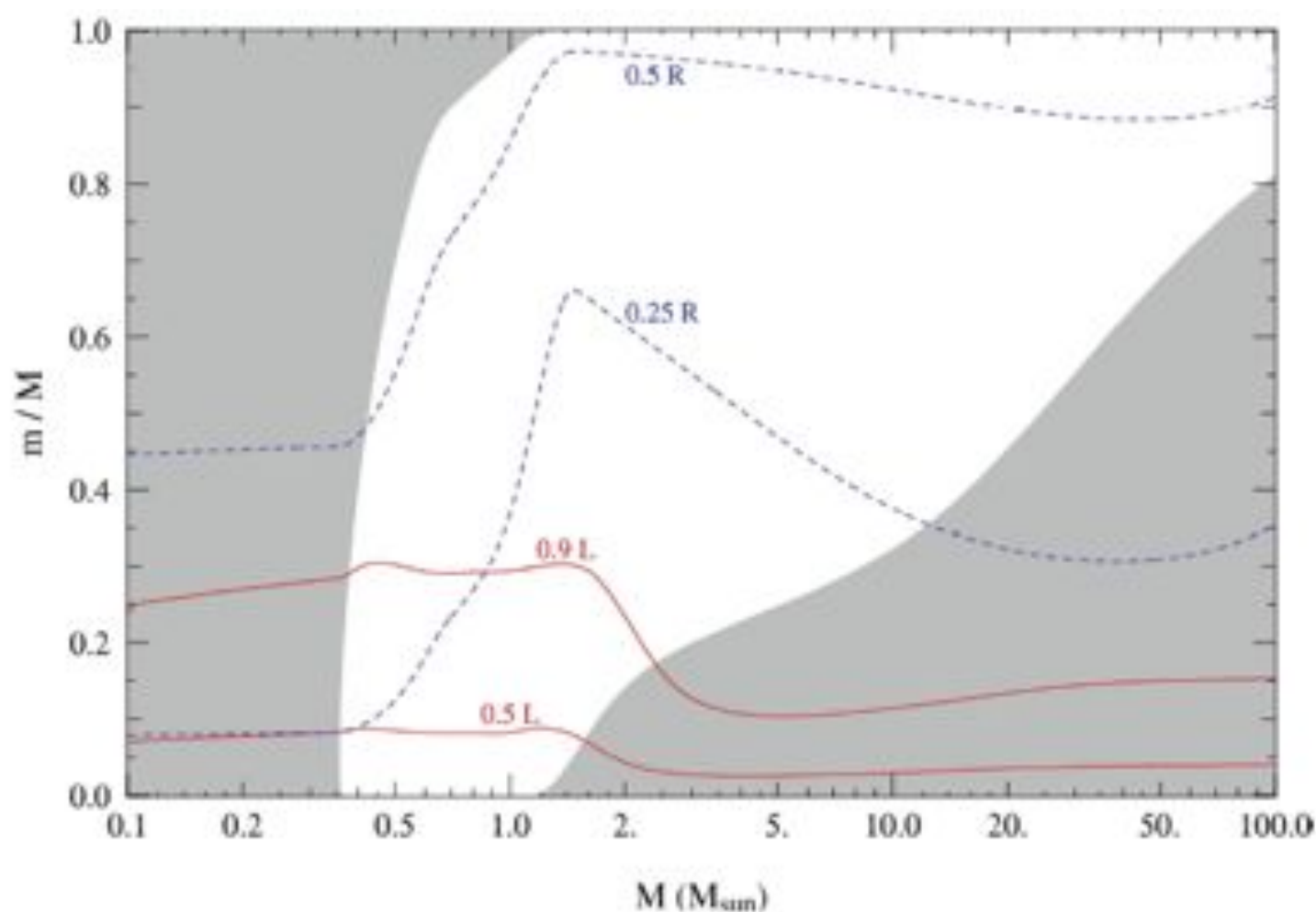


Figure 9.8. Occurrence of convective regions (gray shading) on the ZAMS in terms of fractional mass coordinate m/M as a function of stellar mass, for detailed stellar models with a composition $X = 0.70$, $Z = 0.02$. The solid (red) lines show the mass shells inside which 50% and 90% of the total luminosity are produced. The dashed (blue) lines show the mass coordinate where the radius r is 25% and 50% of the stellar radius R . (After KIPPENHAHN & WEIGERT.)

Implications of homology

- As hydrogen burns in the center of the star, μ rises. The central temperature and luminosity will both rise..

$$T_c \propto \frac{\mu M}{R} \propto \mu M^{0.57} \text{ (pp)} \quad \text{or} \quad \mu^{1/3} M^{0.19} \text{ (CNO)} \quad e - \text{scattering } \kappa$$

$$L \propto \mu^4 \text{ } e - \text{scattering } \kappa \quad L \propto \mu^{7.256} \text{ (pp)} \quad \mu^{7.769} \text{ (CNO) Kramers } \kappa$$

- The density evolution is not properly reflected because the sun's outer layers evolve non-homologously.
- Stars of lower metallicity will have somewhat smaller radii and bluer colors.

$$R = \text{const} \left(\varepsilon_0 \kappa_0 \right)^{\frac{1}{3+\nu-s+3}} \quad s = 0, 7/2 \quad \text{for } e\text{-scattering, Kramers}$$

$$\nu = 4, 17 \quad \text{for } pp, \text{ CNO}$$

10⁹ years – “isochrones”

“env” are conditions at the base of the convective envelope if there is one

Mass	Tc	roc	etac	Menv	Renv/R	Tenv	flag
0.100	4.396E+06	5.321E+02	3.78	0.0000	0.00000	4.396E+06	0
0.130	5.490E+06	3.372E+02	1.96	0.0000	0.00000	5.490E+06	0
0.160	6.120E+06	2.484E+02	1.15	0.0000	0.00000	6.119E+06	0
0.200	6.678E+06	1.826E+02	0.49	0.0000	0.00000	6.677E+06	0
0.250	7.370E+06	1.422E+02	-0.02	0.0000	0.00000	7.369E+06	0
0.300	7.807E+06	1.133E+02	-0.41	0.0000	0.00000	7.808E+06	0
0.400	8.479E+06	7.813E+01	-0.98	0.0237	0.08784	7.851E+06	0
0.500	8.901E+06	7.153E+01	-1.16	0.2883	0.54073	4.593E+06	0
0.600	9.537E+06	7.302E+01	-1.25	0.4558	0.61232	3.803E+06	0
0.700	1.030E+07	7.523E+01	-1.35	0.6057	0.65363	3.222E+06	0
0.800	1.126E+07	7.835E+01	-1.46	0.7371	0.67965	2.835E+06	0
0.900	1.232E+07	8.219E+01	-1.56	0.8547	0.69772	2.627E+06	0
1.000	1.345E+07	8.659E+01	-1.66	0.9722	0.72340	2.302E+06	0
1.100	1.455E+07	8.963E+01	-1.75	1.0864	0.75981	1.855E+06	0
1.200	1.603E+07	9.832E+01	-1.85	1.1965	0.81750	1.246E+06	0
1.300	1.745E+07	1.026E+02	-1.96	1.2995	0.87643	7.524E+05	0
1.400	1.877E+07	1.027E+02	-2.09	1.4000	0.92522	4.121E+05	0
1.500	1.974E+07	9.869E+01	-2.23	1.5000	0.96242	1.984E+05	0
1.600	2.058E+07	9.373E+01	-2.39	1.6000	0.98761	7.253E+04	0
1.700	2.141E+07	8.955E+01	-2.55	1.7000	0.99140	5.515E+04	0
1.800	2.232E+07	8.822E+01	-2.70	1.8000	0.99068	5.390E+04	0
1.900	2.369E+07	9.350E+01	-2.84	1.9000	0.99004	5.362E+04	0
2.000	2.476E+07	5.130E+02	-1.27	2.0000	0.98780	5.242E+04	1
2.200	2.523E+07	4.656E+02	-1.41	2.2000	0.98660	5.216E+04	1
2.500	2.625E+07	3.782E+02	-1.69	2.5000	0.98610	5.109E+04	1
2.700	2.739E+07	3.086E+02	-1.97	2.7000	0.98580	5.085E+04	1
3.000	2.692E+07	4.346E+02	-1.58	3.0000	0.98420	5.056E+04	1
3.500	2.867E+07	4.214E+02	-1.72	3.5000	0.98300	5.029E+04	1
4.000	2.972E+07	2.322E+02	-2.40	4.0000	0.98460	4.947E+04	1
5.000	3.483E+07	3.636E+01	-4.50	5.0000	0.99460	1.705E+04	1
6.000	3.464E+07	2.000E+02	-2.79	6.0000	0.94480	1.825E+05	1
7.000	3.822E+07	1.860E+02	-3.01	7.0000	0.94200	1.971E+05	1

flag = 1
means the
model is
invalid at
the late
time selected

The sun - past and future

Central density
rises as $T_c^{1/3}$

Time (10^9 years)	Luminosity ($L_\odot$)	Radius ($R_\odot$)	T_{central} (10^6 °K)
Past			
0	0.7688	0.872	13.35
0.143	0.7248	0.885	13.46
0.856	0.7621	0.902	13.68
1.863	0.8156	0.924	14.08
2.193	0.8352	0.932	14.22
3.020	0.8855	0.953	14.60
3.977	0.9522	0.981	15.12
Now			
4.587	1.000	1.000	15.51
Future			
5.506	1.079	1.035	16.18
6.074	1.133	1.059	16.65
6.577	1.186	1.082	17.13
7.027	1.238	1.106	17.62
7.728	1.318	1.143	18.42
8.258	1.399	1.180	18.74
8.7566	1.494	1.224	18.81
9.805	1.760	1.361	19.25

