Hydrogen Burning on the Main Sequence and Homology

GK 14
Pols 7.7.4
Prialnik 7

http://www.ucolick.org/~woosley/indexay112.html

Generalities

Assuming constant density

$$\rho = \frac{3M}{4\pi R^3}$$

the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = \frac{-GM(r)\rho}{r^2}$$

can be integrated to give the central pressure

$$P_c = \frac{GM\rho}{2R}$$ (an underestimate for stars since ρ not consant)

Then if ideal gas pressure dominates (it does on the main sequence)

$$\frac{\rho N_A k T_c}{\mu} = \frac{GM\rho}{2R} \Rightarrow T_c = \frac{GM\mu}{2N_A kR} \quad i.e.,\ T_c \propto \frac{\mu M}{R}$$

[For a solar mass, radius, and central composition this gives a central temperature of close to 10 million K]

Recall: How is μ defined?

$$P = nkT = \frac{\rho N_A kT}{\mu} \qquad n = n_e + n_i$$

$$\mu^{-1} = \frac{n_e}{\rho N_A} + \frac{n_i}{\rho N_A} = \sum Z_i \frac{X_i}{A_i} + \sum \frac{X_i}{A_i} = \sum (1+Z_i)\frac{X_i}{A_i}$$

eg. Pure ionized hydrogen

$$\mu^{-1} = (1+1)\bullet 1 = 2 \qquad P = 2\rho N_A kT \quad \mu = 0.5$$

Pure ionized helium

$$\mu^{-1} = (1+2)\bullet \frac{1}{4} = \frac{3}{4} \qquad P = \frac{4}{3}\rho N_A kT \quad \mu = 1.333$$

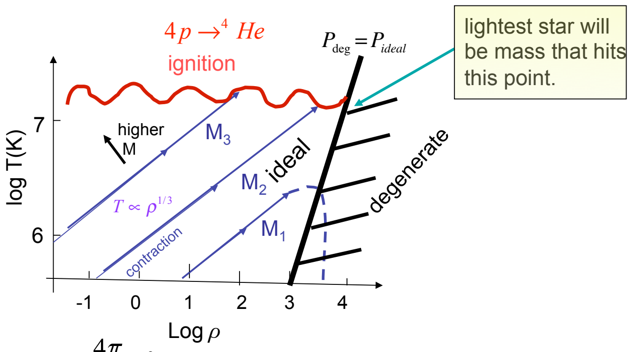
Burning hydrogen to helium increases μ

Generalities

$$T_c \propto \frac{\mu M}{R} \qquad \rho \propto \frac{M}{R^3} \Rightarrow T_c \propto \frac{\mu M}{M^{1/3}} \rho_c^{1/3} = \mu M^{2/3} \rho^{1/3}$$

That is, at least for spheres of constant density, as a star (or protostar) in hydrostatic equilibrium contracts its central temperature rises as the cube root of the density. It also says that stars (or protostars) will have a higher temperature at a given density if their mass is bigger.

In the absence of nuclear reactions the contraction occurs at a rate needed to balance the luminosity of the star (Kelvin-Helmholz evolution). The Virial theorem says that half the work goes into radiation and half into heat.



lightest star will be mass that hits this point.

$$T \propto \frac{M}{R} \qquad M \sim \frac{4\pi}{3} R^3 \rho$$

$$\Rightarrow R \propto \left(\frac{M}{\rho}\right)^{1/3}$$

$$T \propto \frac{M\rho^{1/3}}{M^{1/3}}$$

$$\boxed{T \propto M^{2/3}\rho^{1/3}}$$

$T \propto \rho^{1/3}$ for a given M and

T at a given ρ is higher

for bigger M

This gives the blue lines in the plot

Hydrogen Burning Reactions –

Core hydrogen burning defines “Main Sequence”

pp1

$$p(p,e^+\nu_e)^2H(p,\gamma)^3He(^3He,2p)^4He$$

$$\varepsilon \propto \rho X_h^2 T^4$$

CNO-1

$$^{12}\mathrm{C}(p,\gamma)^{13}\mathrm{N}(e^+\nu)^{13}\mathrm{C}(p,\gamma)^{14}\mathrm{N}(p,\gamma)^{15}\mathrm{O}(e^+\nu)^{15}\mathrm{N}\ (p,\alpha)^{12}\mathrm{C}$$

$$\varepsilon \propto \rho X_H X(CNO)\, T^{18}$$

Minimum Mass Star

Solve for condition that ideal gas pressure and degeneracy pressure are equal at 10^7 K.

$$P_{deg} \approx P_{ideal}$$

$$1.69\,\rho N_A kT \approx 1.00\times10^{13}\ (\rho Y_e)^{5/3}$$

(assuming 75% H, 25% He by mass)

At 10^7 K, this becomes

$$1.40\ \times10^8\ \rho(10^7) \approx 8.00\times10^{12}\ \rho^{5/3} \qquad (\text{taking } Y_e = 0.875)$$

which may be solved for the density to get $\rho \approx 2300$ gm cm^{-3}

The total pressure at this point is

$$P_{tot} \approx \frac{1}{2}\left(P_{deg} + P_{ideal}\right) \approx \frac{1}{2}(2P_{ideal}) \approx P_{ideal}$$

$$\approx 1.40\times10^8\,(2300)\left(10^7\right) \approx 3.2\times10^{18}\ \text{dyne cm}^{-2}$$

$$= \left(\frac{GM\rho}{2R}\right)$$

But $R = \left(\dfrac{3M}{4\pi\rho}\right)^{1/3}$ i.e., $\rho = \dfrac{M}{4/3\ \pi R^3}$

Combining terms we have

$$3.2x10^{18} \approx \frac{(GM\rho)(4\pi\rho)^{1/3}}{2(3M)^{1/3}}$$

$$M^{2/3} \approx \frac{2(3.2x10^{18})(3^{1/3})}{G\rho^{4/3}(4\pi)^{1/3}}$$

and using again $\rho \approx 2300$ gm cm^{-3}

$$M \approx 8.7x10^{31}\ \text{gm}$$

or 0.044 solar masses.

For constant density

$$P = \left(\frac{GM\rho}{2R}\right)$$

$$R = \left(\frac{3M}{4\pi\rho}\right)^{1/3}$$

A more detailed calculation gives 0.08 solar masses.
Protostars lighter than this can never ignite nuclear reactions.
They are known as brown dwarfs (or planets if the mass is less than 13 Jupiter masses, or about 0.01 solar masses.
[above 13 Jupiter masses, some minor nuclear reactions occur that do not provide much energy - “deuterium burning”

Similar mass limits exist for helium burning ignition (0.5 Msun) and carbon burning ignition (8 Msun)

From these considerations we expect some tendencies:

1. The central temperature of more massive main sequence stars to be hotter (unless R increases more than linearly with M on the main sequence and it doesn't)
2. That the actual radius of the star will depend on the form of the energy generation. Until nuclear energy generation is specified, R is undetermined, though L may be.
3. Stars will get hotter in their centers when they use up a given fuel – unless they become degenerate
4. More massive stars will arrive at a given temperature (e.g. ignition) at a lower central density

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One also expects L *roughly* $\propto M^3$ *for main sequence stars*

$$\text{Luminosity} \approx \frac{\text{Heat content in radiation}}{\text{Time for heat to leak out}} = \frac{E_{radiation}}{\tau_{diffusion}}$$

True even if star is not supported by P_{rad}
Note this is not the total heat content, just the radiation.

$$E_{radiation} \approx \frac{4}{3}\pi R^3 aT^4 \propto R^3T^4 \propto \frac{R^3M^4}{R^4} = \frac{M^4}{R}$$

$$\tau_{diffusion} \approx \frac{R^2}{l_{mfp}\,c} \qquad l_{mfp} = \frac{1}{\kappa\rho} \qquad \kappa \text{ is the "opacity" in cm}^2\text{ gm}^{-1}$$

Assume κ is a constant

$$\text{M} \approx \frac{4}{3}\pi R^3 \rho \Rightarrow \rho \approx \frac{3\text{M}}{4\pi R^3}$$

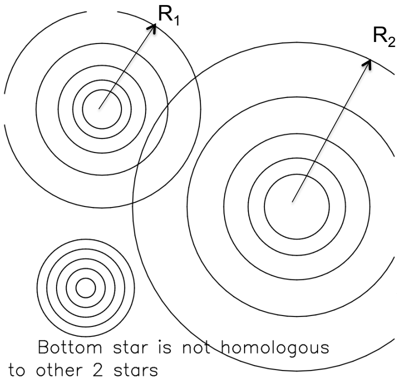
$$l_{mfp} \propto \frac{R^3}{\text{M}} \qquad \tau_{diffusion} \propto \frac{R^2\text{M}}{R^3} = \frac{\text{M}}{R}$$

$$\text{L} \propto \frac{\text{M}^4}{R} \Big/ \frac{\text{M}}{R} = \text{M}^3$$

But one can do better

- Polytropes
- Actually solve the structure equations on a computer (e.g., MESA)
- Homology

Homology relations



Consider 2 stellar models with mass M_1 and M_2 and radius R_1 and R_2

$$\text{Let } x = \frac{m_1}{M_1} = \frac{m_2}{M_2} \qquad 0 \le x \le 1$$

be a mass coordinate such that x = 1 at the surface. The two models are said to be homologous if

$$\frac{r_1(x)}{R_1} = \frac{r_2(x)}{R_2} \quad or \quad \frac{r_1(x)}{r_2(x)} = \frac{R_1}{R_2}$$

This slide from JS Pineda shows circles indicating the radius that encloses 20% mass increments of two stars that are homologous and one that is not

