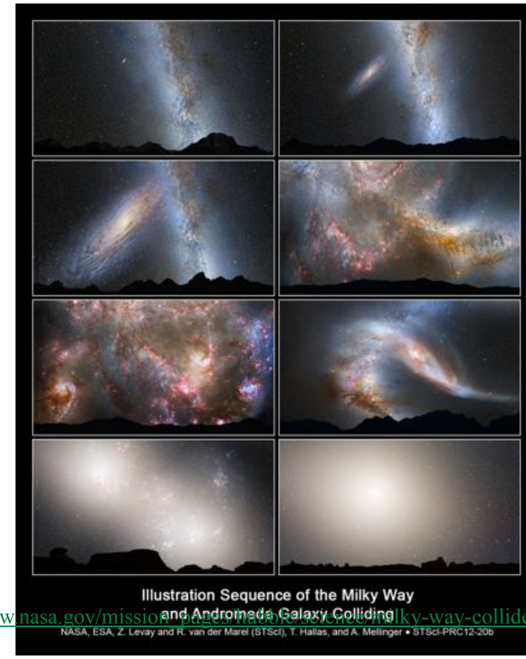
Lecture 2

The Hertzsprung-Russell Diagram
Blackbody Radiation
and Stellar Mass Determination

Glatzmaier and Krumholz 2
Prialnik 1.4
Pols 1

Andromeda and Milky Way collide in 4 billion years. Approaching us at 300 km/s (Doppler shift)

HST astrometry plus Doppler (plus computer)



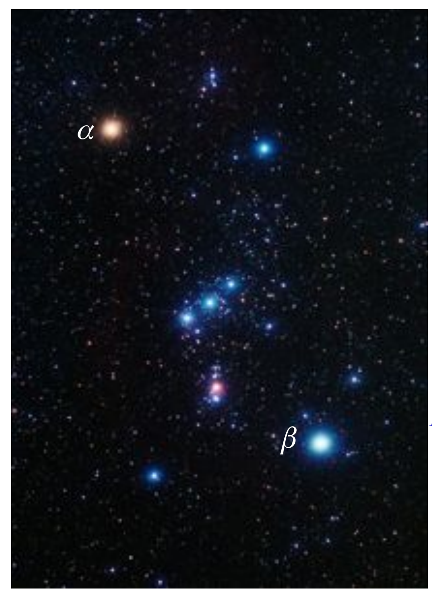
http://www.nasa.gov/mission_pages/hubble/science/milky-way-collide.html

Back to the stars

In addition to location, brightness, and time variability a key way we learn about stars is by analyzing their light. At the most basic level we can analyze their color, which turns out to be determined by their broad band emission across wavelength (black body emission)

In more detail we learn (a lot) more from analyzing their spectra (transitions in individual atoms).

E.g., ORION



Even with the unaided eye you should be able to discern a difference in color between Betelgeuse and Rigel

Betelgeuse and Rigel are $\alpha-$ and β-Orionis

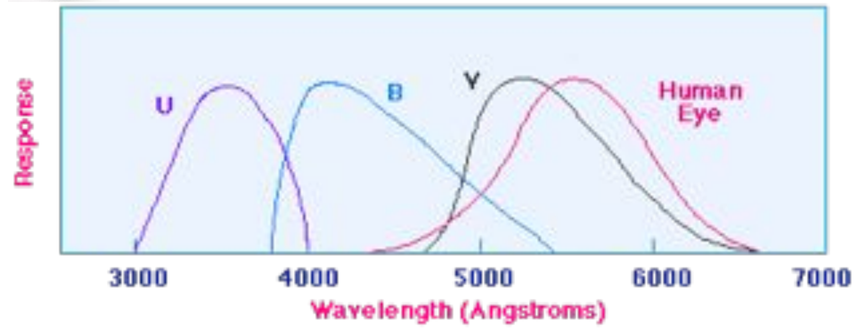
Astronomers historically have measured the color of a star by the difference in its brightness (magnitude) in two images, one with a blue filter (B) and another with a visual filter (V). (i.e., $B = m_B$; $V = m_V$)

This difference, denoted (B-V), is a crude measure of the temperature.

> *Note that the "bluer" the object, the smaller B will be (small magnitudes mean greater fluxes), so small or more negative (B-V) means bluer and hence hotter temperature.*

Blue filter

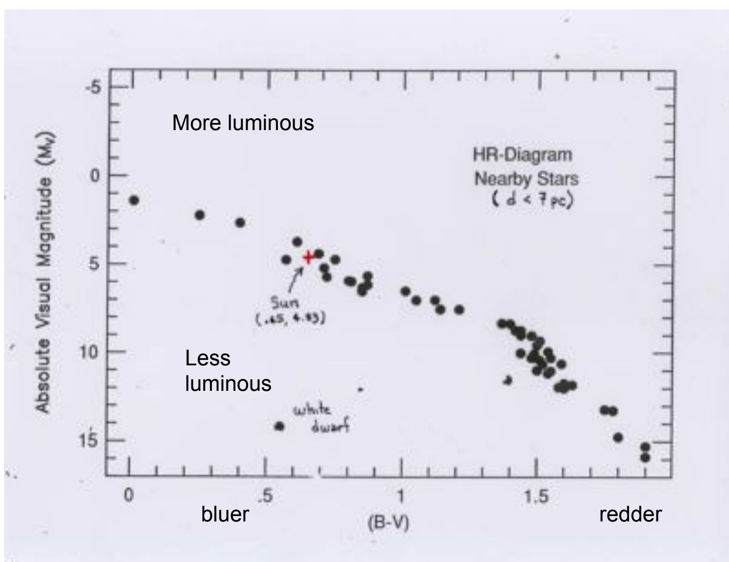
FREQUENTLY USED FILTERS ON THE TELESCOPE (there are many more)



$$1\ \mathring{A} = 1\ \text{Angstrom} = 10^{-8}\ \text{cm}$$

Today there are many more filters, including especially infrared, but these are representative

For local stars – a "volume limited sample"



What does this mean?