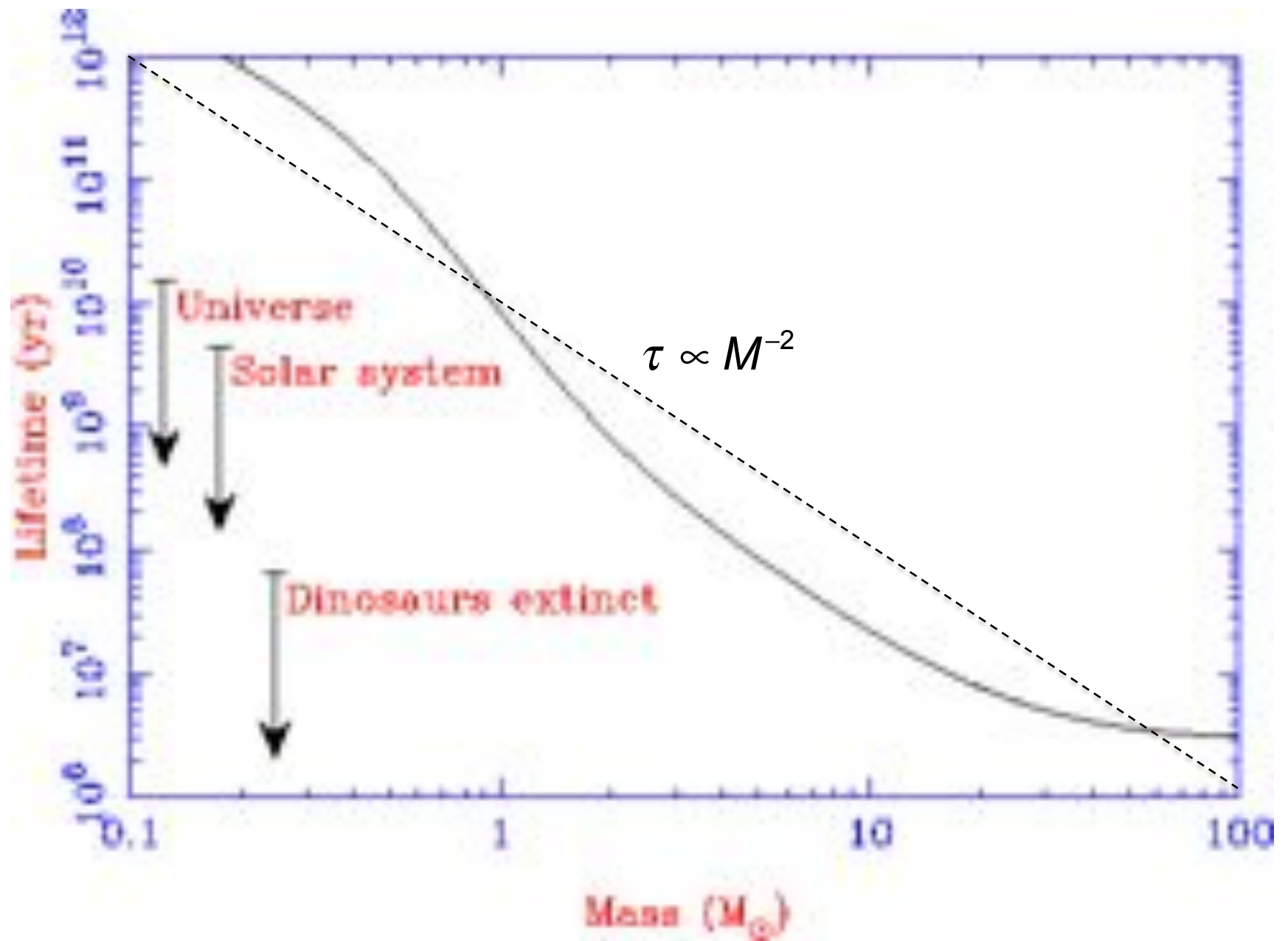
Zero age main
sequence

Oceans gone

CNO dominates

Red Giant

* Adapted from Turck-Chiése et al. (1988).
Composition $X = 0.7046$, $Y = 0.2757$, $Z = 0.0197$.
Present values are $R_\odot$ and $L_\odot$.
** For time t before the present age $t_\odot = 4.6 \times 10^9$ years.