Then for example the mass conservation equation can be written for anywhere inside star number 1: $dm_1 = 4\pi\, r_1^2\, \rho_1\, dr_1$

$$\frac{dr_1}{dm_1} = \frac{1}{4\pi r_1^2 \rho_1} \qquad x = \frac{m_1}{M_1} \qquad \Rightarrow \frac{dr_1}{dx} = \frac{M_1}{4\pi r_1^2 \rho_1}$$

and since $r_1 = r_2 \left(\frac{R_1}{R_2}\right)$ $\qquad \left(\frac{R_1}{R_2}\right)\frac{dr_2}{dx} = \frac{dr_1}{dx}$

$$\left(\frac{R_1}{R_2}\right)\frac{dr_2}{dx} = \frac{M_1}{4\pi r_1^2 \rho_1} = \frac{M_1}{4\pi r_2^2 \rho_1}\left(\frac{R_2}{R_1}\right)^2 = \frac{M_2}{4\pi r_2^2 \rho_2} \cdot \left[\frac{\rho_2}{\rho_1}\frac{M_1}{M_2}\left(\frac{R_2}{R_1}\right)^2\right]$$

$$\frac{dr_2}{dx} = \frac{M_2}{4\pi r_2^2 \rho_2} \cdot \left[\frac{\rho_2}{\rho_1}\frac{M_1}{M_2}\left(\frac{R_2}{R_1}\right)^3\right]$$

$$\frac{dr_2}{dx} = \frac{M_2}{4\pi r_2^2 \rho_2} \cdot \left[\frac{\rho_2}{\rho_1}\frac{M_1}{M_2}\left(\frac{R_2}{R_1}\right)^3\right]$$

but mass conservation for star 2 implies $\frac{dr_2}{dx} = \frac{M_2}{4\pi r_2^2 \rho_2}$, so

$$\left[\frac{\rho_2}{\rho_1}\frac{M_1}{M_2}\left(\frac{R_2}{R_1}\right)^3\right] = 1 \Rightarrow \frac{\rho_2(x)}{\rho_1(x)} = \frac{M_2}{M_1}\left(\frac{R_2}{R_1}\right)^{-3} \qquad \boxed{\rho(x) \propto \frac{M}{R^3}}$$

This must hold for any mass shell $0 \leq x \leq 1$ and for x = 0

$$\boxed{\frac{\rho_{c2}}{\rho_{c1}} = \frac{M_2}{M_1}\left(\frac{R_2}{R_1}\right)^{-3} = \frac{\bar{\rho}_2}{\bar{\rho}_1}}$$

i.e.,

$$dm = 4\pi r^2 \rho\ dr \Rightarrow \rho(x) = \frac{M}{R^3}$$

for any value of x $\quad 0 \leq x \leq 1$

In practice this is equivalent to replacing dm with M and r and dr with R. This only works because of the assumption of homology. Does not work e.g., for red giants, but pretty good for main sequence stars.

Similarly using the HE equation $\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$ which is

(dm = $4\pi r^2 \rho dr$) $\qquad \frac{dP}{dr} = -\frac{Gm\rho}{r^2}$ in Lagrangian coordinates,

(Pols *p*.104) shows

$$\boxed{P(x) \propto \frac{M^2}{R^4} \propto \frac{\rho(x)}{R}}$$

Again, this is the same result one gets by replacing *dm*, m(r), and r in the differential equation by their full star counterparts.

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

Putting this together with $\rho(x) \propto \frac{M}{R^3} \Rightarrow R \propto \left(\rho / M\right)^{1/3}$,one gets a "new" result

$$\boxed{P(x) \propto M^{2/3}\rho(x)^{4/3}} \quad (\text{i.e., } P_c = const\ M^{2/3}\rho_c^{4/3})$$

which we have actually seen several times before, e.g., when talking about polytropes. (polytropes of the same index n are homologous). Taking $P \propto \rho T$ recovers $T_c \propto M^{2/3}\, \rho_c^{1/3}$

and the whole set for radiative stars supported by ideal gas pressure

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho} \qquad \rho \propto \frac{M}{R^3} \qquad 1)$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \qquad \mathrm{P} \propto \frac{M^2}{R^4} \propto \frac{M\rho}{R} \qquad 2)$$

$$\frac{dT}{dm} = -\frac{3}{4ac}\frac{\kappa}{T^3}\frac{L(r)}{(4\pi r^2)^2} \qquad L \propto \frac{R^4T^4}{\kappa M} \qquad 3)$$

$$\frac{dL(m)}{dm} = \varepsilon \qquad L \propto M\varepsilon \qquad 4)$$

$$P = P_0\rho T / \mu \qquad \mathrm{P} \propto \frac{\rho T}{\mu} \qquad 5)$$

$$\varepsilon = \varepsilon_0 \rho T^{\nu} \qquad \varepsilon \propto \rho T^{\nu} \qquad 6)$$

$$\kappa = \kappa_0 \rho^a T^b \qquad \kappa \propto \rho^a T^b \qquad 7)$$

These are 7 equations in 9 unknowns.

$$\rho,\ T,\ \mu,\ P,\ L, R, M, \varepsilon, \kappa$$

Once can solve for any one of them in terms of at most two others. e.g. L as f(μ,M)

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e.g. ideal gas and constant opacity

$$\rho \propto \frac{M}{R^3} + \mathrm{P} \propto \frac{M^2}{R^4} \qquad \Rightarrow P \propto \frac{M^2\rho^{4/3}}{M^{4/3}} = M^{2/3}\rho^{4/3}$$

$$+\ P \propto \frac{\rho T}{\mu} \qquad \Rightarrow T \propto \frac{\mu P}{\rho} \propto \frac{\mu}{\rho} M^{2/3}\rho^{4/3} \propto \frac{\mu M}{R}$$

$$+L \propto \frac{R^4T^4}{\kappa M} \qquad \Rightarrow L \propto \frac{\mu^4 M^4}{\kappa M} = \frac{\mu^4 M^3}{\kappa}$$

$$+L \propto M\varepsilon \text{ and } \varepsilon = \varepsilon_0 \rho T^{\nu} \qquad \Rightarrow \frac{\mu^4 M^3}{\kappa} \propto M \frac{M}{R^3}\left(\frac{\mu M}{R}\right)^{\nu}$$

$$R^{3+\nu} \propto M^{\nu+2-3}\mu^{\nu-4}\kappa$$

$$\mathrm{R} \propto M^{\left(\frac{\nu-1}{\nu+3}\right)}\mu^{\left(\frac{\nu-4}{\nu+3}\right)}\kappa^{\left(\frac{1}{\nu+3}\right)}$$

These have been evaluated for constant κ, e.g., electron scattering, but the generalization to $\kappa = \kappa_0\rho^a T^b$ is straightforward.

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from previous page

$$\mathrm{R} \propto M^{\left(\frac{\nu-1}{\nu+3}\right)}\mu^{\left(\frac{\nu-4}{\nu+3}\right)}\kappa^{\left(\frac{1}{\nu+3}\right)}$$

e.g. pp cycle ($\nu = 4$) and electron scattering κ=constant

$$\mathrm{R} \propto M^{3/7}$$

while for the *CNO* cycle ($\nu = 18$) and electron scattering κ=constant

$$\mathrm{R} \propto \mu^{2/3} M^{17/21}$$

If one further includes the density and temperature variation of κ other relations result. E.g. if $\kappa = \kappa_0 \rho T^{-7/2}$ and pp-energy generation dominates

$$L \propto \mu^{7.5}\frac{M^{5.5}}{R^{1/2}} \qquad \text{(left to the student)}$$

and

$$R \propto \mu^{\frac{\nu-7.5}{\nu+2.5}} M^{\frac{\nu-3.5}{\nu+2.5}}$$

Note that the relevant values of e.g., κ and μ, are averages for the whole star, not just the photosphere

e.g. $\nu = 4$ $\quad R \propto \mu^{-0.54}\ M^{0.0769}$ and $\mathrm{L} \propto \mu^{7.77} M^{5.46}$

Aside:

The Kramer's opacity solution is not particularly useful because when the opacity becomes high the star becomes convective and the simplest homology arguments rely on the assumption of transport by radiative diffusion.

Still the prediction that L becomes sensitive to a power of M steeper than 3 at low mass is generally true.

In general, for main sequence stars, the radius is weakly dependent on the mass. Given these relations one can also estimate how the central temperature and density will vary on the main sequence. For illustration, the electron scattering case (κ = constant)

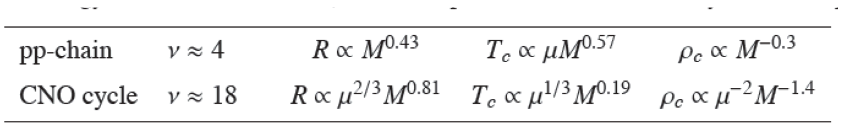
$$T_c \propto \frac{\mu M}{R} \propto \mu M^{0.57} \ (pp) \quad \text{or} \quad \mu^{1/3} M^{0.19} \ (CNO)$$

$$\rho_c \propto \frac{M}{R^3} \propto M^{-0.29} \ (pp) \quad \text{or} \quad \mu^{-2} M^{-1.43} \ (CNO)$$

since $R \propto M^{3/7} \, (pp) \ or \ \mu^{2/3} M^{17/21} \, (CNO) \qquad \frac{3}{7} = 0.43 \qquad \frac{17}{21} = 0.81$

That is the central temperature will increase with mass while the central density decreases

Summary for constant opacity
and ideal gas



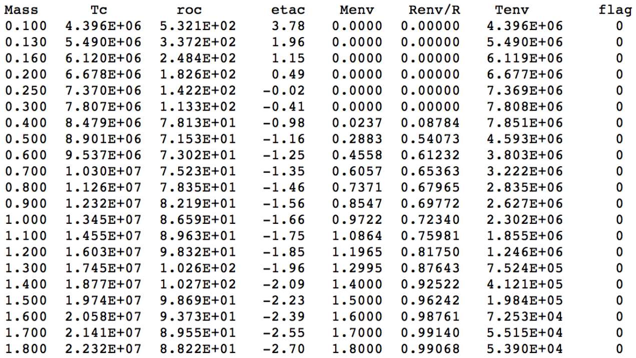
pp-chain	$\nu \approx 4$	$R \propto M^{0.43}$	$T_c \propto \mu M^{0.57}$	$\rho_c \propto M^{-0.3}$
CNO cycle	$\nu \approx 18$	$R \propto \mu^{2/3} M^{0.81}$	$T_c \propto \mu^{1/3} M^{0.19}$	$\rho_c \propto \mu^{-2} M^{-1.4}$