- Most, though not all "stars" lie on a well defined "line"
- The line is not straight
- There are many more points to the right of the sun than to the left

To better understand this result, we need to know what color means and what sets the color of a star

As we will discuss more later, stars are blackbodies. This is because they are extremely optically thick and the light is trapped within for a long time, eventually coming into equilibrium with its surroundings. Then the distribution of energy with wavelength or frequency is given by the Planck function:

$$B_\lambda(T)d\lambda = \frac{2hc^2}{\lambda^5}\frac{d\lambda}{e^{hc/(\lambda k_B T)}-1}$$

$$d\lambda = -\frac{c}{\nu^2}d\nu$$

$$B_\nu(T)d\nu = \frac{2h\nu^3}{c^2}\frac{d\nu}{e^{h\nu/(k_B T)}-1}$$

This distribution gives the power emitted from the surface, per unit "everything" – i.e., projected area of emitting surface, per unit solid angle, per unit (frequency,wavelength)

Aside

We will see this Planck function several times later so it is worth taking a look at it. It has 4 parts

1) The distribution function, aka the Bose Einstein distribution function $\left(e^{\frac{h\nu}{kT}}-1\right)^{-1}$. It is common to see exponentials like this in ditribution functions. They say how particles with energy (hν here)distribute themselves among states whose characteristic scale is is kT. The -1 is peculiar to spin 0 particles that can have any number in a given state - subject to energy conservation.

2) This number distribution function is multiplied by hν, the energy of a particle

3) And then by $4\pi(\nu/c)^2\, d(\nu/c) = 4\pi k^2 dk$, where k, the wave number ($1/\lambda$) is the "phase space" for paking so many photons into a given volume

4) And by 2 because there are two polarization states for the photon

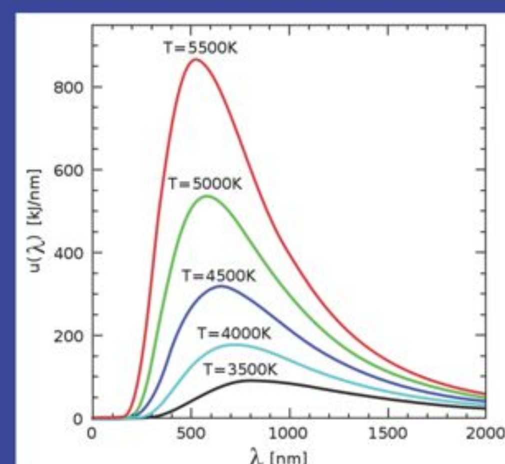
This gives $(2)(4\pi\frac{\nu^2 d\nu}{c^3})(h\nu)\left(\frac{1}{e^{h\nu/kT}-1}\right)$

or $u_\nu d\nu = \frac{8\pi h\nu^3 d\nu}{c^3\left(e^{h\nu/kT}-1\right)}$ and $u_\nu = \frac{8\pi h\nu^3}{c^3\left(e^{h\nu/kT}-1\right)}$ erg cm^{-3} Hz^{-1}

$B_\nu = \frac{c}{4\pi}u_\nu = \frac{2h\nu^3}{c^2\left(e^{h\nu/kT}-1\right)}$ erg cm^{-2} s^{-1}Hz^{-1}Ster^{-1}

http://disciplinas.stoa.usp.br/pluginfile.php/48089/course/section/16461/qsp_chapter10-plank.pdf

https://www.youtube.com/watch?v=syQbWP-7WC4

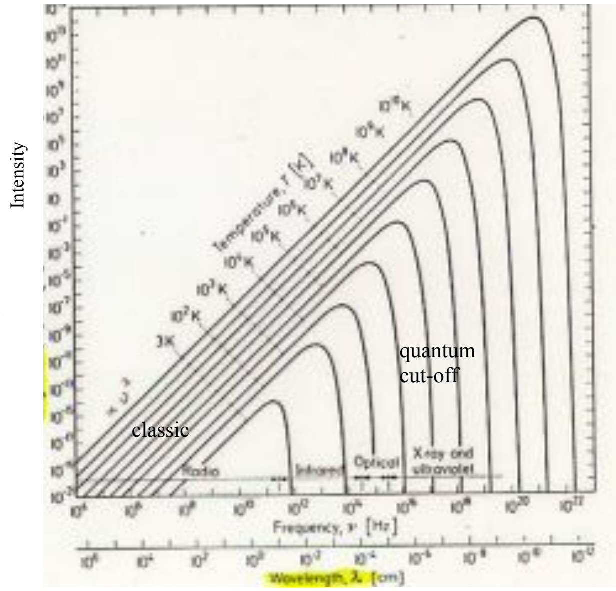


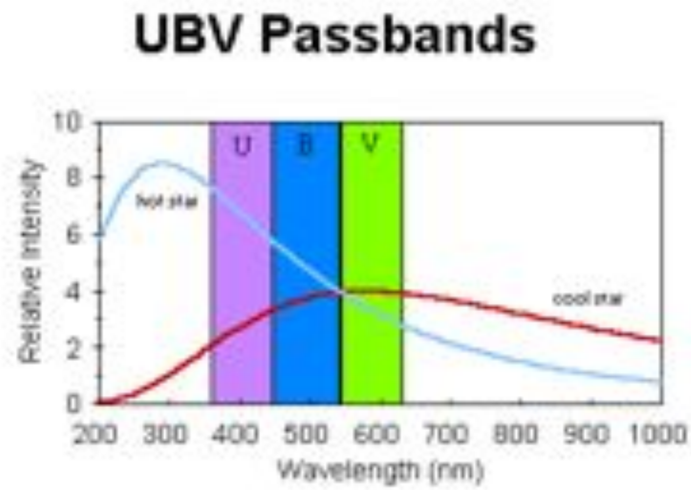
The Planck function for temperatures from 3500 - 5500 K (graphic stolen from wikipedia)

Blackbody (Thermal) Radiation

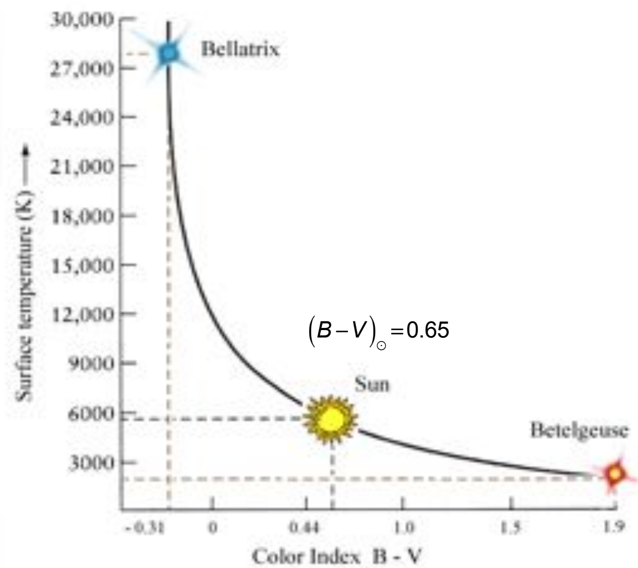
As T rises:

- more radiation at all wavelengths
- shift of peak emission to shorter wavelength
- greater total emission (area under the curve)