luminosity

$t = 0$

10 Myr

100 Myr

1 Gyr

10 Gyr

25 Gyr

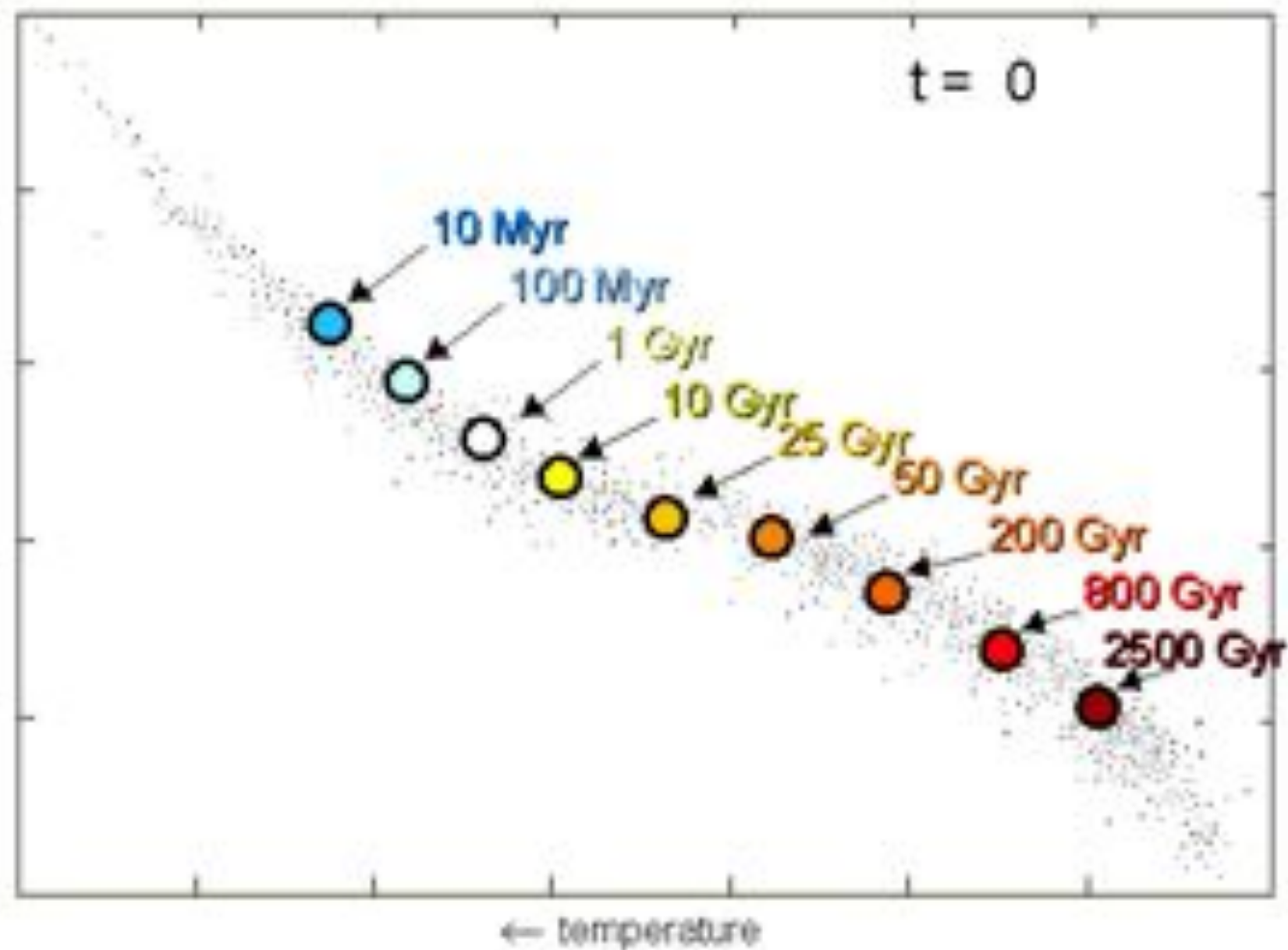
50 Gyr

200 Gyr

800 Gyr

2500 Gyr

← temperature

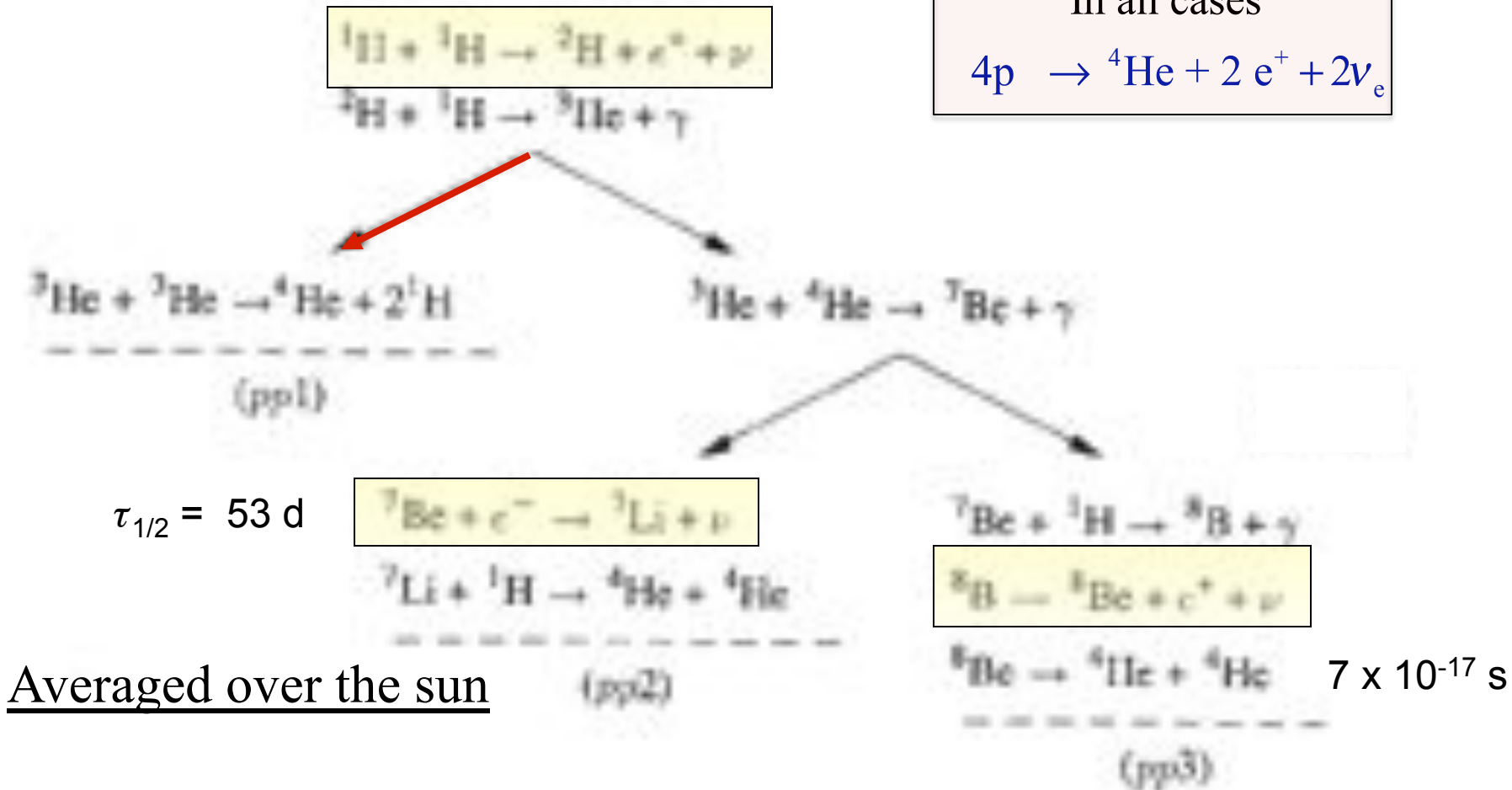


*Interlude: The Solar
Neutrino “Problem”*

Hydrogen Burning on the Main Sequence

In all cases

$$4p \rightarrow {}^4\text{He} + 2e^+ + 2\nu_e$$



Averaged over the sun

- pp1 85%
- pp2 15%
- pp3 0.02%

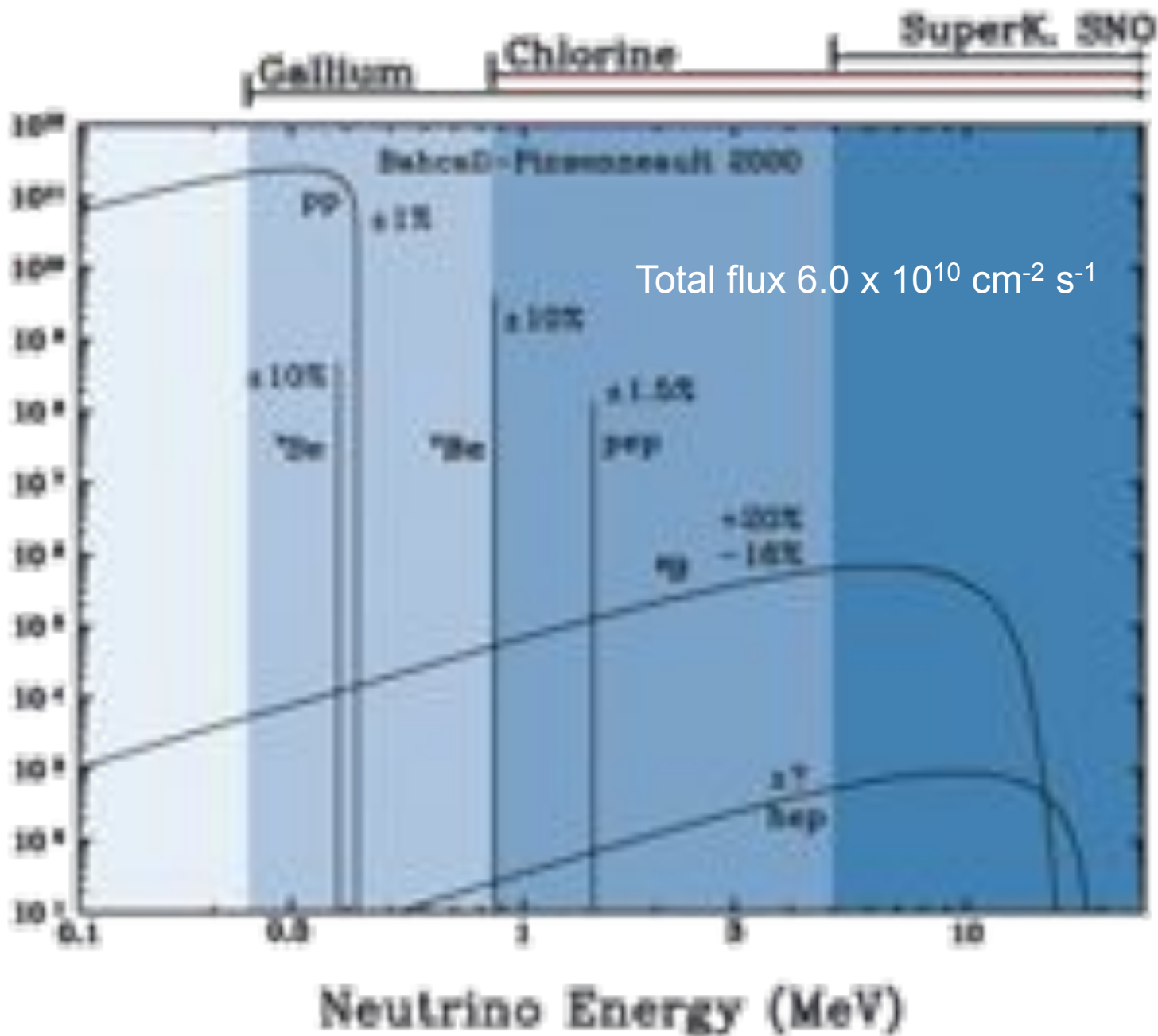
$$T_{\text{central}} = 15.7 \text{ Million K}$$

Neutrino Energies

Species	Average energy	Maximum energy	
p+p	0.267 MeV	0.420 MeV	
${}^7\text{Be}$	0.383 MeV	0.383 MeV	10%
	0.861	0.861	90%
${}^8\text{B}$	6.735 MeV	15 MeV	

In the case of ${}^8\text{B}$ and p+p, the energy is shared with a positron hence there is a spread. For ${}^7\text{Be}$ the electron capture goes to two particular states in ${}^7\text{Li}$ and the neutrino has only two energies

Neutrino Flux MeV^{-1}



Since 1965, experiments have operated to search for and study the neutrinos produced by the sun - in order to:

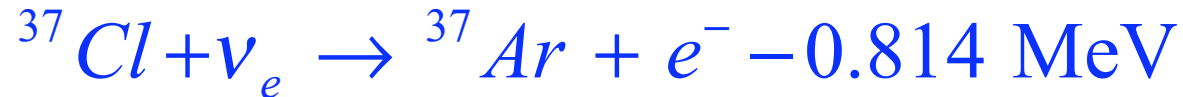
- Test solar models
- Determine the central temperature of the sun

The flux of neutrinos from ${}^8\text{B}$ is sensitive to T^{18}

- Learn new particle physics

DETECTORS

The chlorine experiment – Ray Davis – 1965 - ~1999



i.e., a neutron inside of ^{37}Cl is turned into a proton by a weak interaction involving an incident neutrino

^{37}Cl

^{37}Ar

17 p 18 n

18 p 17 n



Homestake Gold Mine
Lead, South Dakota

4850 feet down

tank 20 x 48 feet
615 tons (3.8×10^5 liters)
 C_2Cl_4

Threshold 0.814 MeV

Half-life $^{37}\text{Ar} = 35.0$ days

Neutrino sensitivity
 ^7Be , ^8B

8×10^{30} atoms of Cl

Nobel Prize 2002

Other Detectors

The gallium experiments (GALLEX and SAGE) –
1991 – 1997 and 1990 – 2001

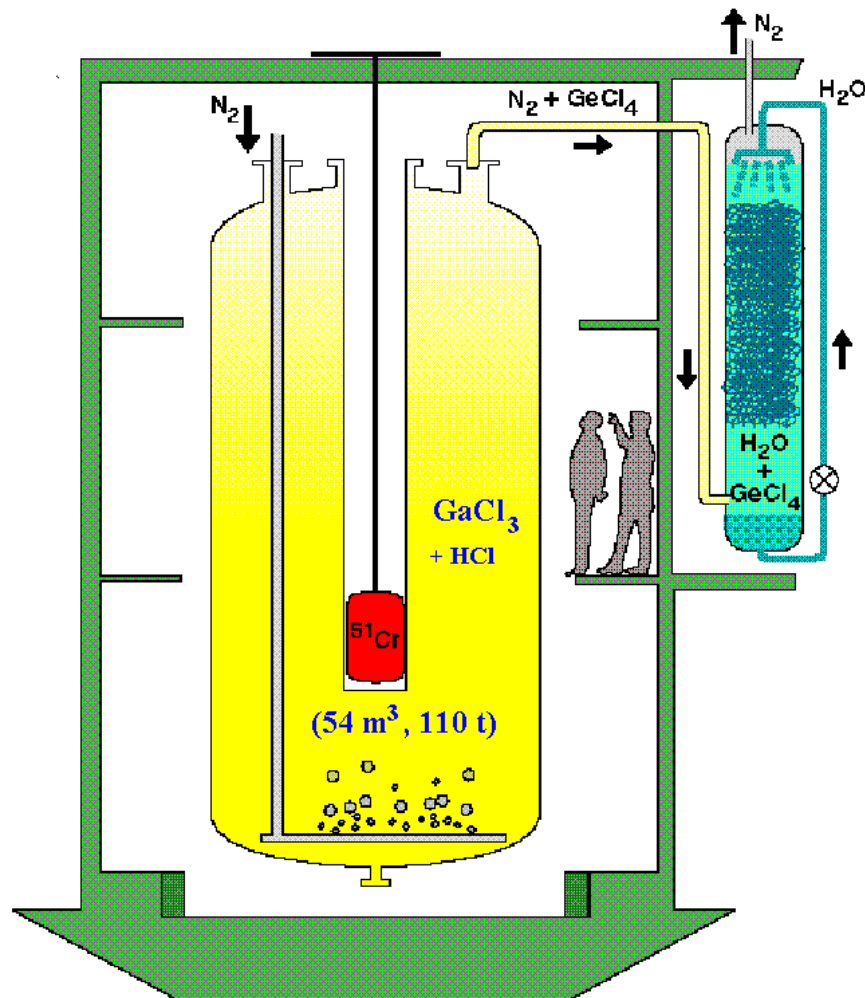


Kamiokande II - 1996 – 2001



Inelastic scattering of neutrinos on electrons in
water. Threshold 9 MeV. Scattered electron emits
characteristic radiation.