It turns out that the pp chain dominates above 1.3 solar masses (for solar metallicity)

In general R slowly rises with M on the main sequence, central T rises and central density declines

10^9 years – "isochrones"

"env" are conditions at the base of the convective envelope if there is one



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Mass      Tc         roc        etac    Menv    Renv/R     Tenv        flag
0.100  4.396E+06  5.321E+02     3.78   0.0000  0.00000   4.396E+06      0
0.130  5.490E+06  3.372E+02     1.96   0.0000  0.00000   5.490E+06      0
0.160  6.120E+06  2.484E+02     1.15   0.0000  0.00000   6.119E+06      0
0.200  6.678E+06  1.826E+02     0.49   0.0000  0.00000   6.677E+06      0
0.250  7.370E+06  1.422E+02    -0.02   0.0000  0.00000   7.369E+06      0
0.300  7.807E+06  1.133E+02    -0.41   0.0000  0.00000   7.808E+06      0
0.400  8.479E+06  7.813E+01    -0.98   0.0237  0.08784   7.851E+06      0
0.500  8.901E+06  7.153E+01    -1.16   0.2883  0.54073   4.593E+06      0
0.600  9.537E+06  7.302E+01    -1.25   0.4558  0.61232   3.803E+06      0
0.700  1.030E+07  7.523E+01    -1.35   0.6057  0.65363   3.222E+06      0
0.800  1.126E+07  7.835E+01    -1.46   0.7371  0.67965   2.835E+06      0
0.900  1.232E+07  8.219E+01    -1.56   0.8547  0.69772   2.627E+06      0
1.000  1.345E+07  8.659E+01    -1.66   0.9722  0.72340   2.302E+06      0
1.100  1.455E+07  8.963E+01    -1.75   1.0864  0.75981   1.855E+06      0
1.200  1.603E+07  9.832E+01    -1.85   1.1965  0.81750   1.246E+06      0
1.300  1.745E+07  1.026E+02    -1.96   1.2995  0.87643   7.524E+05      0
1.400  1.877E+07  1.027E+02    -2.09   1.4000  0.92522   4.121E+05      0
1.500  1.974E+07  9.869E+01    -2.23   1.5000  0.96242   1.984E+05      0
1.600  2.058E+07  9.373E+01    -2.39   1.6000  0.98761   7.253E+04      0
1.700  2.141E+07  8.955E+01    -2.55   1.7000  0.99140   5.515E+04      0
1.800  2.232E+07  8.822E+01    -2.70   1.8000  0.99068   5.390E+04      0
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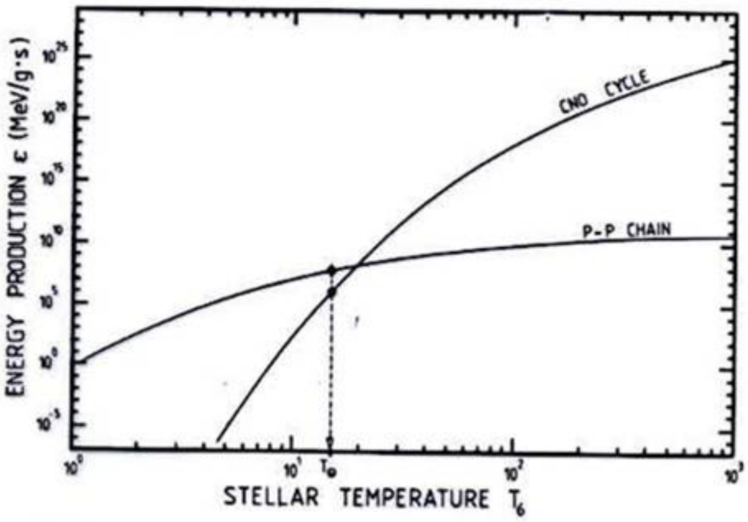
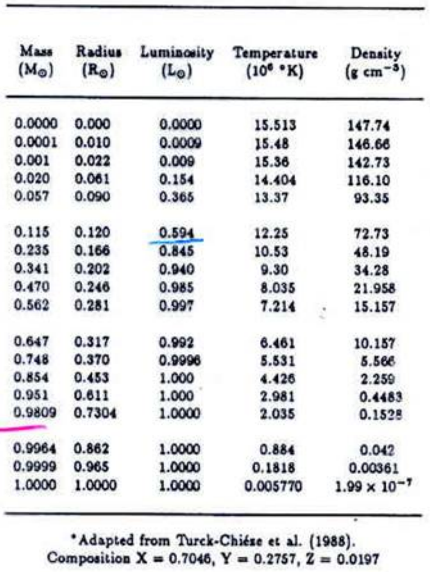
http://www.astro.ulb.ac.be/~siess/pmwiki/pmwiki.php/WWWTools/Isochrones



The sun (Model is a bit old; best T_c now is 15.71)

Radiative



Mass ($M_\odot$)	Radius ($R_\odot$)	Luminosity ($L_\odot$)	Temperature (10^6 °K)	Density (g cm^{-3})
0.0000	0.000	0.0000	15.513	147.74
0.0001	0.010	0.0009	15.48	146.66
0.001	0.022	0.009	15.36	142.73
0.020	0.061	0.154	14.404	116.10
0.057	0.090	0.365	13.37	93.35
0.115	0.120	0.594	12.25	72.73
0.235	0.166	0.845	10.53	48.19
0.341	0.202	0.940	9.30	34.28
0.470	0.246	0.985	8.035	21.958
0.562	0.281	0.997	7.214	15.157
0.647	0.317	0.992	6.461	10.157
0.748	0.370	0.9996	5.531	5.566
0.854	0.453	1.000	4.426	2.259
0.951	0.611	1.000	2.981	0.4483
0.9809	0.7304	1.0000	2.035	0.1528
0.9964	0.862	1.0000	0.884	0.042
0.9999	0.965	1.0000	0.1818	0.00361
1.0000	1.0000	1.0000	0.005770	1.99×10^{-7}

*Adapted from Turck-Chiéze et al. (1988).
Composition X = 0.7046, Y = 0.2757, Z = 0.0197

Convective

More Massive Main Sequence Stars

	$10 M_\odot$	$25 M_\odot$
X_H	0.32	0.35
L	$3.74 x 10^{37}$ erg s^{-1}	$4.8 x 10^{38}$ erg s^{-1}
T_{eff}	24,800 (B)	36,400 (O)
Age	16 My	4.7 My
T_{center}	$33.3 x 10^6$ K	$38.2\ x 10^6$ K
ρ_{center}	8.81 g cm^{-3}	3.67 g cm^{-3}
τ_{MS}	23 My	7.4 My
R	$2.73 x 10^{11}$ cm	$6.19 x\ 10^{11}$ cm
P_{center}	$3.13 x 10^{16}$ dyne cm^{-2}	$1.92 x 10^{16}$ dyne cm^{-2}
$\% P_{radiation}$	10%	33%

Surfaces stable (radiative, not convective); inner roughly 1/3 of mass is convective.

Suppose radiation pressure dominates and opacity is constant (very massive stars)

$L \propto \frac{R^4T^4}{\kappa M}$ if energy transport by radiative diffusion

(actually as M goes up, convection increasingly dominates)

$$P \propto \frac{M^2}{R^4} \propto T^4 \text{ so } R^4T^4 \propto M^2$$

$$L \propto \frac{R^4T^4}{\kappa M} \propto \frac{M}{\kappa}$$

It can in fact be shown that extremely massive stars approach the "Eddington limit" (though this is not the best way to derive it)

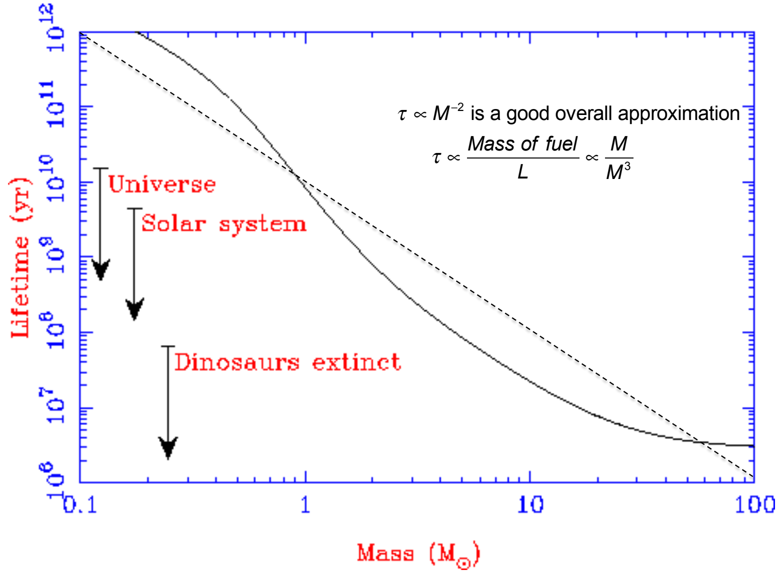
$$L_{Ed} = \frac{4\pi GMc}{\kappa} \approx 1.3 \times 10^{38}\left(\frac{M}{M_\odot}\right)\left(\frac{0.34\ cm^2\ g^{-1}}{\kappa}\right) erg\ s$$

There is also a lower limit to the lifetime of an extremely massive star given by the Eddington luminosity and the assumption that the (fully convective) star burns its entire mass