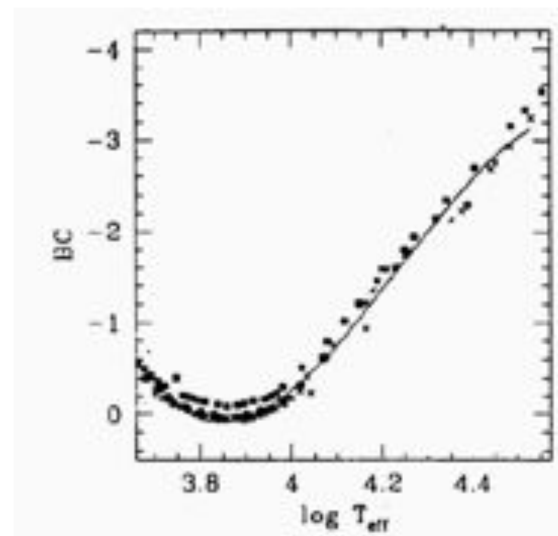




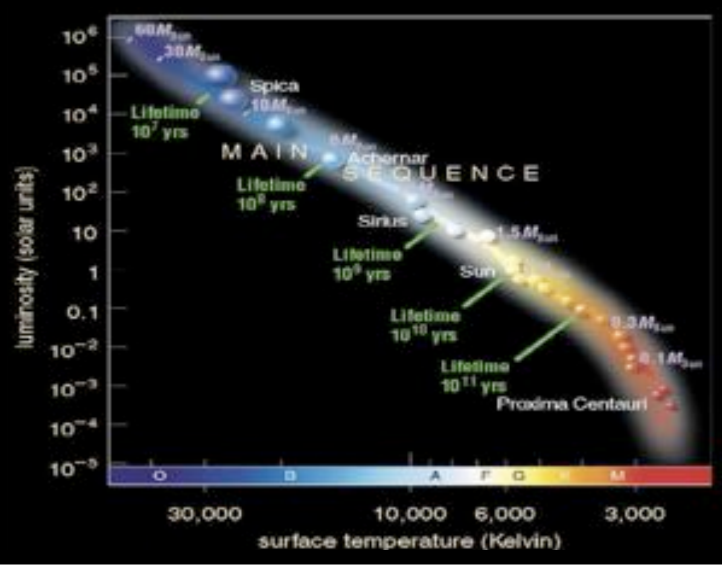
A SMALLER COLOR INDEX MEANS A HOTTER STAR

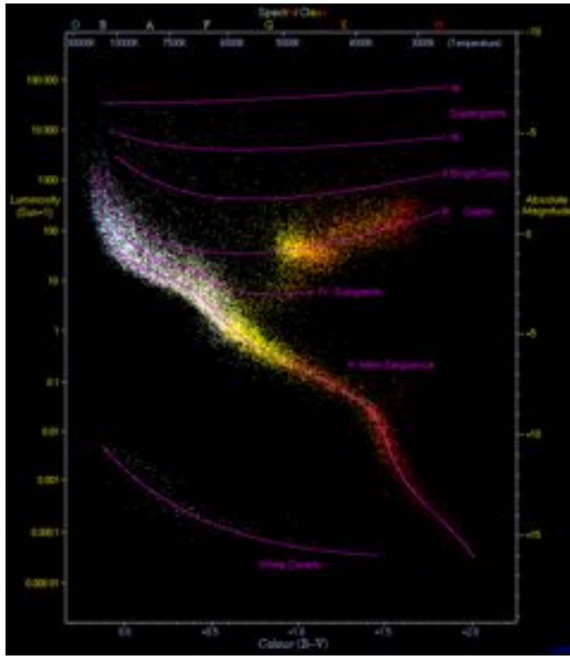


Can also get the bolometric correction from the Planck function



Color index can be converted to temperature and absolute magnitude
to luminosity to give the Hertzsprung-Russel diagram in more physical units





Hertzsprung–Russell diagram with 22,000 nearby stars from the Hipparcos catalog supplemented with 1000 stars from other catalogs.

Absolute bolometric magnitude has been converted into luminosity,

Note that the stars prefer to congregate in well defined strips

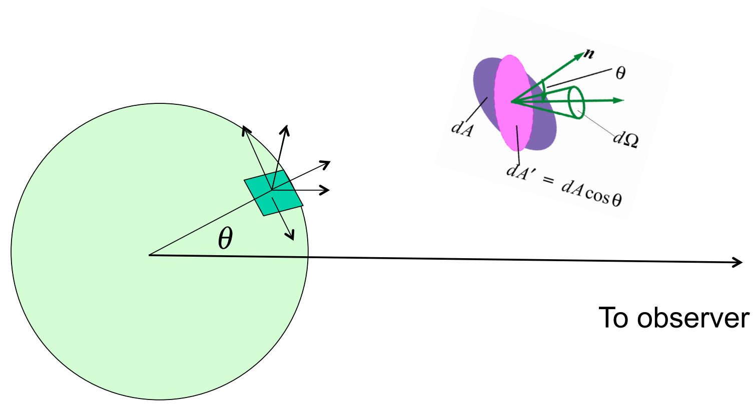
What does it mean?

As we will discuss further in lectures 4 – 6, stars are very optically thick and their radiation is trapped inside them for a long time. The emergent light, to good accuracy thus follows the Planck function which describes radiation that is in equilibrium with its surroundings:

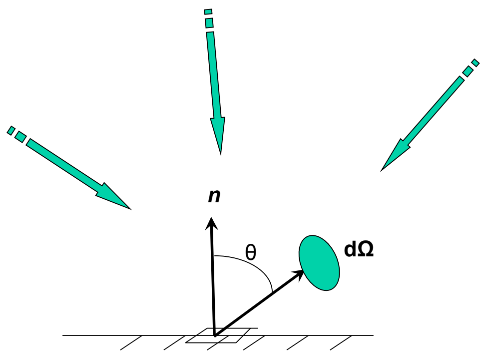
$$B_\nu(T)d\nu = \frac{2h\nu^3}{c^2}\frac{d\nu}{e^{h\nu/(k_BT)}-1} \quad \text{erg cm}^{-2}\,\text{s}^{-1}\,\text{Hz}^{-1}\,\text{steradian}$$

1) Integrating over solid angle and frequencies (a double integral) will give the total power radiated per square cm by a blackbody

2) The frequency, or wavelength where most of the power comes out can be obtained by setting the derivative to zero.