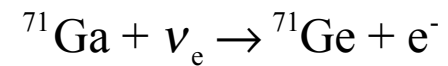
GALLEX



In Gran Sasso Tunnel – Italy

3300 m water equivalent

30.3 tons of gallium in GaCl_3 -
 HCl solution

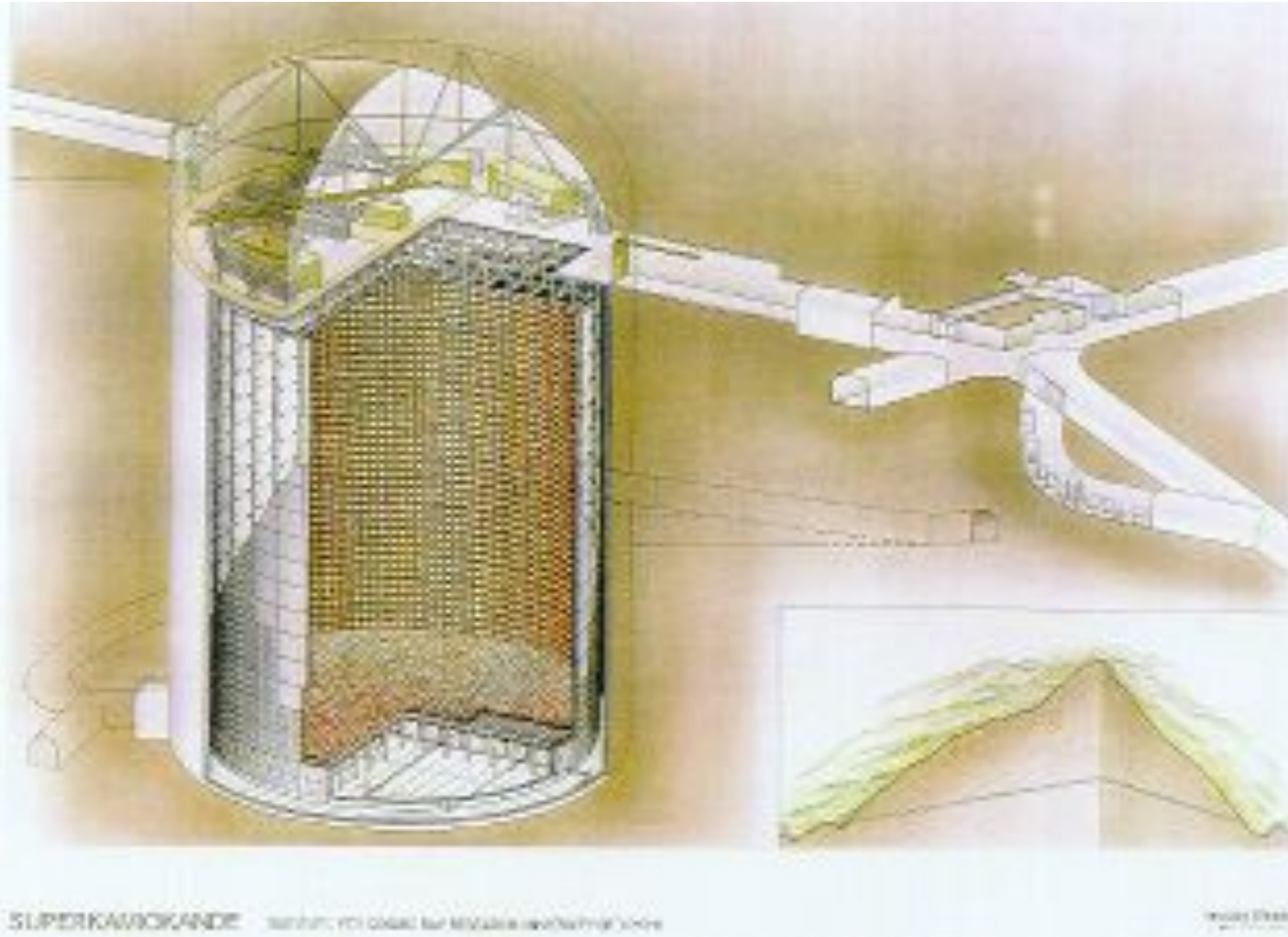


Threshold 0.233 MeV

Sees pp, ^7Be , and ^8B .

Calibrated using radioactive ^{51}Cr neutrino source

Kamiokande II (in Japanese Alps) 1996 - 2001



Depth 1 km

Detector H₂O

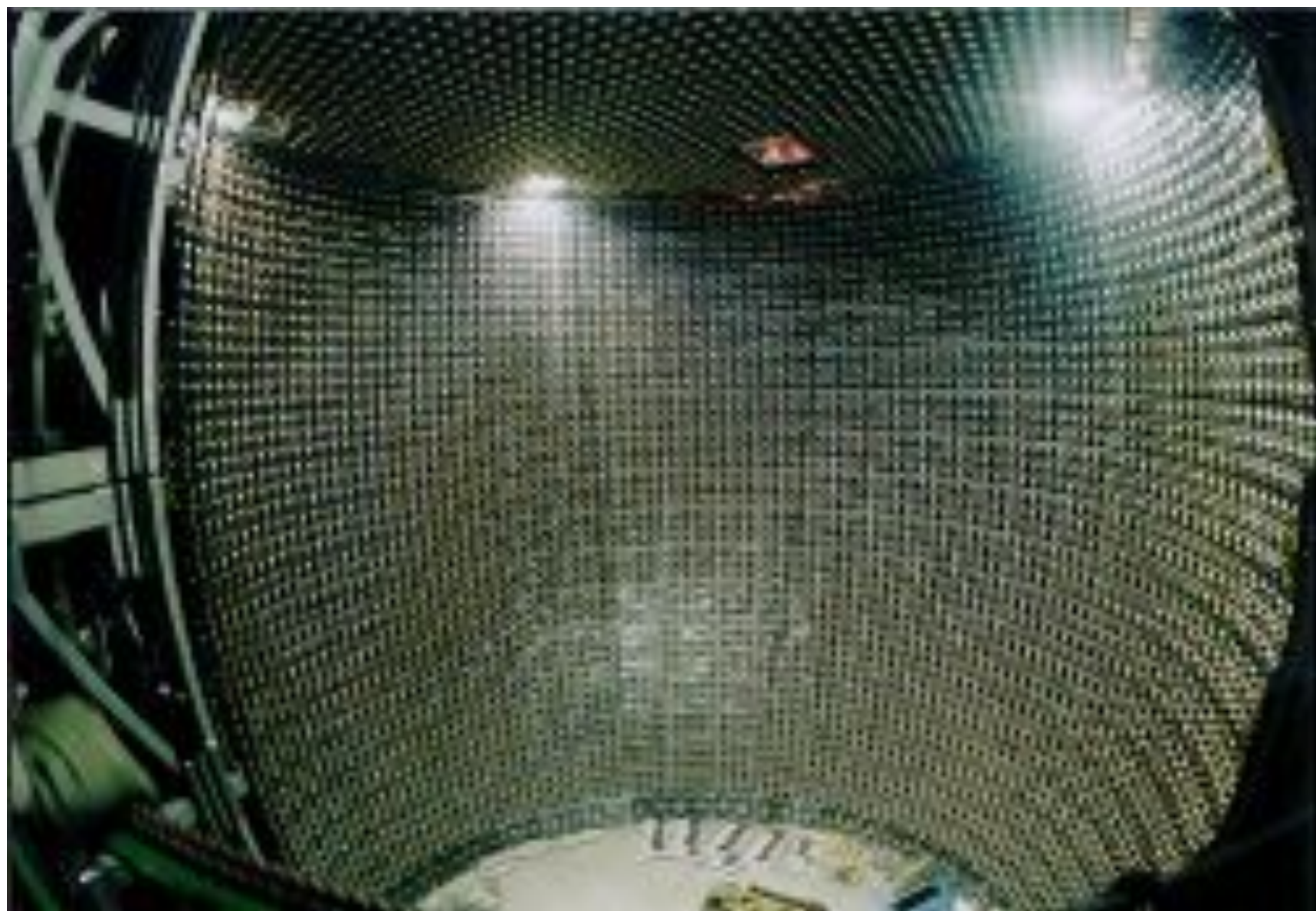
Threshold 9 MeV

Sensitive to ⁸B

20'' photomultiplier
tubes

Measure Cerenkov
light

2.3×10^{32} electrons

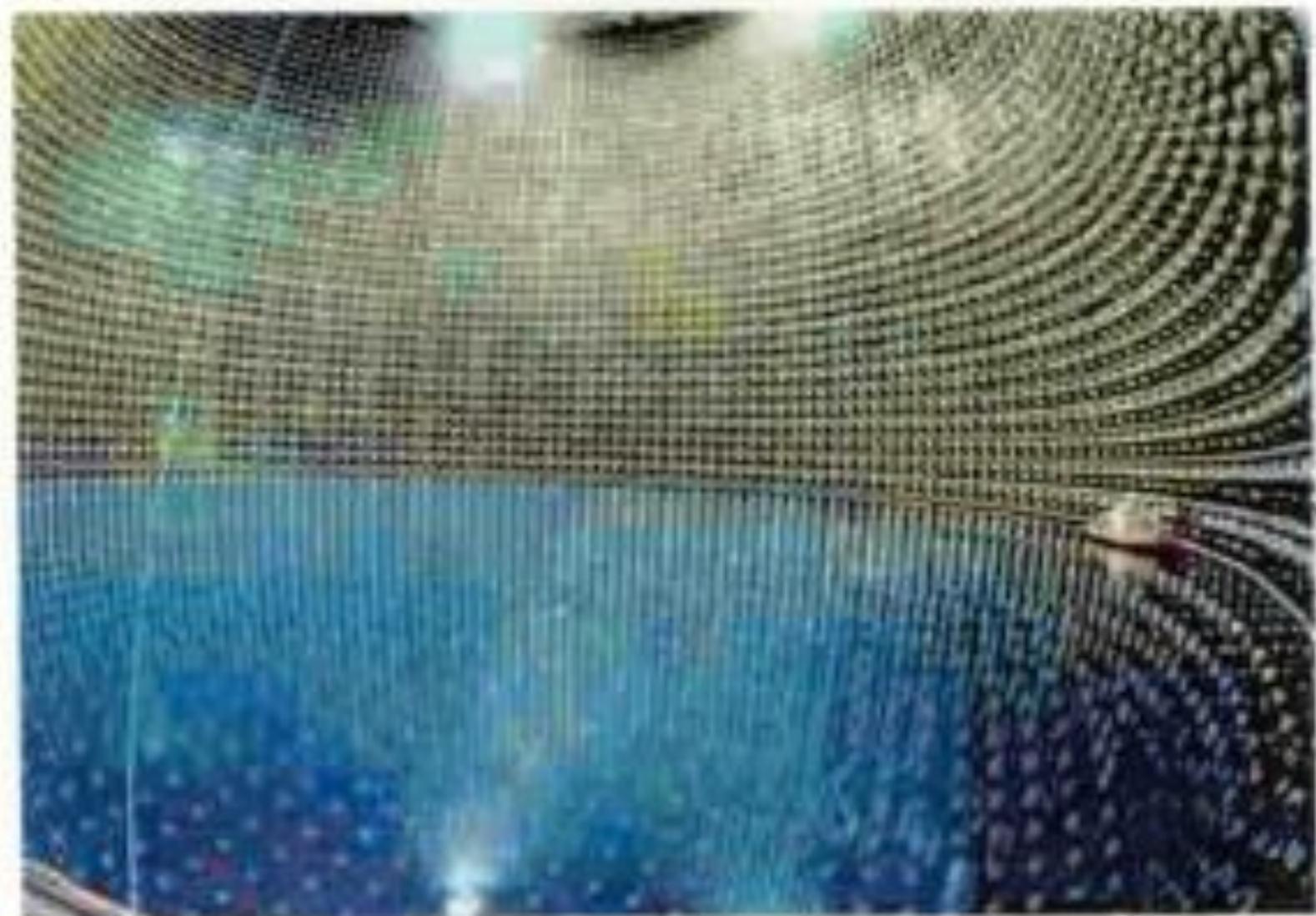


Super-Kamiokande (Japan)