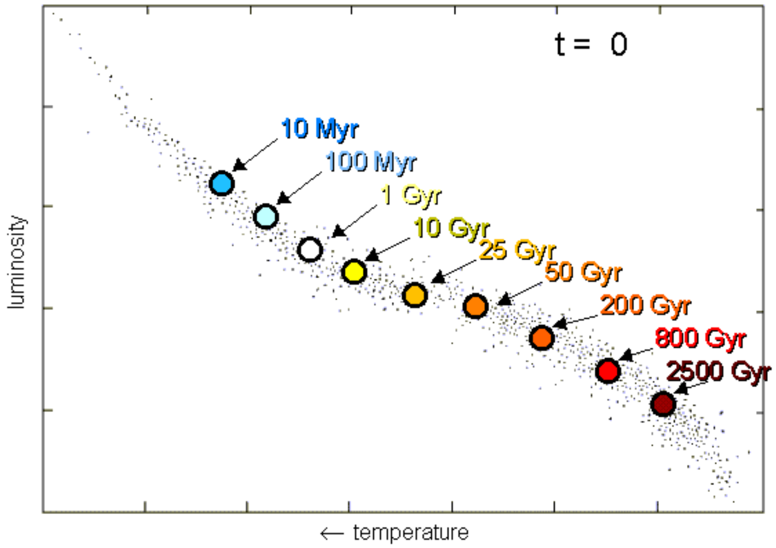
$$\tau_{Ed} = \frac{Mq}{L_{Ed}} = \frac{\left(6.8\times10^{18}\,erg\,g^{-1}\right)\left(2\times10^{33}\,g\right)}{1.3\times10^{38}\,erg\;s^{-1}}$$

$$= 3.3\ million\ years$$

Very massive stars approach these luminosities and lifetimes



http://www.astro.soton.ac.uk/~pac/PH112/notes/notes/node100.html



More implications of homology

- The mass luminosity relation L = f(M), varies with mass. For lighter stars on the pp cycle with Kramers opacity L is predicted to be proportional to $M^{5.46}$ though convection complicates the interpretation. For stars where electron scattering dominates it is M^3. For very high masses where radiation pressure becomes important, L becomes proportional to M.

  This is consistent with what is seen (The observed mass-luminosity relation for stars lighter than about 0.5 solar masses is not consistent with homology because the convective structure of the star, neglected here.

Homology works well for massive main sequence stars
but does not give the mass luminosity relation correctly below 1 Msun

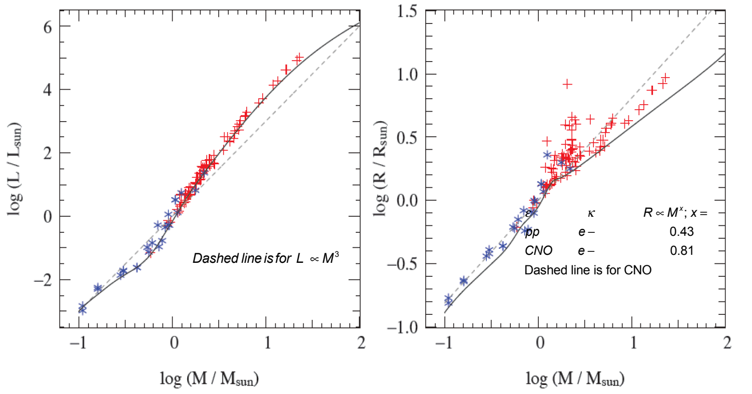
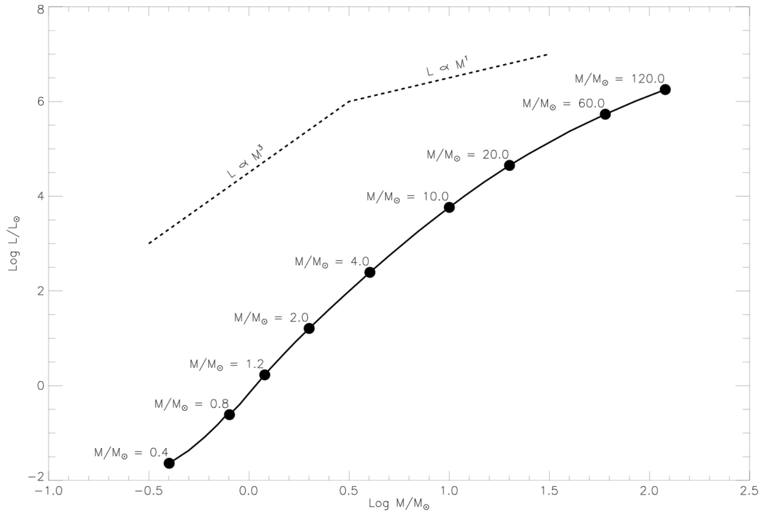


Figure 9.5. ZAMS mass-luminosity (left) and mass-radius (right) relations from detailed structure models with $X = 0.7, Z = 0.02$ (solid lines) and from homology relations scaled to solar values (dashed lines). For the radius homology relation, a value $\nu = 18$ appropriate for the CNO cycle was assumed (giving $R \propto M^{0.81}$); this does not apply to $M < 1\,M_\odot$ so the lower part should be disregarded. Symbols indicate components of double-lined eclipsing binaries with accurately measured M, R and L, most of which are MS stars.



Implications of homology- continued

- The Kelvin helmholtz time scale $\tau_{KH} = \dfrac{\alpha G M^2}{RL}$ will be shorter for more massive stars. They will not only live shorter lives but be born more quickly
- Lower mass stars with Kramers opacity will have higher opacity (because of their lower T and larger ρ) especially near their surfaces and will tend to be convective there.
- Higher mass stars will shine by the CNO cycle and will therefore have more centrally concentrated energy generation. They will thus have convective cores.
- And to restate the obvious, massive stars with their higher luminosities will have shorter lifetimes.

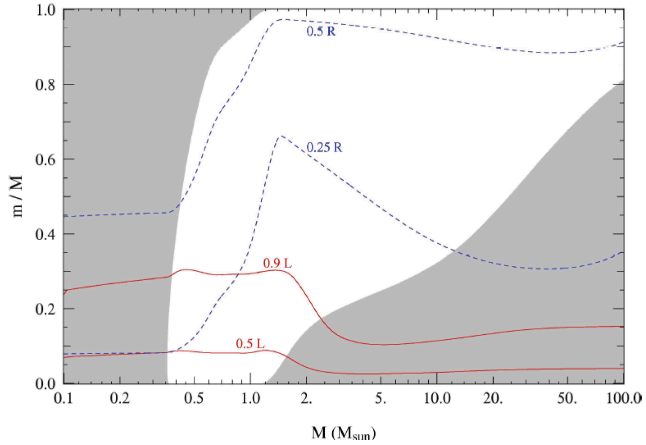
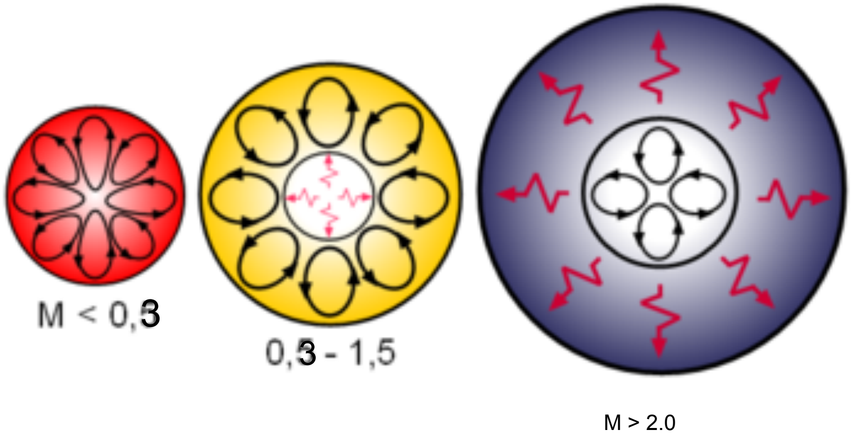


Figure 9.8. Occurrence of convective regions (gray shading) on the ZAMS in terms of fractional mass coordinate m/M as a function of stellar mass, for detailed stellar models with a composition $X = 0.70, Z = 0.02$. The solid (red) lines show the mass shells inside which 50% and 90% of the total luminosity are produced. The dashed (blue) lines show the mass coordinate where the radius r is 25% and 50% of the stellar radius R. (After KIPPENHAHN & WEIGERT.)

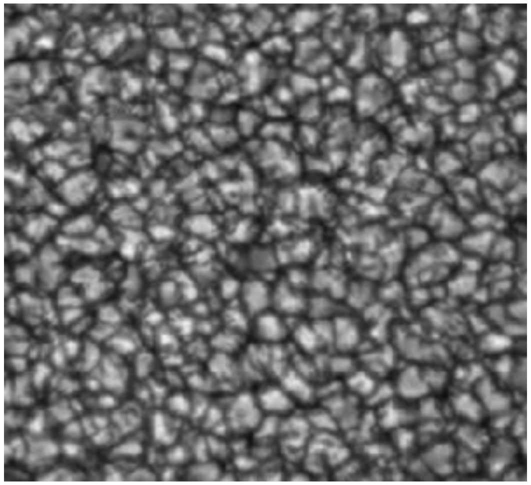


Whether the surface of the star is convective or not has important effects on its evolution and appearance.

Convection coupled with differential rotation can generate magnetic fields that energize surface activity like winds, flares, sunspots, coronal emission, etc.