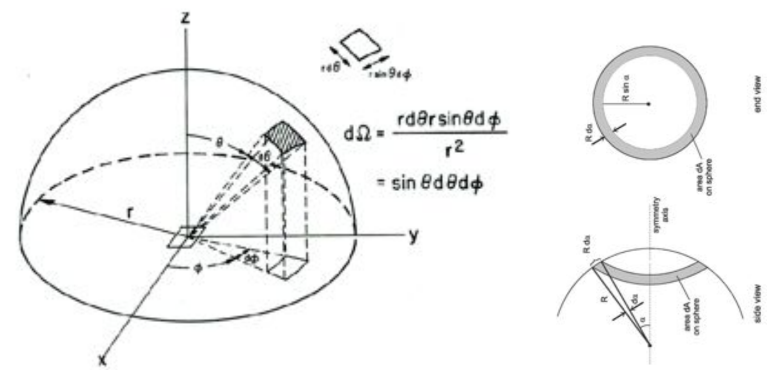


B_ν, the Planck radiance, is isotropic, but one must correct for the orientation of different regions of the stellar surface with respect to the observer. The emission from each cm^2 should be multipled by Cos θ



Perhaps it's easier to visualize photons falling from the sky
from all directions on a flat area, dA,
on the surface of the Earth

Solid angle as a function of ϕ and θ



$$2\int_0^{2\pi}\int_0^{\pi/2} d\Omega = 4\pi$$

The total power emitted at all angles and frequencies is thus:

$$P = \int_0^\infty d\nu \int_{\substack{hemi\\sphere}} d\Omega\, B_\nu Cos\theta \qquad d\Omega = \sin\theta\, d\theta\, d\phi \qquad d\,Cos\theta = -Sin\theta\, d\theta$$

$$P = \int_0^\infty d\nu \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\, B_\nu\, Cos\theta Sin\theta = \int_0^\infty B_\nu\, d\nu \int_0^{\pi/2} d\theta \int_0^{2\pi} d\phi\;\; Cos\theta Sin\theta$$

$$= -2\pi \int_0^\infty B_\nu\, d\nu \int_1^0 Cos\theta (dCos\theta) = 2\pi\left(0 - \frac{1}{2}\right)\int_0^\infty B_\nu\, d\nu$$

$$= \pi \int_0^\infty B_\nu\, d\nu$$

$$\int_0^\infty B_\nu\, d\nu = \int_0^\infty \frac{2h\nu^3}{c^2}\frac{d\nu}{e^{\frac{h\nu}{kT}} - 1} = 2\frac{k^3T^3}{h^2c^2}\int_0^\infty \left(\frac{h\nu}{kT}\right)^3 \frac{d\nu}{e^{\frac{h\nu}{kT}} - 1}$$

$$= 2\frac{k^4T^4}{h^3c^2}\int_0^\infty x^3 \frac{dx}{e^x - 1} \qquad x = \frac{h\nu}{kT}$$

$$= 2\frac{k^4T^4}{h^3c^2}\left(\frac{\pi^4}{15}\right)$$

Adding the additional factor of π from the integral over angles

$$P = 2\frac{k^4}{h^3c^2}\left(\frac{\pi^5}{15}\right)T^4 \equiv \sigma T^4$$

The maximum occurs where $\frac{dB_\lambda}{d\lambda} = 0$, which is given by

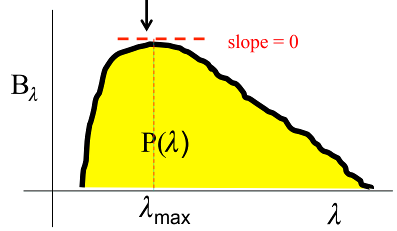
$$\lambda_{max} = \frac{hc}{x}\frac{1}{kT} \quad \text{where x is the solution of}$$

$$\frac{xe^x}{e^x - 1} - 5 = 0 \quad \text{or } x = 4.96511423\ldots$$

$$\lambda_{max} = \frac{0.28978 \text{ cm}}{T}$$

$$= \frac{2.8978 \times 10^7 \ A}{T}$$

This is known as Wien's Law and it relates the "color" of a star to its temperature



Some blackbody examples:

1) A spherical object with no internal heat source but shining because of stored up heat

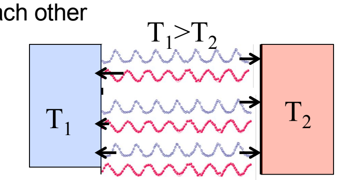
$$\frac{d}{dt}\left((heat\ capacity)*mass*T\right) = -4\pi r^2 \sigma T^4$$

e.g., a white dwarf

2) A main sequence star in thermal equilibrium

$$\int_0^M \varepsilon_{nuc}\, dm = 4\pi r^2 \sigma T^4$$

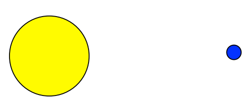
3) Two parallel planes emitting black body radiation at each other



Both planes emit and absorb radiation perfectly. T_1 and T_2 evolve depending on the situation (sizes, masses, etc)

Some blackbody examples:

4) Planets



Planet absorbs some small fraction of the power emitted by a star, reprocesses it into heat and radiates as a black body with a temperature quite different from that of the star.

The interior of the planet does not participate on relevant time scales but the surface and atmosphere quickly acquire a temperature as needed to satisfy balanced lower – energy absorbed = energy radiated.

Since stars are good blackbodies and are to good approximation, and are spheres, their luminosities are to good accuracy

$$L = 4\pi R_{star}^2 \sigma T_{photo}^4$$

This provides both a tool for measuring stellar radii (if we can measure the temperature and luminosity) and a physical basis for understanding the systematics of plots of luminosity vs temperature, i.e., the Herzsprung-Russell Diagram

The sun vs a 5790 K blackbody

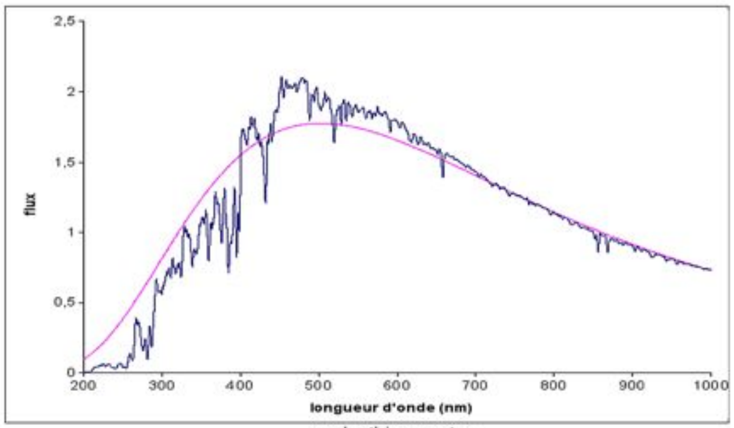
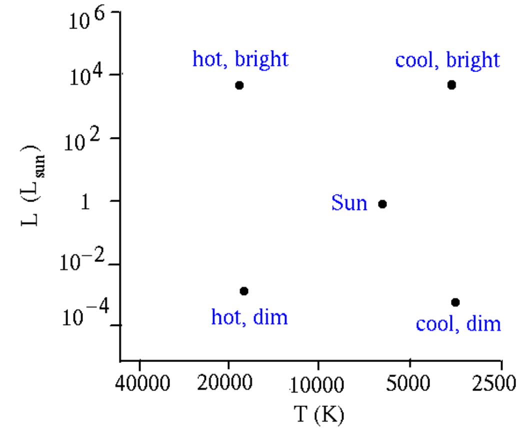