50,000 tons of water

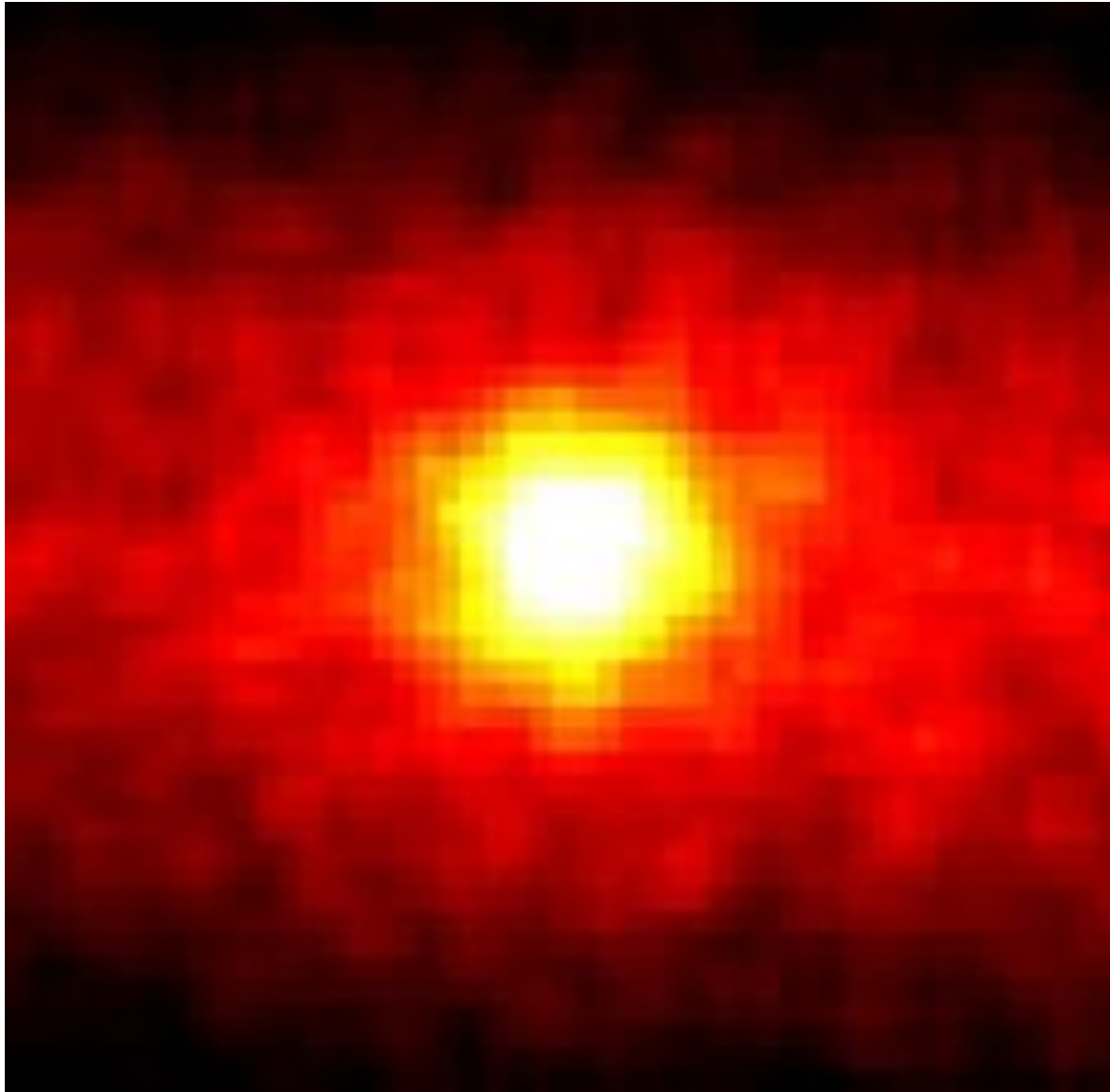
11,146 20" light detectors

0.6 mi
down



The Sun - 1999

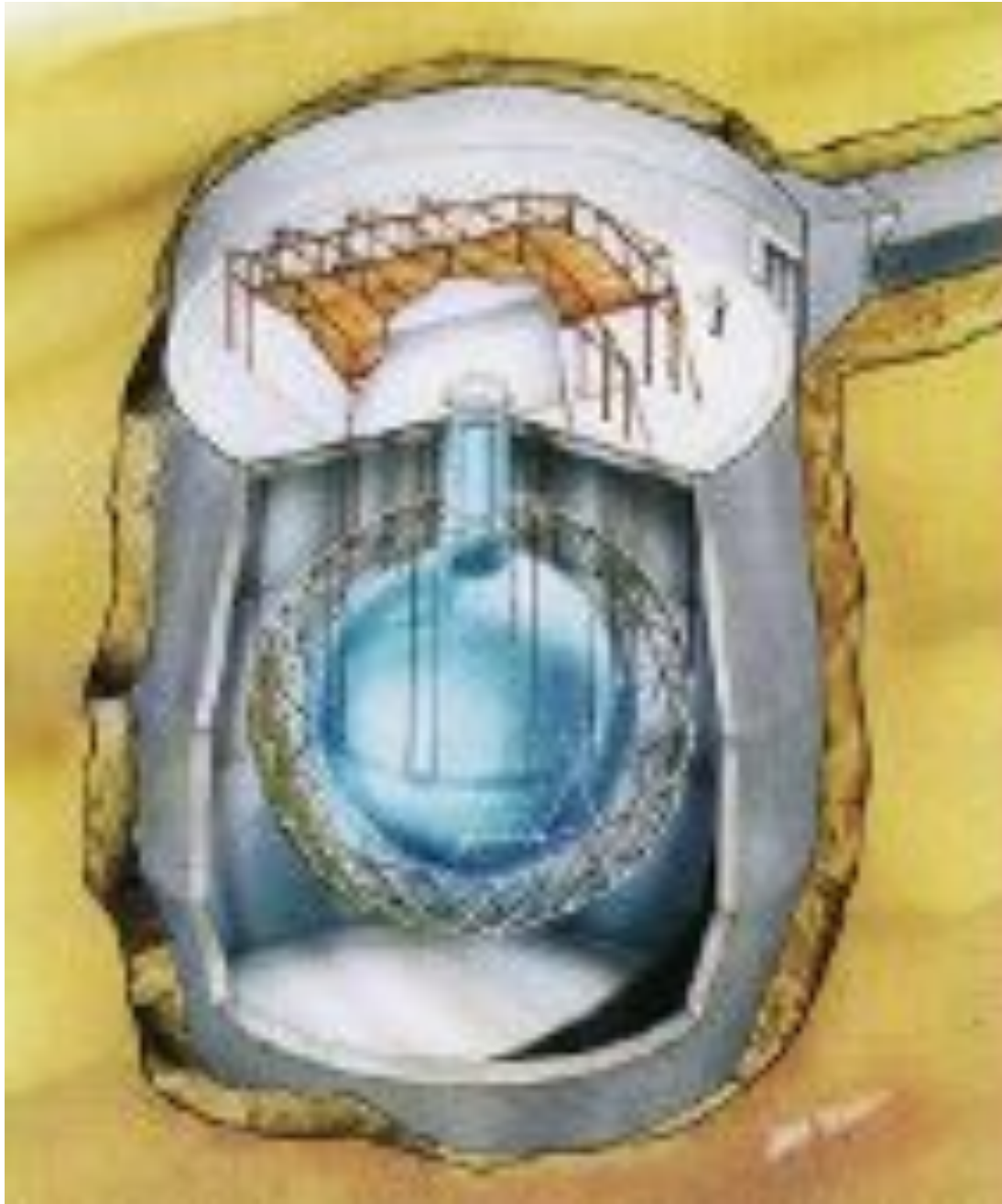
(First picture in neutrinos)



This “picture” was taken using data from the Kamiokande 2 neutrino observatory. It contains data from 504 nights (and days) of observation. The observatory is about a mile underground.

Each pixel is about a degree and the whole frame is $90^\circ \times 90^\circ$.

And finally, the Sudbury Neutrino Observatory



6800 ft down

1000 tons
 D_2O .