These winds may play a role in braking the rotation rate of the star over time. The sun rotates at only about 2 km/s at its equator but a massive O or B star may rotate at 100 – 200 km/s.

http://www3.kis.uni-freiburg.de/~pnb/granmovtext1.html



June 5, 1993

Matter rises in the centers of the granules, cools then falls down. Typical granule size is 1300 km. Lifetimes are 8-15 minutes. Horizontal velocities are 1 – 2 km s^{-1}. The movie is 35 minutes in the life of the sun

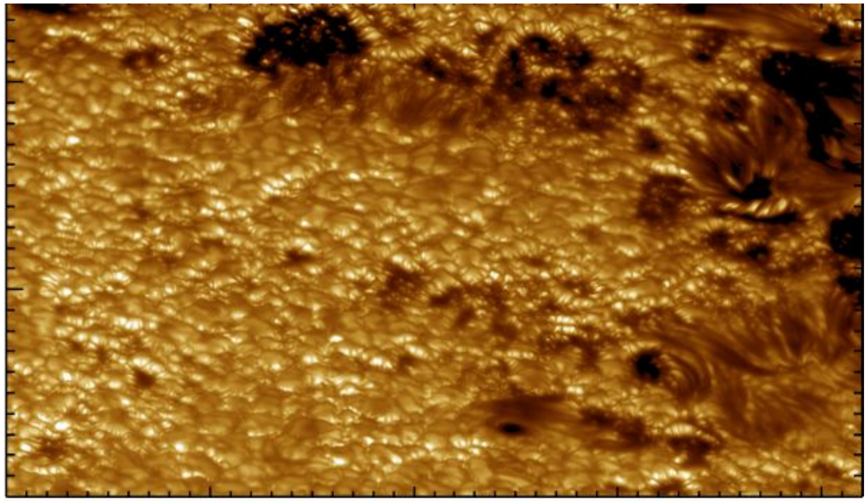
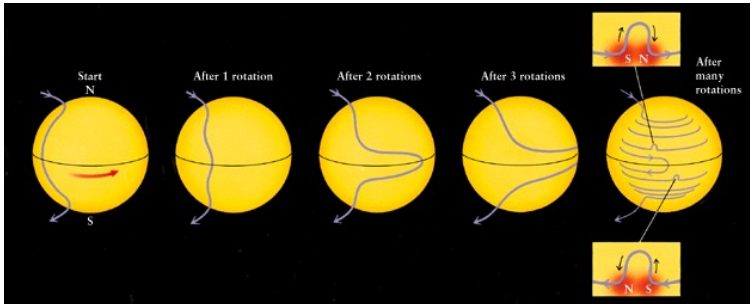


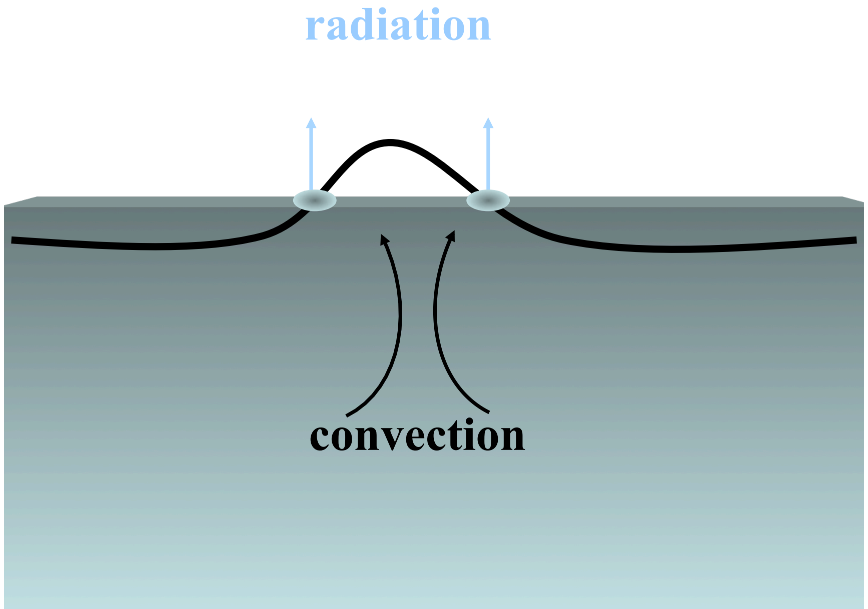
Image of an active solar region taken on July 24, 2002 near the eastern limb of the Sun.

http://www.boston.com/bigpicture/2008/10/the_sun.html
http://www.uwgb.edu/dutchs/planets/sun.htm

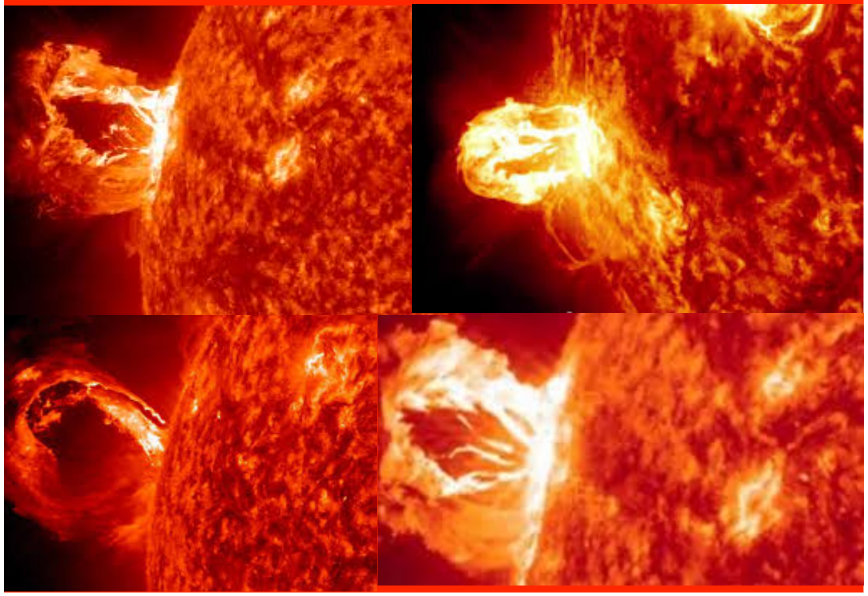
Rotation:
26.8 d at equator
31.8 d at 75° latitude

This differential rotation exists only in the convection zone. The radiative core rotates rigidly.





Solar Flares



THE HR DIAGRAM

$$L = 4\pi R^2 \sigma T_{eff}^4 \propto R^2 T_{eff}^4$$

and in the simplest case (constant opacity; ideal gas)

$$L \propto \frac{\mu^4 M^3}{\kappa} \quad \mathrm{R} \propto M^{3/7}\ (pp);\ \mathrm{R} \propto \mu^{2/3} M^{17/21}\ (CNO)$$

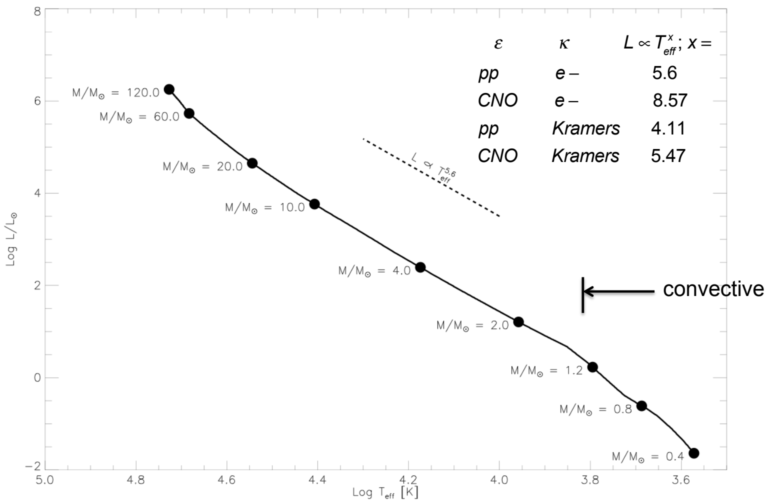
$$\mu^4 M^3 \propto M^{6/7} T_{eff}^4 \Rightarrow M^{15/7} \propto \left(\frac{T_{eff}}{\mu}\right)^4$$

$$M \propto \left(\frac{T_{eff}}{\mu}\right)^{28/15} \qquad \mathrm{R} \propto M^{3/7} \propto \left(\frac{T_{eff}}{\mu}\right)^{84/105}$$

$$L \propto R^2 T_{eff}^4 \propto \mu^{-168/105}\ T_{eff}^{588/105} = \mu^{-1.6}\ T_{eff}^{5.6} \qquad (pp)$$

Simiarly for CNO it can be shown

$$L \propto \mu^{-1.786}\ T_{eff}^{8.571} \qquad (CNO)$$



ε	κ	$L \propto T_{eff}^{x}$; $x =$
pp	*e* –	5.6
CNO	*e* –	8.57
pp	*Kramers*	4.11
CNO	*Kramers*	5.47

convective

Implications of homology for end of H-burning

- As hydrogen burns in the center of the star, μ rises. The central temperature and luminosity will thus both rise.

$$T_c \propto \frac{\mu M}{R} \propto \mu M^{0.57} \;\; (pp) \quad \text{or} \quad \mu^{1/3} M^{0.19} \;\; (CNO) \qquad e-scattering\ \kappa$$

$$L \propto \mu^4 \quad e-scattering\ \kappa \qquad L \propto \mu^{7.256} \;\; (pp) \;\; \mu^{7.769} \;\; (CNO) \text{ Kramers } \kappa$$

- The density evolution is not properly reflected because the sun's outer layers evolve non-homologously.
- Stars of lower metallicity with have somewhat smaller radii and bluer colors.

$$R = const\left(\varepsilon_0 \kappa_0\right)^{\frac{1}{3+\nu-s+3}} \quad s = 0,\ 7/2 \quad \text{for e-scattering, Kramers}$$

$$\nu = 4,\ 17 \quad \text{for pp, CNO}$$

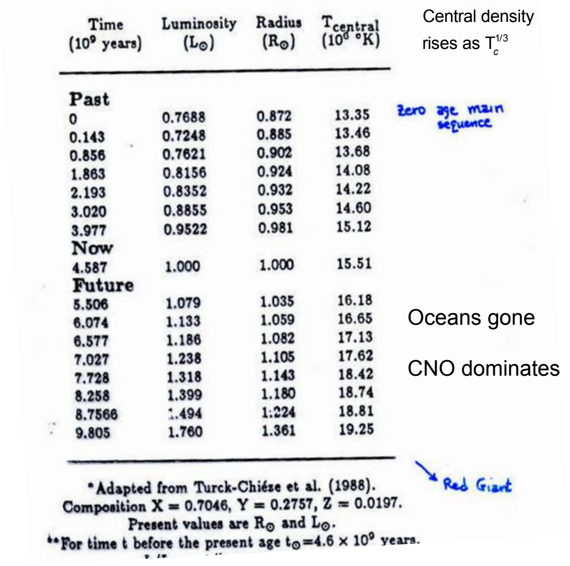
Evolution on the main sequence

The composition is not constant on the main sequence because hydrogen is changing especially in the center. This has two consequences

- As hydrogen decreases μ increases. Since the luminosity depends on μ to some power, the luminosity increases
- To keep the luminosity slightly rising as hydrogen decreases the central temperature must rise (slightly).