Fig. 1 : in black, the spectrum of the Sun (G2 V, Teff = 5790 K) ; in purple, the blackbody with T = 5790 K with radius = 1 solar radiu

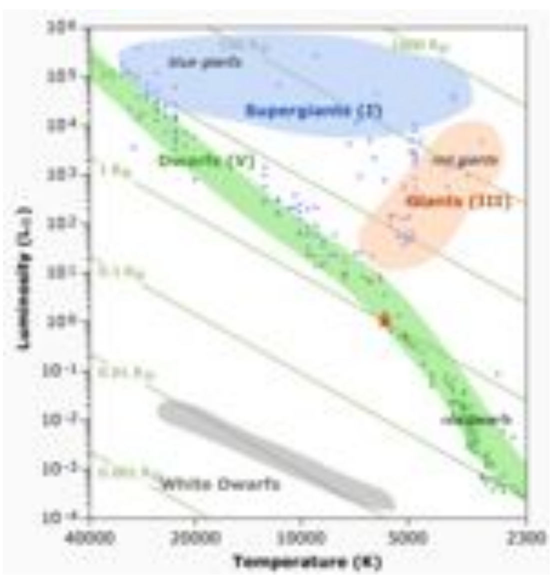


$L = 4\pi R_{star}^2 \sigma T_{photo}^4$

e.g. If L is high and T is small, R must be big

low high small

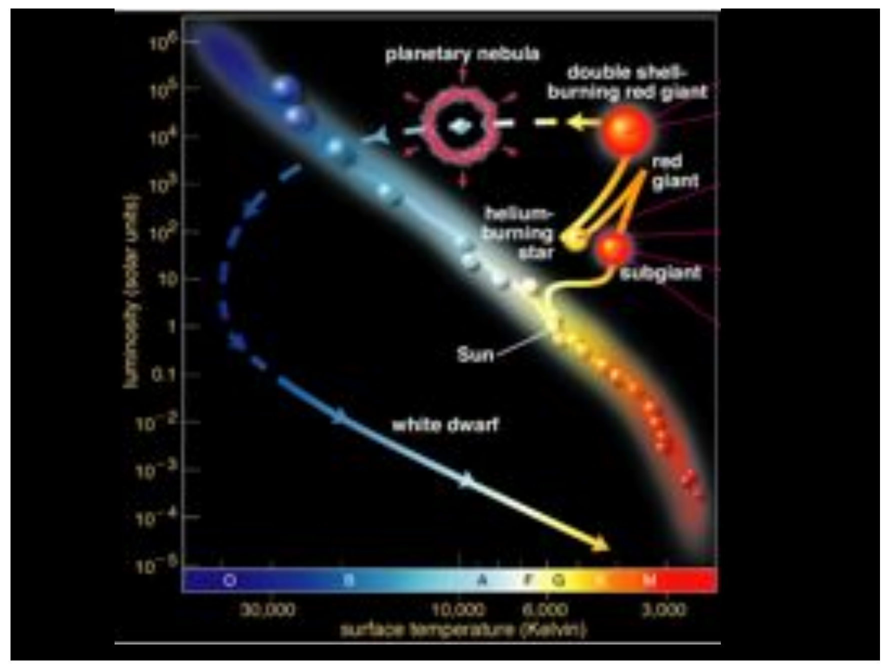
etc



On the main sequence more massive stars have bigger radii.

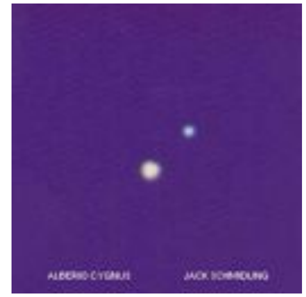
It turns out that the radius does not increase quite as rapidly as the mass. The total range of variation is about 100.

But there are other kinds of stars with bigger and smaller radii.

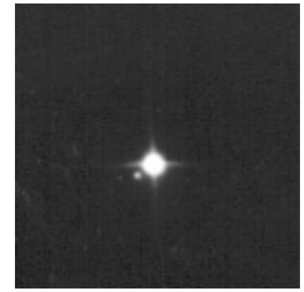


Getting Masses in Binary Systems

Binary and Multiple Stars
(about one-hird to one-half of all stars)



Beta-Cygnus (also known as Alberio)
Separation 34.6”. Magnitudes 3.0 and 5.3.
Yellow and blue. 380 ly away.
P > 75000 y. The brighter yellow component is also a (close) binary. P ~ 100 yr.



Alpha Ursa Minoris (Polaris)
Separation 18.3”. Magnitudes 2.0 and 9.0. Now known to be a triple.
Separation ~2000 AU for distant pair.

Polaris

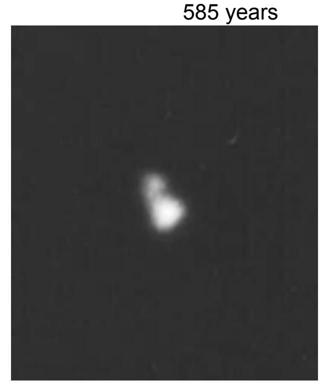
1.2 Msun Polaris Ab
Type F6 - V
4.5 Msun Polaris A
Cepheid

Period 30 yr

Polaris B is
F3 - V

$> 10^5$ years

585 years



1165 years

Epsilon Lyra – a double double.
The stars on the left are separated by 2.3” about 140 AU; those on the right by 2.6”. The two pairs are separated by about 208” (13,000 AU separation, 0.16 ly between the two pairs, all about 162 ly distant). Each pair would be about as bright as the quarter moon viewed from the other.

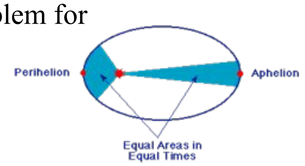
CLASSES OF BINARIES

- Visual binaries in which the stars and their orbital elements are well resolved (in principle if one waits long enough)
- Spectroscopic binaries – in which the presence of a companion can be inferred by the periodic Doppler shift exhibited in one or both spectra
- Eclipsing spectroscopic binaries in which the spectrum shows evidence for binarity and the light curve shows periodic eclipses or partial occultations.