20 m diameter

Sudbury,
Canada

Threshold 5 MeV

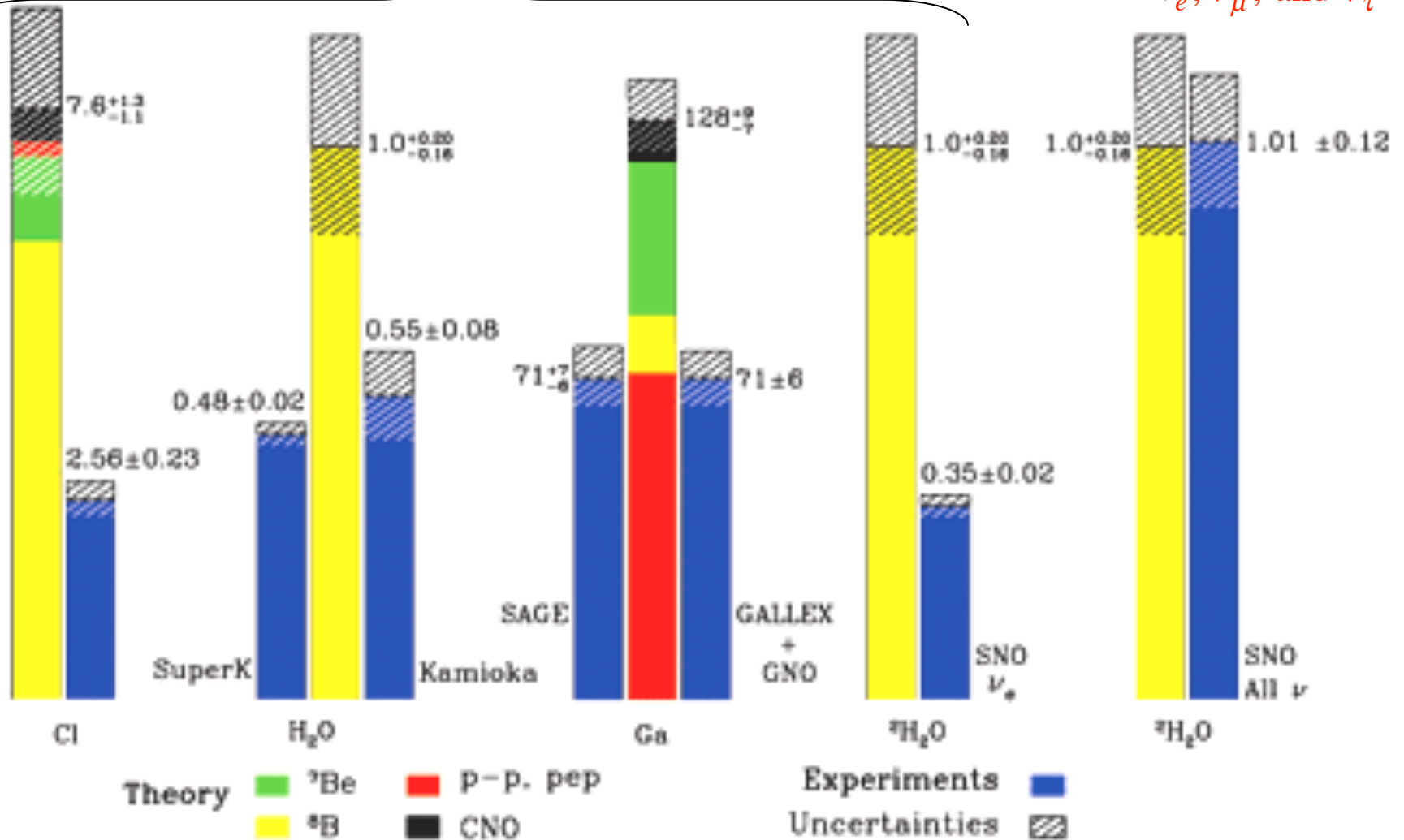
Sees 8B decay
but can see all
three kinds
of neutrinos

ν_e, ν_μ, ν_τ

Total Rates: Standard Model vs. Experiment Bahcall-Pinsonneault 2000

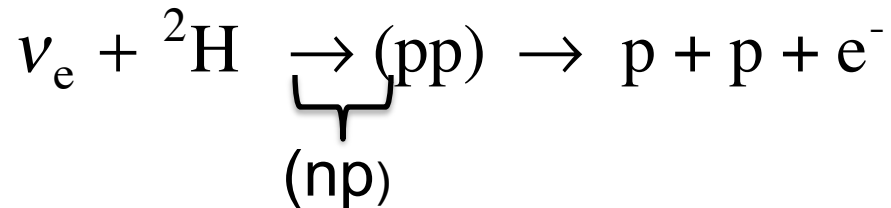
Only sensitive to ν_e

Sensitive to
 ν_e, ν_μ , and ν_τ

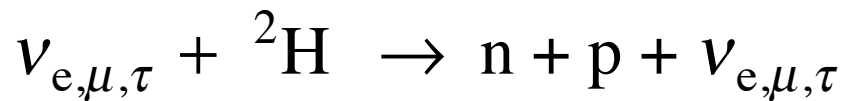


Neutrino interactions with heavy water $D_2O = {}^2H_2O$

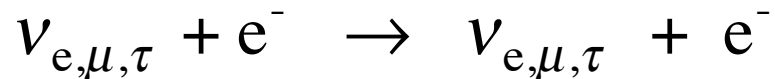
Electron neutrino



All neutrinos with energy above $2.2 \text{ MeV} = BE({}^2H)$



add salt to increase sensitivity to neutrons,



Results from SNO – 2002 (turned off in 2006)

The flux of electron flavored neutrinos above 5 MeV (i.e., only pp3 = ^8B neutrinos) is

$$1.76 \pm 0.1 \times 10^6 \text{ cm}^{-2} \text{ s}^{-1}$$

But the flux of μ and τ flavored neutrinos is

$$3.41 \pm 0.64 \times 10^6 \text{ cm}^{-2} \text{ s}^{-1}$$

Nobel Prize in Physics - 2002

Standard Solar Model ^8B neutrinos

$$5.05^{+1.01}_{-0.81} \times 10^6 \text{ neutrinos cm}^{-2} \text{ s}^{-1}$$

Particle physics aside:

Three Generations
of Matter (Fermions)

	I	II	III	
mass→	2.4 MeV	1.27 GeV	171.2 GeV	0
charge→	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	0
spin→	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	1
name→	u up	c charm	t top	γ photon
Quarks	4.8 MeV $-\frac{1}{3}$ $\frac{1}{2}$ d down	104 MeV $-\frac{1}{3}$ $\frac{1}{2}$ s strange	4.2 GeV $-\frac{1}{3}$ $\frac{1}{2}$ b bottom	0 0 1 g gluon
	<2.2 eV 0 $\frac{1}{2}$ ν_e electron neutrino	<0.17 MeV 0 $\frac{1}{2}$ ν_μ muon neutrino	<15.5 MeV 0 $\frac{1}{2}$ ν_τ tau neutrino	91.2 GeV 0 1 Z weak force
	0.511 MeV -1 $\frac{1}{2}$ e electron	105.7 MeV -1 $\frac{1}{2}$ μ muon	1.777 GeV -1 $\frac{1}{2}$ τ tau	80.4 GeV ± 1 1 W weak force
Leptons				Bosons (Forces)

emitted by pp-cycle
cosmology limits
the sum of the 3
neutrino masses
to < 1 eV

The explanation of the solar neutrino “problem” is apparently *neutrino flavor mixing*.

http://en.wikipedia.org/wiki/Neutrino_oscillation

A flux that starts out as pure electron-”flavored” neutrinos at the middle of the sun ends up at the earth as a mixture of electron, muon, and tauon flavored neutrinos in comparable proportions.

The transformation occurs in the sun and is complete by the time the neutrinos leave the surface. The transformation affects the highest energy neutrinos the most (MSW-mixing).

Such mixing requires that the neutrino have a very small but non-zero rest mass. This is different than in the so called “standard model” where the neutrino is massless. The mass is less than about 10^{-5} times that of the electron. (Also observed in earth’s atmosphere and neutrinos from reactors).

New physics.... (plus we measure the central temperature of the sun very accurately – 15.71 million K)

More Massive Main Sequence Stars

	$10 M_{\odot}$	$25 M_{\odot}$
X_H	0.32	0.35
L	$3.74 \times 10^{37} \text{ erg s}^{-1}$	$4.8 \times 10^{38} \text{ erg s}^{-1}$
T_{eff}	24,800 (B)	36,400 (O)
Age	16 My	4.7 My
T_{center}	$33.3 \times 10^6 \text{ K}$	$38.2 \times 10^6 \text{ K}$
ρ_{center}	8.81 g cm^{-3}	3.67 g cm^{-3}
τ_{MS}	23 My	7.4 My
R	$2.73 \times 10^{11} \text{ cm}$	$6.19 \times 10^{11} \text{ cm}$
P_{center}	$3.13 \times 10^{16} \text{ dyne cm}^{-2}$	$1.92 \times 10^{16} \text{ dyne cm}^{-2}$
$\% P_{radiation}$	10%	33%

Surfaces stable (radiative, not convective); inner roughly 1/3 of mass is convective.

Evolution on the main sequence

The composition is not constant on the main sequence because hydrogen is changing especially in the center. This has two consequences

- As hydrogen decreases μ increases. Since the luminosity depends on μ to some power, the luminosity increases
- To keep the luminosity slightly rising as hydrogen decreases the central temperature must rise (slightly). This has important secondary consequences.

Since μ_c increases more than T_c increases (due to the high sensitivity of ε to T), and since the pressure is due to ideal gas, $\frac{P_c}{\rho_c} \propto \frac{T}{\mu}$ must decrease. Thus P_c must

decline or ρ_c must increase or both. Which alternative dominates depends on the relative changes of μ and T and hence on whether the star is burning by the pp cycle with $\varepsilon \propto T^4$ ($M < 1.5M_\odot$) or CNO cycle with $\varepsilon \propto T^{18}$.

Since ρ_c varies roughly as T_c^3 , it too cannot increase much, so especially for stars burning by the CNO cycle, P_c must decrease. This is accomplished by an expansion of the overlying layers - and the star in general. Note the non-homologous aspect. ρ goes up in the center but declines farther out. For stars burning by the pp chain, the changes in ρ and T are bigger so P does not have to change so much as μ goes up.