The sun - past and future

Central density rises as $T_c^{1/3}$



Time (10^9 years)	Luminosity (L$_\odot$)	Radius (R$_\odot$)	T$_{central}$ (10^6 °K)	
Past				
0	0.7688	0.872	13.35	Zero age main sequence
0.143	0.7248	0.885	13.46	
0.856	0.7621	0.902	13.68	
1.863	0.8156	0.924	14.08	
2.193	0.8352	0.932	14.22	
3.020	0.8855	0.953	14.60	
3.977	0.9522	0.981	15.12	
Now				
4.587	1.000	1.000	15.51	
Future				
5.506	1.079	1.035	16.18	
6.074	1.133	1.059	16.65	Oceans gone
6.577	1.186	1.082	17.13	
7.027	1.238	1.105	17.62	
7.728	1.318	1.143	18.42	CNO dominates
8.258	1.399	1.180	18.74	
8.7566	1.494	1:224	18.81	
9.805	1.760	1.361	19.25	

↘ Red Giant

*Adapted from Turck-Chiéze et al. (1988).
Composition X = 0.7046, Y = 0.2757, Z = 0.0197.
Present values are R$_\odot$ and L$_\odot$.
**For time t before the present age t$_\odot$=4.6 × 10^9 years.

Since μ_c increases more than T_c increases (due to the high sensitivity of ε to T), and since the pressure is due to ideal gas, $\frac{P_c}{\rho_c} \propto \frac{T}{\mu}$ must decrease. Thus P_c must decline or ρ_c must increase or both. Which alternative dominates depends on the relative changes of μ and T and hence on whether the star is burning by the pp cycle with $\varepsilon \propto T^4$ ($M < 1.5M_\odot$) or CNO cycle with $\varepsilon \propto T^{18}$. Since ρ_c varies roughly as T_c^3, it too cannot increase much, so especially for stars burning by the CNO cycle, P_c must decrease. This is accomplished by an expansion of the overlying layers - and the star in general. Note the non-homologous aspect. ρ_c goes up in the center but declines farther out. For stars burning by the pp chain, the changes in ρ and T are bigger so P does not have to change so much as μ goes up.

Combining – L increases and R increases. Path moves up on the HR diagram

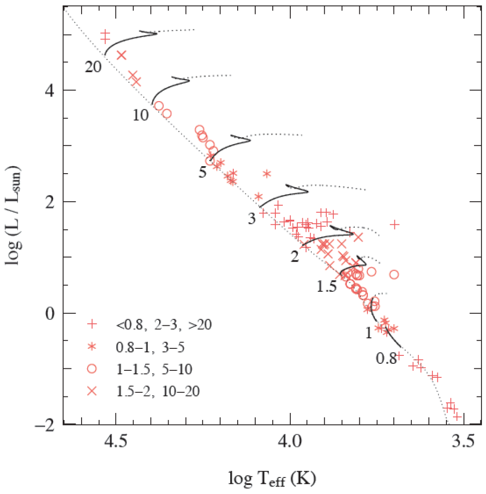


Figure 9.9. Evolution tracks in the H-R diagram during central hydrogen burning for stars of various masses, as labelled (in $M_\odot$), and for a composition $X = 0.7, Z = 0.02$. The dotted portion of each track shows the continuation of the evolution after central hydrogen exhaustion; the evolution of the 0.8 $M_\odot$ star is terminated at an age of 14 Gyr. The thin dotted line in the ZAMS. Symbols show the location of binary components with accurately measured mass, luminosity and radius (as in Fig. 9.5). Each symbol corresponds to a range of measured masses, as indicated in the lower left corner (mass values in $M_\odot$).

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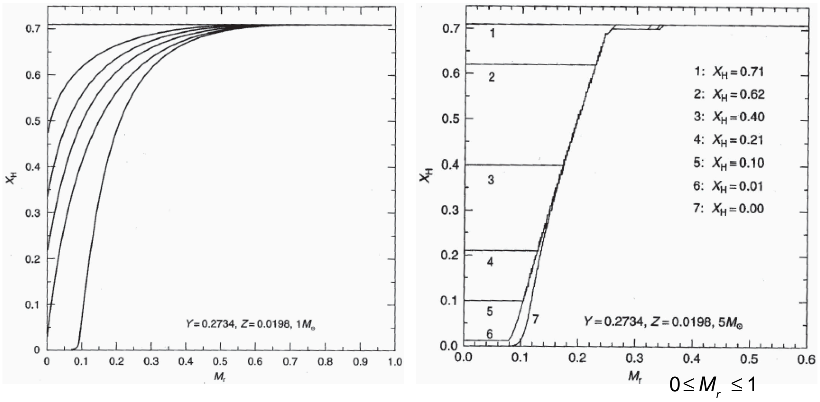


$0 \leq M_r \leq 1$

Figure 9.10. Hydrogen abundance profiles at different stages of evolution for a 1 $M_\odot$ star (left panel) and a 5 $M_\odot$ star (right panel) at quasi-solar composition. Figures reproduced from SALARIS & CASSISI.

Once the hydrogen depleted core exceeds the Schonberg Chandrasekhar mass, about 8% of the mass of the star, that depleted (isothermal) core can no longer support its own weight and begins to contract rapidly. This causes vigorous hydrogen shell burning that expans the star to red giant proportions

Schonberg Chandrasekhar mass

In the hydrogen depleted core there are no sources of nuclear energy, but the core's surface is kept warm by the overlying hydrogen burning, so that it does not radiate and therefore cannot contract, at least not quickly (on a Kelvin Helmholtz time scale). In these circumstances the core becomes *isothermal.*

L = 0 implies dT/dr = 0

A full star with constant temperature is unstable. With ideal gas pressure, hydrostatic equilibrium would have to be provided entirely by the density gradient, which would be very steep. Such a star (n = 1 polytrope) would not stable because $\gamma < 4/3$. n = 1 polytropes in fact have have either infinite radius or infinite central density. They are not physical

Schonberg Chandrasekhar mass

A star can be stable however if only a certain fraction of its inner core is isothermal. Even with dT/dr = 0 it can sustain a certain pressure at its edge. Once that pressure is exceeded however. the core must contract and develop a temperature gradient. That means it must radiate and evolve, i.e., shrink further.

The derivation is not given here but see Pols 9.1 and especially GK Chap 16

$$\frac{M_c}{M} = 0.37\left(\frac{\mu_{env}}{\mu_c}\right)^2$$

For μ_{env} =0.59 and μ_{He} =1.3, the limit is 0.08. When hydrogen has been depleted in the inner 8% of the stars mass, the helium core begins to contract and hydrogen shell burning is accelerated. The star becomes a red giant.

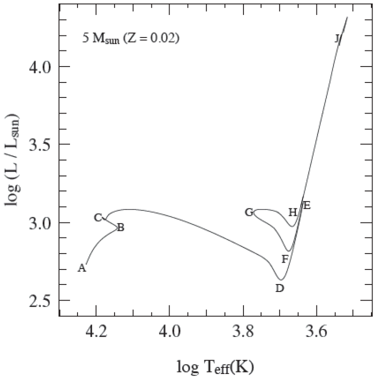
In high and intermediate mass stars, the hydrogen depleted core is usually initially smaller that the SC mass but the core grows by hydrogen shell burning. After exceeding the SC mass, H shell burning accelerates and the star moves quite rapidly to the right in the HR diagram

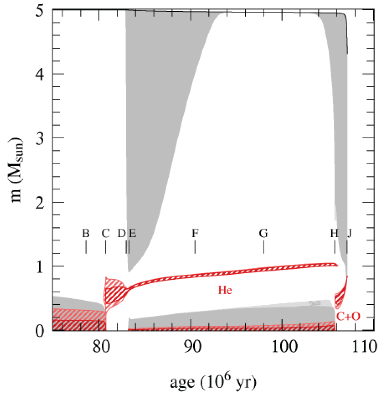
For lower mass stars, like the sun, the He core may become degenerate before exceeding the SC mass (which then becomes irrelevant). Their evolution off the main sequence is more “steady”

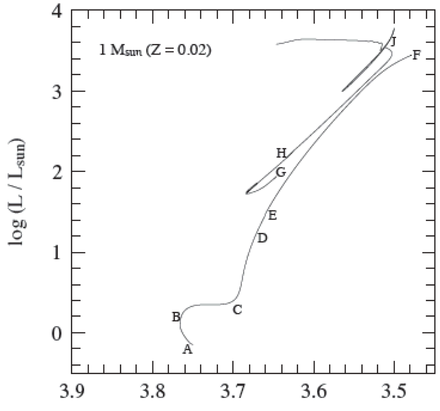
5 Solar Masses

A H ignition
B H = 0.03 – rapid contraction
C H depletion in center

C -> D Very fast towards end HR gap.
D He core now bigger than SC mass
H shell narrows
E Red giant formation