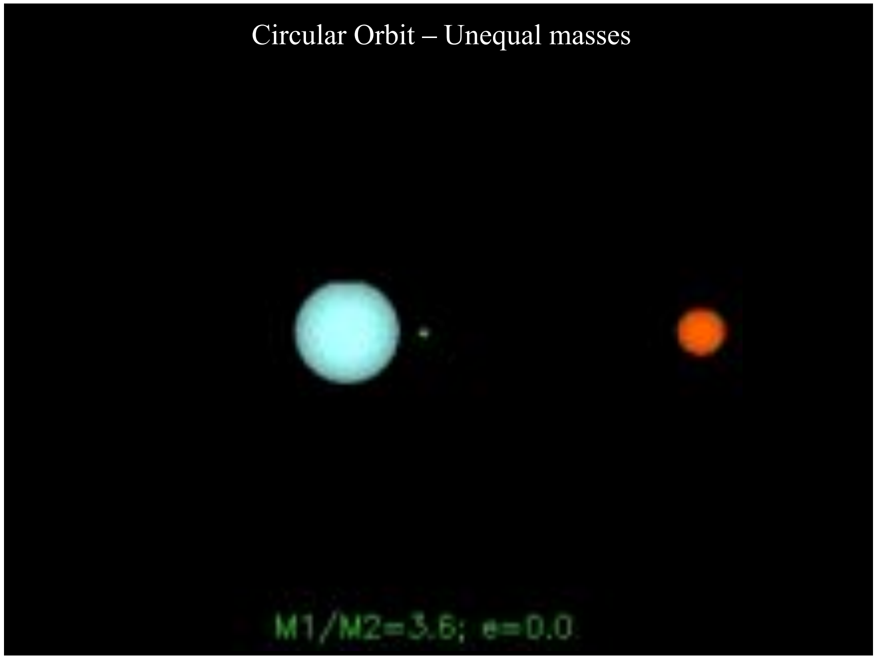
KEPLER'S LAWS

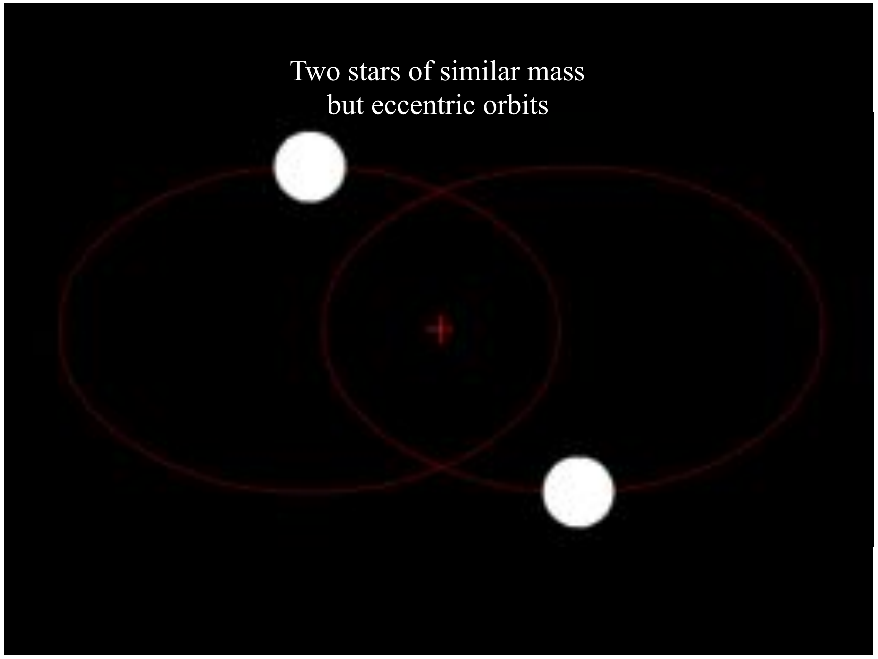
- Solution to the central force problem for a $1/r^2$ force, i.e., gravity

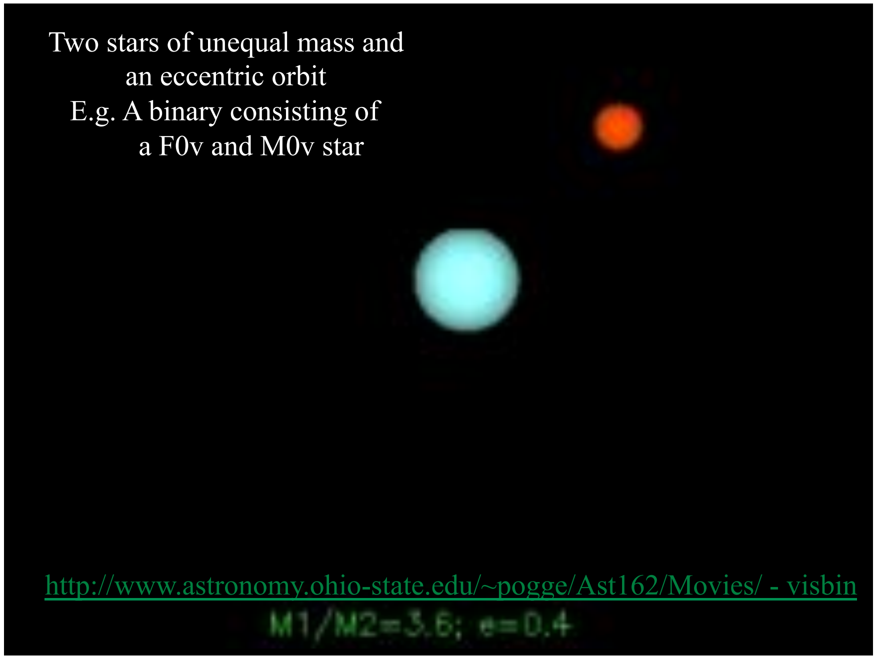
$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}$$



- Orbits are ellipses with central force at one focus of the ellipse
- A lines connecting the central force to the orbiting body sweeps out equal areas in equal times
- The square of the period is proportional to the cube of the semi-major axis. $P^2 = \frac{4\pi^2}{GM}a^3$
- Can be generalized to binary stars







Some things to note:

- The system has only one period. The time for star A to go round B is the same as for B to go round A.
- A line connecting the centers of A and B always passes through the center of mass of the system.
- The orbits of the two stars are similar ellipses with the center of mass at a focal point for both ellipses.
- At each point in time, the product of the mass of one star times its distance to the center of mass is equal to a similar product for the other star.

However:

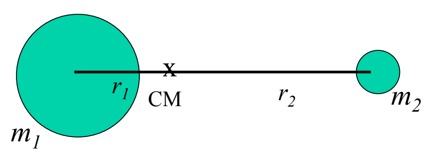
The actual separation between the stars is obviously not constant in the general case.