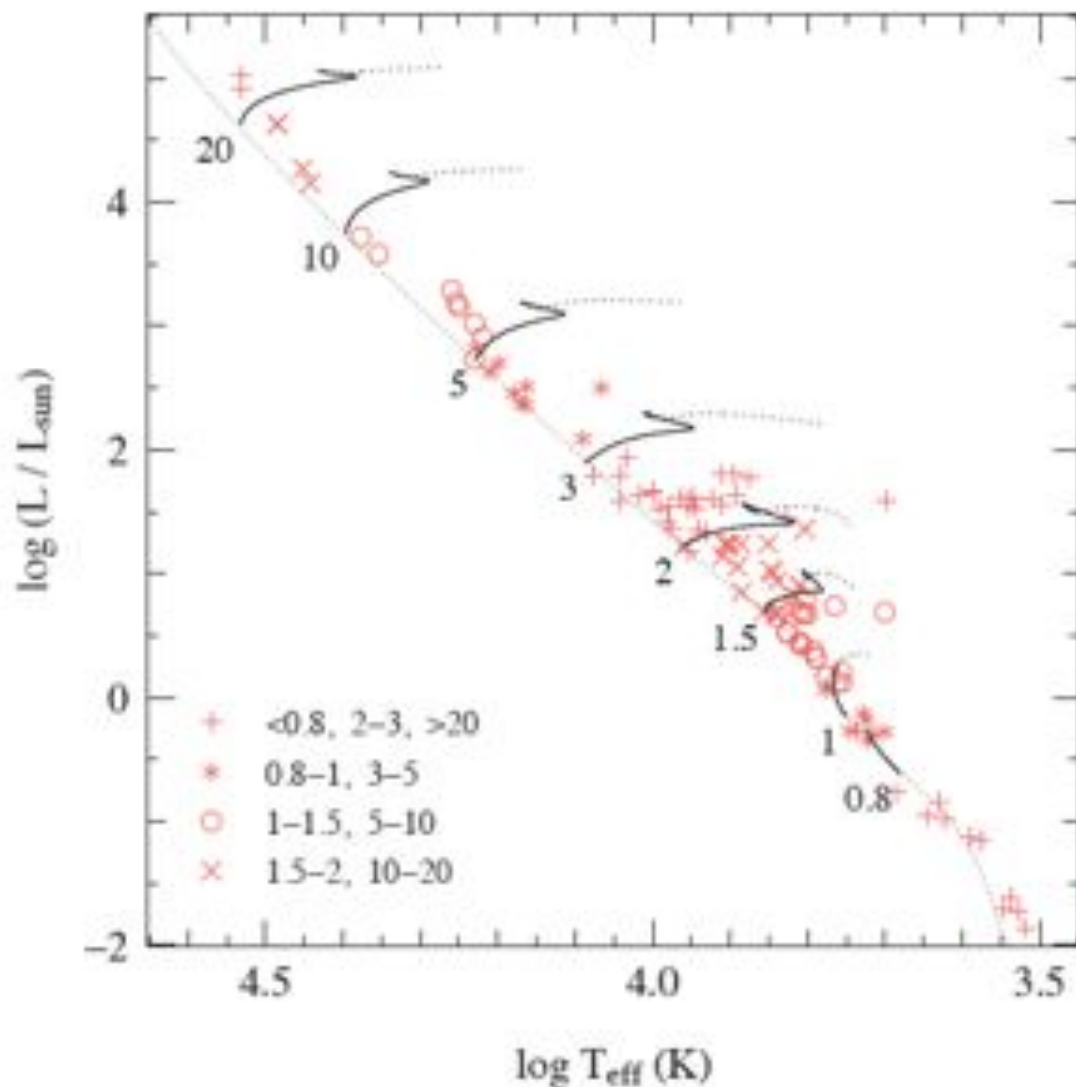


Figure 9.9. Evolution tracks in the H-R diagram during central hydrogen burning for stars of various masses, as labelled (in $M_{\odot}$), and for a composition $X = 0.7, Z = 0.02$. The dotted portion of each track shows the continuation of the evolution after central hydrogen exhaustion; the evolution of the $0.8 M_{\odot}$ star is terminated at an age of 14 Gyr. The thin dotted line in the ZAMS. Symbols show the location of binary components with accurately measured mass, luminosity and radius (as in Fig. 9.5). Each symbol corresponds to a range of measured masses, as indicated in the lower left corner (mass values in $M_{\odot}$).

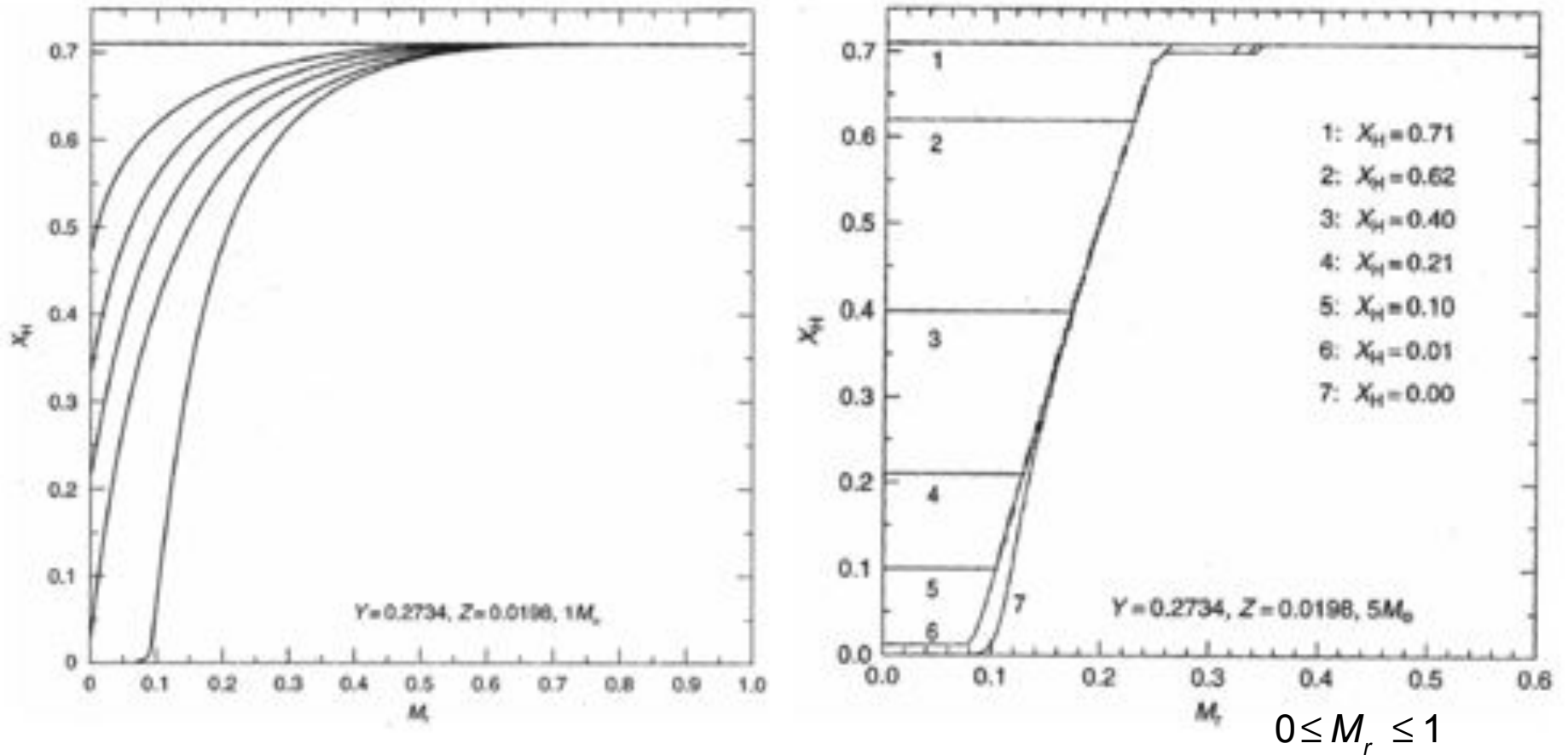


Figure 9.10. Hydrogen abundance profiles at different stages of evolution for a $1 M_\odot$ star (left panel) and a $5 M_\odot$ star (right panel) at quasi-solar composition. Figures reproduced from SALARIS & CASSISI.

Post-main sequence evolution segregates into three cases based upon the mass of the star

- Low mass stars – lighter than 2 (or 1.8) solar masses. Develop a degenerate helium core after hydrogen burning and ignite helium burning in a “flash”
- Intermediate mass stars – 2 – 8 solar masses. Ignite helium burning non-degenerately but do not ignite carbon
- Massive stars – over 8 solar masses. Ignite carbon burning and in most cases heavier fuels as well (8 – 10 is a complex transition region) and go on to become supernovae.

Schonberg Chandrasekhar mass

In the hydrogen depleted core there are no sources of nuclear energy. Further its surface is kept warm by the overlying hydrogen burning, so that the core does not radiate and contract, at least not quickly (on a Kelvin Helmholtz time scale). In these circumstances the core becomes *isothermal*.

A full star with constant temperature is unstable. With ideal gas pressure, hydrostatic equilibrium would have to be provided entirely by the density gradient. This is not possible because $\gamma < 4/3$.

Schonberg Chandrasekhar mass

However, one can stably have an isothermal core inside a larger star, provided that core does not exceed some fraction of the total mass. That fraction can be approximately derived assuming hydrostatic equilibrium and ideal gas pressure. Evaluate the Virial Theorem out to some critical radius R_c where the enclosed mass is M_c .

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \quad \text{or} \quad 4\pi r^3 \frac{dP}{dm} = -\frac{Gm}{r} \quad \text{but}$$

$$4\pi r^3 \frac{dP}{dm} = \frac{d(4\pi r^3 P)}{dm} - 12\pi r^2 P \frac{dr}{dm} = \frac{d(4\pi r^3 P)}{dm} - \frac{3P}{\rho} \quad \text{since} \quad \frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

Integrating over just the mass of the core we have

$$\int_0^{M_c} \frac{d(4\pi r^3 P)}{dm} dm - \int_0^{M_c} \frac{3P}{\rho} dm = - \int_0^{M_c} \frac{Gm}{r} dm$$

$$4\pi r^3 P \Big|_0^{M_c} - \int_0^{M_c} \frac{3P}{\rho} dm = - \Omega_c$$

Schonberg Chandrasekhar mass

$$4\pi r^3 P \Big|_0^{M_c} - \int_0^{M_c} \frac{3P}{\rho} dm = -\Omega_c$$

$$4\pi R_c^3 P(R_c) - \int_0^{M_c} \frac{3P}{\rho} dm = -\Omega_c \quad \frac{P}{\rho} = \frac{N_A k T}{\mu_c}$$

but $T = \text{constant in the core} = T_c$ so

$$\int_0^{M_c} \frac{3P}{\rho} dm = \frac{3N_A k T_c M_c}{\mu_c} = 2 U(\text{core})$$

$$4\pi R_c^3 P(R_c) = \frac{3N_A k T_c M_c}{\mu_c} - \frac{\alpha G M_c^2}{R_c}$$

$$P(R_c) = \frac{3N_A k T_c M_c}{4\pi R_c^3 \mu_c} - \frac{\alpha G M_c^2}{4\pi R_c^4}$$

Schonberg Chandrasekhar mass

$$P(R_c) = \frac{3N_A k T_c M_c}{4\pi R_c^3 \mu_c} - \frac{\alpha G M_c^2}{4\pi R_c^4} \quad \text{agrees with GK}$$