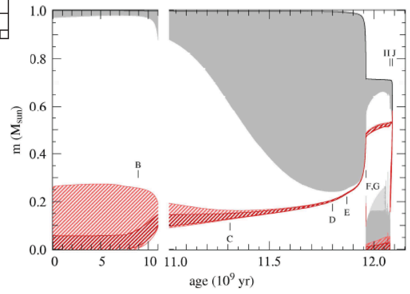






One Solar Mass

A H ignition
B H depeltion at center
C narrowing of H shell, exceed SC mass. RG formation. He core has become degenerate



Post-main sequence evolution segregates into three cases based upon the mass of the star

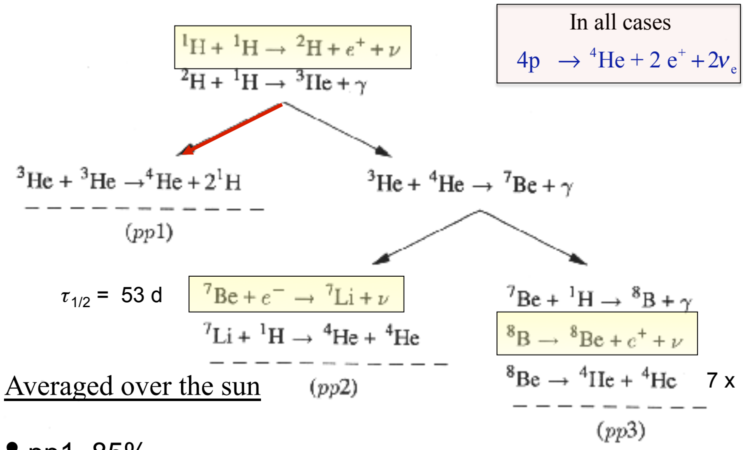
- Low mass stars – lighter than 2 (or 1.8) solar masses. Develop a degenerate helium core after hydrogen burning and ignite helium burning in a "flash"
- Intermediate mass stars – 2 – 8 solar masses. Ignite helium burning non-degenerately but do not ignite carbon
- Massive stars – over 8 solar masses. Ignite carbon burning and in most cases heavier fuels as well (8 – 10 is a complex transition region) and go on to become supernovae.

The Solar Neutrino "Problem"

Hydrogen Burning on the Main Sequence

In all cases

$4p \to {}^4He + 2\,e^+ + 2\nu_e$



$^1H + {}^1H \to {}^2H + e^+ + \nu$

$^2H + {}^1H \to {}^3He + \gamma$

$^3He + {}^3He \to {}^4He + 2\,{}^1H$

(pp1)

$^3He + {}^4He \to {}^7Be + \gamma$

$\tau_{1/2}$ = 53 d

$^7Be + e^- \to {}^7Li + \nu$

$^7Li + {}^1H \to {}^4He + {}^4He$

(pp2)

$^7Be + {}^1H \to {}^8B + \gamma$

$^8B \to {}^8Be + e^+ + \nu$

$^8Be \to {}^4He + {}^4He$ 7 x 10^{-17} s

(pp3)

Averaged over the sun

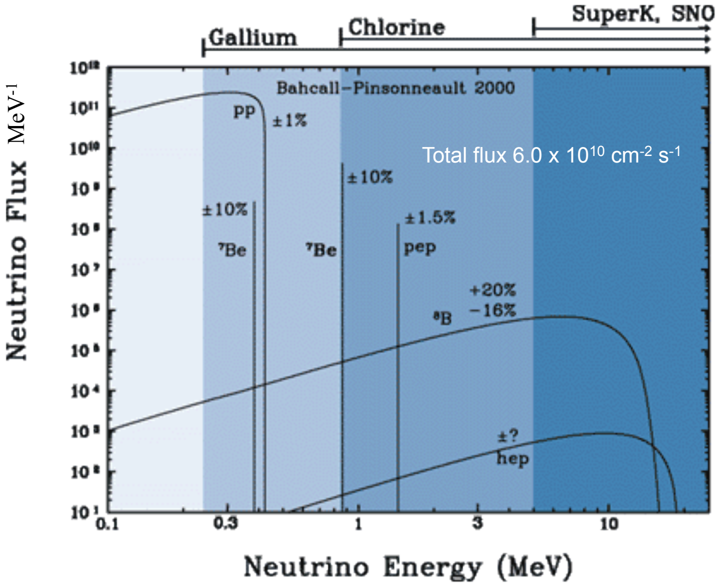
- pp1 85%
- pp2 15%
- pp3 0.02%

$T_{central}$ = 15.7 Million K

Neutrino Energies

Species	Average energy	Maximum energy	
p+p	0.267 MeV	0.420 MeV	
^{7}Be	0.383 MeV 0.861	0.383 MeV 0.861	10% 90%
^{8}B	6.735 MeV	15 MeV	

In the case of ^{8}B and p+p, the energy is shared with a positron hence there is a spread. For ^{7}Be the electron capture goes to two particular states in ^{7}Li and the neutrino has only two energies



Since 1965, experiements have operated to search for and study the neutrinos produced by the sun - in order to:

- Test solar models
- Determine the central temperature of the sun

  The flux of neutrinos from ^{8}B is sensitive to T^{18}

- Learn new particle physics

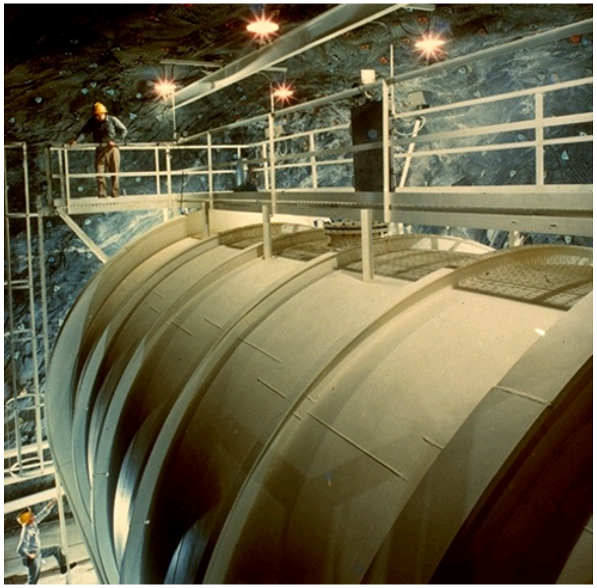
DETECTORS

The chlorine experiment – Ray Davis – 1965 - ~1999

$$^{37}Cl+\nu_e \rightarrow {}^{37}Ar + e^- - 0.814 \text{ MeV}$$

i.e., a neutron inside of ^{37}Cl is turned into a proton by a weak interaction involving an incident neutrino

^{37}Cl	^{37}Ar
17 p 18 n	18 p 17 n



Homestake Gold Mine
Lead, South Dakota

4850 feet down

tank 20 x 48 feet
615 tons (3.8 x 10^5 liters)
C_2Cl_4

Threshold 0.814 MeV

Half-life ^{37}Ar = 35.0 days

Neutrino sensitivity
^{7}Be, ^{8}B

8 x 10^{30} atoms of Cl

Nobel Prize 2002

Other Detectors

The gallium experiments (GALLEX and SAGE) – 1991 – 1997 and 1990 – 2001

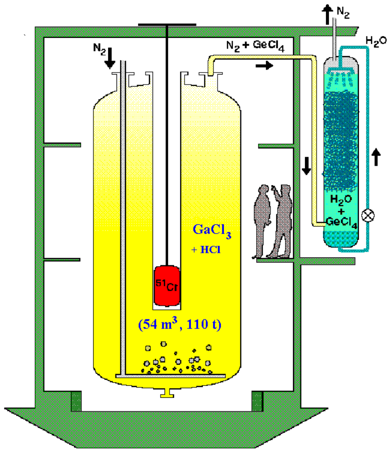
$$^{71}Ga + \nu_e \rightarrow {}^{71}Ge + e^- - 0.233\text{ MeV}$$

Kamiokande II - 1996 – 2001

$$e^- + \nu_e \rightarrow e^- + \nu_e$$

Inelastic scattering of neutrinos on electrons in water. Threshold 9 MeV. Scattered electron emits characteristic radiation.

GALLEX



In Gran Sasso Tunnel – Italy

3300 m water equivalent

30.3 tons of gallium in $GaCl_3$-HCl solution

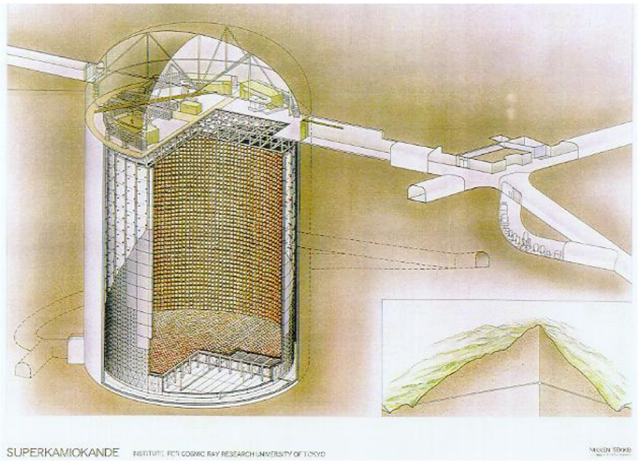
$^{71}Ga + \nu_e \rightarrow {}^{71}Ge + e^-$

Threshold 0.233 MeV

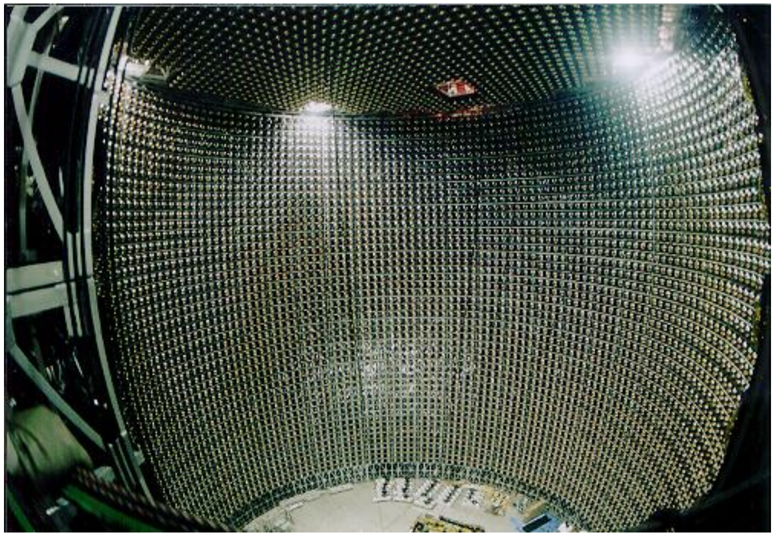
Sees pp, ^{7}Be, and ^{8}B.