While not obvious, the separation at closest approach is the sum of the semi-major axes of the two elliptical orbits, $a = a_1 + a_2$, times (1-e) where e is the eccentricity. [the eccentricity of an ellipse is one half the distance between the two focii divided by the semimajor axis)

At the most distant point the separation is "a" times (1+e).

For circular orbits e = 0 and the separation is constant.

ASSUME FOR NOW CIRCULAR ORBITS



both stars feel the same gravitational attraction and thus both have the same centrifugal force

$$\frac{m_1 v_1^2}{r_1} = \frac{m_2 v_2^2}{r_2} = \frac{G m_1 m_2}{(r_1 + r_2)^2}$$

$$\frac{2\pi r_1}{v_1} = \frac{2\pi r_2}{v_2} = \text{Period} \qquad \therefore \frac{v_1}{v_2} = \frac{r_1}{r_2}$$

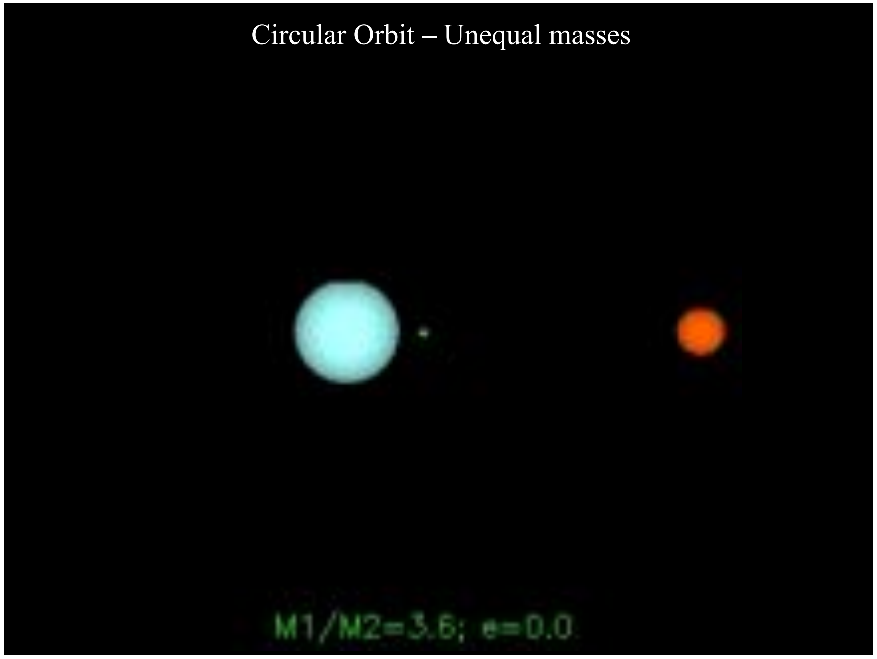
$$\frac{m_1 r_1^2 v_2^2}{r_1 r_2^2} = \frac{m_2 v_2^2}{r_2}$$

$$m_1 r_1 = m_2 r_2$$

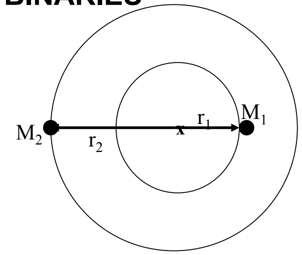
More massive star is closer to the center of mass and moves slower.

Also since the periods are equal $m_1 v_1 = m_2 v_2$

$$\frac{r_1}{r_2} = \frac{m_2}{m_1} = \frac{v_1}{v_2}$$



KEPLER'S THIRD LAW FOR BINARIES



$$\frac{G\cancel{M_1}M_2}{(r_1+r_2)^2} = \frac{\cancel{M_1}v_1^2}{r_1}$$

$$+\quad \frac{GM_1\cancel{M_2}}{(r_1+r_2)^2} = \frac{\cancel{M_2}v_2^2}{r_2}$$

$$\frac{G(M_1+M_2)}{(r_1+r_2)^2} = \frac{v_1^2}{r_1} + \frac{v_2^2}{r_2} = \frac{4\pi^2 r_1^2}{P^2 r_1} + \frac{4\pi^2 r_2^2}{P^2 r_2}$$

$$= \frac{4\pi^2}{P^2}(r_1+r_2)$$

$$\boxed{P^2 = K\,(r_1+r_2)^3} \qquad K = \frac{4\pi^2}{G(M_1+M_2)}$$

$$(M_1+M_2) = \frac{4\pi^2}{GP^2}(r_1+r_2)^3$$

$$M_\odot = \frac{4\pi^2}{G(1\,yr)^2}(AU)^3$$ for the earth

Divide the two equations

$$\frac{M_1+M_2}{M_\odot} = \left(\frac{(r_1+r_2)^3_{AU}}{P^2_{yr}}\right)$$

$$\frac{M_1}{M_2} = \frac{r_2}{r_1} \quad \text{or} \quad \frac{M_1}{M_2} = \frac{v_2}{v_1}$$

If you know r_1, r_2 *in AU*, or v_1, v_2, and P in years you can solve for the two masses.

The general case

(unequal masses eccentric orbits)

Define a coordinate system based on the center of mass

$$M_1\vec{r}_1 + M_2\vec{r}_2 = 0$$

Then $\vec{r}_1 = -\frac{M_2}{M_1}\vec{r}_2$

where $\vec{r}_1$ and $\vec{r}_2$ are the distances *from the center of mass* to stars 1 and 2 respectively. Let $\vec{r} = \vec{r}_2 - \vec{r}_1$ be the vector between the two stars (in magnitude $r = r_1 + r_2$ since they are always in opposite directions).

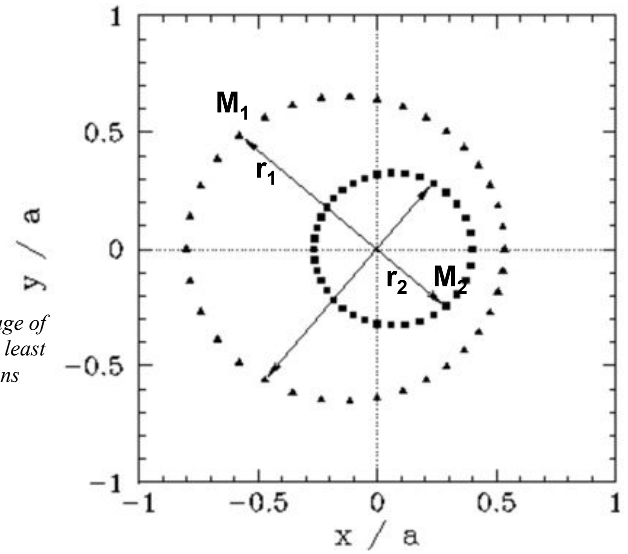
$$\vec{r} = \left(1 + \frac{M_2}{M_1}\right)\vec{r}_2 = \left(\frac{M_1 + M_2}{M_1}\right)\vec{r}_2$$

$$\vec{r}_2 = \left(\frac{M_1}{M_1 + M_2}\right)\vec{r} = \left(\frac{M_2 M_1}{M_1 + M_2}\right)\frac{\vec{r}}{M_2} = \frac{\mu\vec{r}}{M_2}$$

where μ is the "reduced mass" (smaller than M_1 or M_2).