The pressure is a combination of competing terms. It has a maximum given by $\frac{dP(R_c)}{dR_c} = 0$

$$\frac{-9N_A k T_c M_c}{4\pi R_c^4 \mu_c} + \frac{4\alpha G M_c^2}{4\pi R_c^5} = 0$$

$$\frac{-9N_A k T_c}{\mu_c} + \frac{4\alpha G M_c}{R_c} = 0$$

$$R_c = \frac{4\alpha G M_c \mu_c}{9N_A k T_c}$$

Schonberg Chandrasekhar mass

Putting this into our expression for pressure gives

$$\begin{aligned} P(R_c) &= \frac{3N_A k T_c M_c}{4\pi R_c^3 \mu_c} - \frac{\alpha G M_c^2}{4\pi R_c^4} \\ P(R_c) &= \frac{3N_A k T_c M_c (9N_A k T_c)^3}{4\pi (4\alpha G M_c \mu_c)^3 \mu_c} - \frac{\alpha G M_c^2 (9N_A k T_c)^4}{4\pi (4\alpha G M_c \mu_c)^4} \\ &= \left(\frac{3^7}{2^8} - \frac{3^8}{2^{10}} \right) \left(\frac{N_A k T_c}{\mu_c} \right)^4 \frac{1}{\pi \alpha^3 G^3 M_c^2} \\ &= \frac{3^7}{2^{10}} (4 - 3) \left(\frac{N_A k T_c}{\mu_c} \right)^4 \frac{1}{\pi \alpha^3 G^3 M_c^2} = \frac{3^7}{2^{10}} \left(\frac{N_A k T_c}{\mu_c} \right)^4 \frac{1}{\pi \alpha^3 G^3 M_c^2} \end{aligned}$$

The key point here is that as the core mass grows the pressure it is able to support at its boundary declines (as $1/M_c^2$).

Schonberg Chandrasekhar mass

The pressure that the envelope requires in order to be supported does not decrease much as M_c grows.

That pressure, P_{env} , can also be estimated using HE

$$\int_{P_{env}}^0 dP = -P_{env} = - \int_{M_c}^M \frac{Gm}{4\pi r^4} dm$$

Now $m \propto r^3$ and $dm \propto \rho r^2$ and $\rho \downarrow$ as $r \uparrow$, so the integrand is not varying very rapidly, very approximately we can replace

$\frac{Gm}{4\pi r^4}$ by $\frac{Gm}{4\pi R^4}$ where R is the radius of the star, then

$$P_{env} \approx \frac{G}{4\pi R^4} \int_{M_c}^M m dm \approx \frac{GM^2}{8\pi R^4}$$

For a sphere of constant density

$$P_c = \frac{GM\rho}{2R} = \frac{3GM^2}{8\pi R^4}$$

nb independent of M_c if $M_c \ll M$

Pressure balance when the core is providing the maximum pressure at its edge that it is able is then

$$\frac{GM^2}{8\pi R^4} \approx \left(\frac{3^7}{2^{10}} \right) \left(\frac{N_A k T_c}{\mu_c} \right)^4 \frac{1}{\pi \alpha^3 G^3 M_c^2}$$

Schonberg Chandrasekhar mass

$$\frac{GM^2}{8\pi R^4} \approx \left(\frac{3^7}{2^{10}} \right) \left(\frac{N_A k T_c}{\mu_c} \right)^4 \frac{1}{\pi \alpha^3 G^3 M_c^2}$$

Ideal gas implies that the density at the core-envelope interface is given by

$$\rho_{\text{interface}} = \frac{P_{\text{interface}} \mu_{\text{env}}}{N_A k T_c} \sim \bar{\rho} = \frac{3M}{4\pi R^3}$$

So
$$\frac{GM^2}{8\pi R^4} \frac{\mu_{\text{env}}}{N_A k T_c} \sim \frac{3M}{4\pi R^3} \Rightarrow R \sim \frac{GM \mu_{\text{env}}}{6 N_A k T_c}$$

$$\frac{GM^2 (6 N_A k T_c)^4}{8\pi (GM \mu_{\text{env}})^4} \approx \left(\frac{3^7}{2^{10}} \right) \left(\frac{N_A k T_c}{\mu_c} \right)^4 \frac{1}{\pi \alpha^3 G^3 M_c^2}$$

$$\frac{(6)^4}{8M^2 (\mu_{\text{env}})^4} \approx \left(\frac{3^7}{2^{10}} \right) \left(\frac{1}{\mu_c} \right)^4 \frac{1}{\alpha^3 M_c^2}$$

Schonberg Chandrasekhar mass

$$\frac{(6)^4}{8M^2(\mu_{env})^4} \approx \left(\frac{3^7}{2^{10}}\right) \left(\frac{1}{\mu_c}\right)^4 \frac{1}{\alpha^3 M_c^2}$$

$$\frac{M_c}{M} \sim \left(\frac{3^7 \cdot 8}{2^{10}(6)^4 \alpha^3}\right)^{1/2} \left(\frac{\mu_{env}}{\mu_c}\right)^2 = \sqrt{\frac{27}{2048}} \alpha^{-3/2} \left(\frac{\mu_{env}}{\mu_c}\right)^2 \approx 0.115 \left(\frac{\mu_{env}}{\mu_c}\right)^2$$

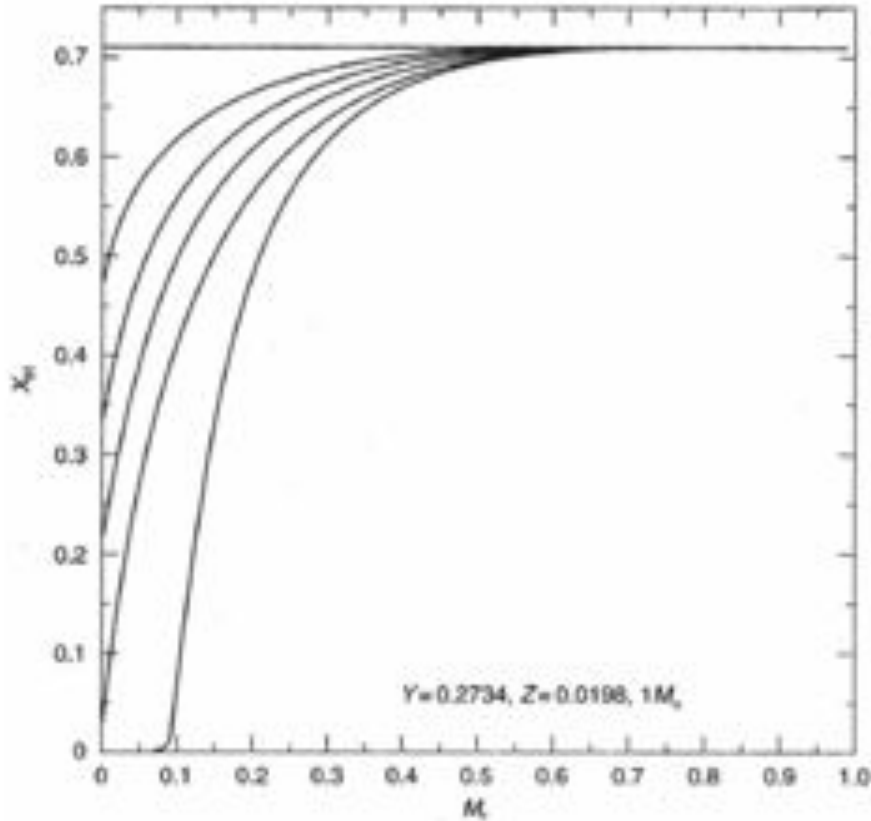
A more accurate derivation gives

$$\frac{M_c}{M} = 0.37 \left(\frac{\mu_{env}}{\mu_c}\right)^2$$

For $\mu_{env} = 0.59$ and $\mu_{He} = 1.3$, the limit is 0.08. When hydrogen has been depleted in the inner 8% of the stars mass, the helium core begins to contract and the star burns hydrogen in a shell. It leaves the main sequence. For quite massive stars ($> 6 M_{\odot}$) or so, the helium core is not very isothermal because of the short Kelvin Helmholtz time, so the equation loses accuracy. For a $15 M_{\odot}$ star, the initial H-depleted core is $2.4 M_{\odot}$, not $1.5 M_{\odot}$.

Hydrogen depletion

1 $M_{\odot}$



5 $M_{\odot}$

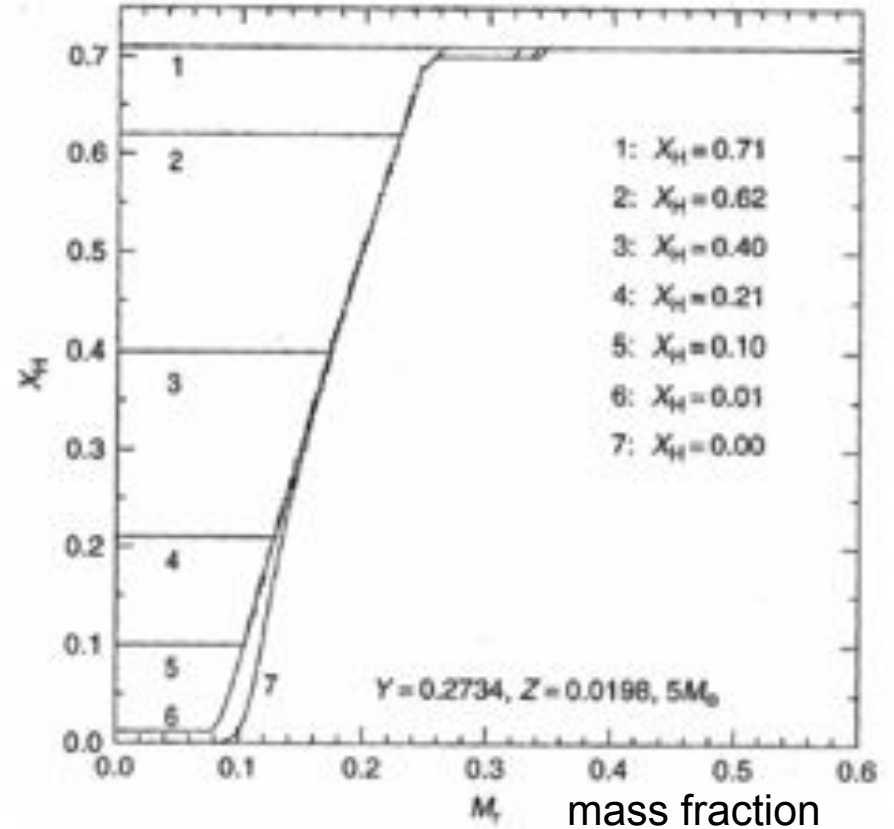


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