Calibrated using radioactive ^{51}Cr neutrino source

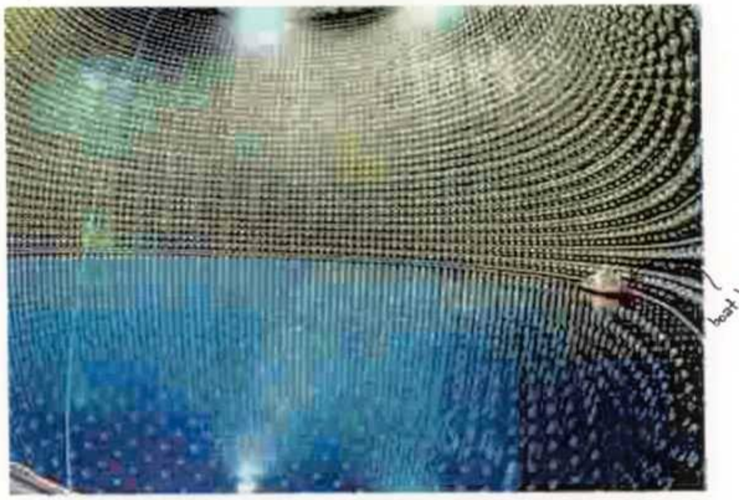
Kamiokande II (in Japanese Alps) 1996 - 2001



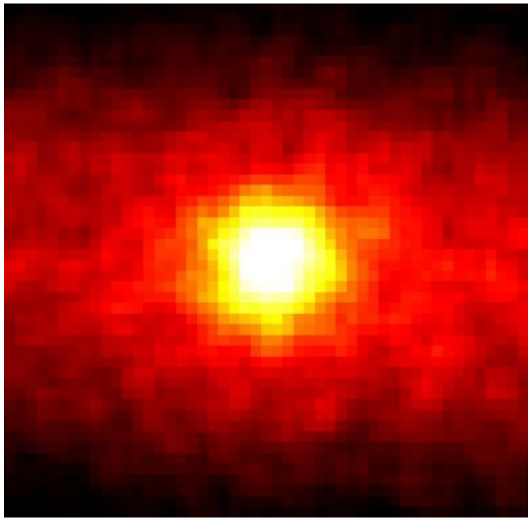
Depth 1 km
Detector H_2O
Threshold 9 MeV
Sensitive to ^{8}B
20'' photomultiplier tubes
Measure Cerenkov light
2.3 x 10^{32} electrons



Super-KamioKande (Japan)
50,000 tons of water
11,146 20" light detectors
0.6 mi
down



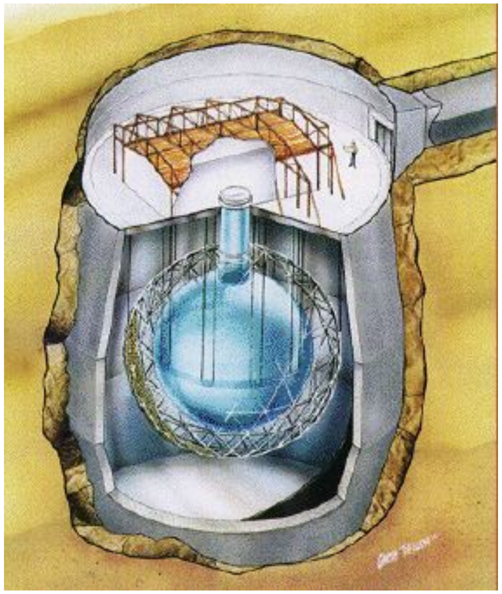
The Sun - 1999
(First picture in neutrinos)



This "picture" was taken using data from the Kamiokande 2 neutrino observatory. It contains data from 504 nights (and days) of observation. The observatory is about a mile underground.

Each pixel is about a degree and the whole frame is 90º x 90º.

And finally, the Sudbury Neutrino Observatory



6800 ft down

1000 tons
D_2O.

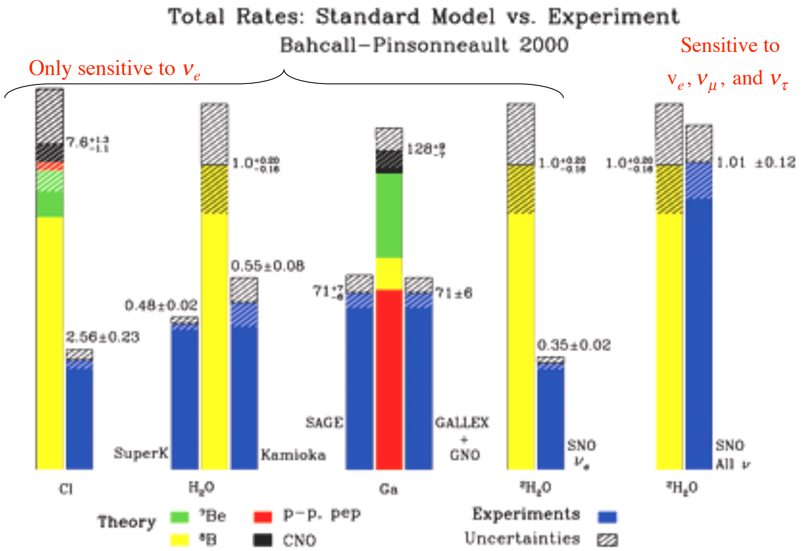
20 m diameter

Sudbury,
Canada

Threshold 5 MeV

Sees ^8B decay but can see all three kinds of neutrinos

ν_e, ν_μ, ν_τ



http://www.sno.phy.queensu.ca/sno/sno2.html - interactions

Neutrino interactions with heavy water $D_2O = {}^2H_2O$

Electron neutrino

$$\nu_e + {}^2\text{H} \underbrace{\rightarrow (\text{pp})}_{(\text{np})} \rightarrow \text{p} + \text{p} + \text{e}^-$$

All neutrinos with energy above 2.2 MeV = BE(^2H)

$$\nu_{e,\mu,\tau} + {}^2\text{H} \rightarrow \text{n} + \text{p} + \nu_{e,\mu,\tau}$$

add salt to increase sensitivity to neutrons,

$$\nu_{e,\mu,\tau} + \text{e}^- \rightarrow \nu_{e,\mu,\tau} + \text{e}^-$$

Results from SNO – 2002 (turned off in 2006)

The flux of electron flavored neutrinos above 5 MeV (i.e., only pp3 = ^{8}B neutrinos) is

$$1.76 \pm 0.1 \times 10^6 \text{ cm}^{-2} \text{ s}^{-1}$$

But the flux of μ and τ flavored neutrinos is

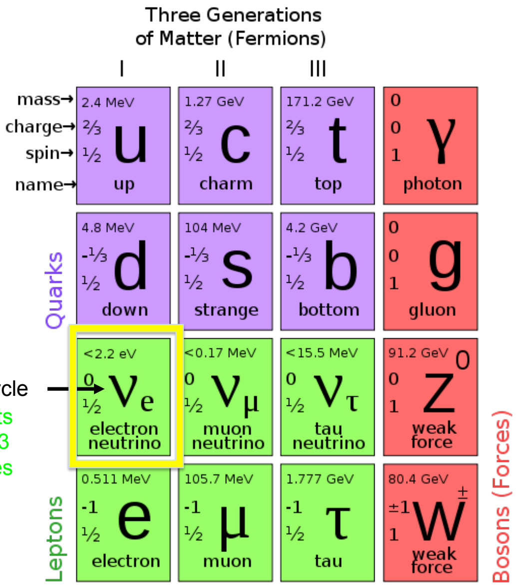
$$3.41 \pm 0.64 \times 10^6 \text{ cm}^{-2} \text{ s}^{-1}$$

Nobel Prize in Physics - 2002

Standard Solar Model ^{8}B neutrinos

$$5.05\,^{+1.01}_{-0.81} \times 10^6 \text{ neutrinos cm}^{-2} \text{ s}^{-1}$$

Particle physics aside:



emitted by pp-cycle

cosmology limits the sum of the 3 neutrino masses to < 1 eV

The explanation of the solar neutrino "problem" is apparently *neutrino flavor mixing*.

http://en.wikipedia.org/wiki/Neutrino_oscillation

A flux that starts out as pure electron-"flavored" neutrinos at the middle of the sun ends up at the earth as a mixture of electron, muon, and tauon flavored neutrinos in comparable proportions.

The transformation occurs in the sun and is complete by the time the neutrinos leave the surface. The transformation affects the highest energy neutrinos the most (MSW-mixing).

Such mixing requires that the neutrino have a very small but non-zero rest mass. This is different than in the so called "standard model" where the neutrino is massless. The mass is less than about 10^{-5} times that of the electron. (Also observed in earth's atmosphere and neutrinos from reactors).

New physics.... (plus we measure the central temperature of the sun very accurately – 15.71 million K)