SImilarly $\vec{r}_1 = -\frac{\mu\vec{r}}{M_1}$ in the frame with the c/m at the origin.

$$\frac{M_1}{M_2} = 0.5,\ e = 0.2$$



"a" = average of greatest and least separations

$$M_2\ddot{\boldsymbol{r}}_2 = -\frac{GM_1M_2}{r^3}\boldsymbol{r} \qquad (1)$$

$$M_1\ddot{\boldsymbol{r}}_1 = \frac{GM_1M_2}{r^3}\boldsymbol{r} \qquad (2)$$

In the circular case $\ddot{r}_2 = \frac{v_2^2}{r_2}$

$$r = r_1 + r_2$$

Multiply (1) by M_1 and (2) by M_2 and subtract (2) from (1)

$$M_1M_2\left(\ddot{\boldsymbol{r}}_2 - \ddot{\boldsymbol{r}}_1\right) = -\frac{GM_1M_2(M_1+M_2)}{r^3}\boldsymbol{r}$$

$$M_1M_2\ddot{\boldsymbol{r}} = -\frac{GM_1M_2(M_1+M_2)}{r^3}\boldsymbol{r}$$

$$\ddot{\boldsymbol{r}} = -\frac{G(M_1+M_2)}{r^3}\boldsymbol{r}$$

Thus we have transformed to an equivalent central force problem in which the mass that appears is the sum of the masses and the relevant vector is the distance between the two stars

$$\vec{f} = \frac{GM_1M_2}{r^3}\vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} = -\frac{G(M_1+M_2)}{r^3}\vec{r}$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$

$$\mu = \frac{M_1M_2}{M_1+M_2}$$

$$\vec{r}_1 = -\frac{\mu}{M_1}\vec{r} \qquad \vec{r}_2 = \frac{\mu}{M_2}\vec{r}$$

http://farside.ph.utexas.edu/teaching/336k/newton/node50.html

The solution to the central force problem, as before, is that the orbit is still an ellipse with distance between the two stars given by

$$r = |\vec{r}_2 - \vec{r}_1| = \frac{a(1-e^2)}{1+e\,Cos\,\theta}$$

e = 0 is the circular case in which $r = r_1 + r_2$

Since $(1-e^2) = (1+e)(1-e)$, the semi-major axis of this ellipse is half the sum of the maximum and minimum separations ($\theta = 0, \pi$)

$$\frac{1}{2}\left(\frac{a(1-e^2)}{1+e} + \frac{a(1-e^2)}{1-e}\right) = \frac{1}{2}\left(a(1-e) + a(1+e)\right) = a$$

$$\frac{1}{2}(r_{min} + r_{max}) = a$$

The orbital period is related to a by

$$P^2 = \frac{4\pi^2}{GM} a^3 \quad \text{where } M = (M_1 + M_2)$$

as we found for circular orbits. Also still $m_1 v_1 = m_2 v_2$ and $m_1 r_1 = m_2 r_2$ though v and r both vary with time now.

The quantities to be measured then are a) the period b) some measure of the semimajor axis and c) an observable ratio, r_1/r_2 or v_1/v_2, that can give the ratio of the masses.

$$r_1 = \frac{M_2}{M_1} r_2 \quad \text{and so} \quad \frac{dr_1}{dt} = \frac{M_2}{M_1}\frac{dr_2}{dt}$$

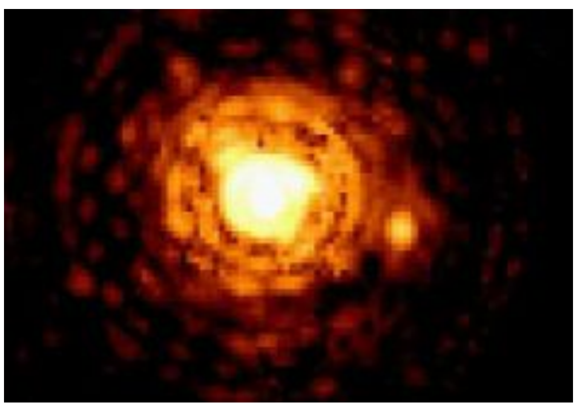
$$M_1 v_1 = M_2 v_2$$

at all times (and at all angles)

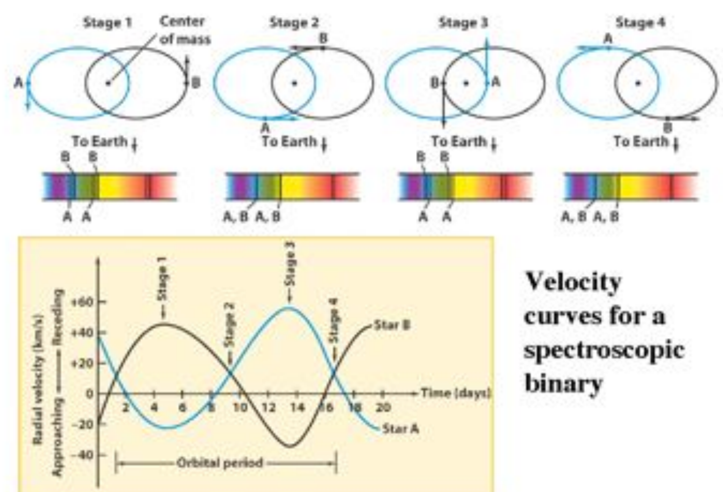
In the ideal case we can look at the orbit face on and measure the closest and most distant separation of the stars at Cos $\theta = 0, \pi$ as discussed on the previous page, $r_{min} + r_{max} = 2a$

Otherwise we have to measure r or v indirectly and also somehow account for inclination.

Spectroscopic binaries



Hubble Space Telescope photo of Gliese 623, two stars separated by 2 AU.



Velocity curves for a spectroscopic binary

$$t(\theta) = \frac{P}{2\pi}\left[2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\frac{\theta}{2}\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}\right]$$

- This tells you the time t it takes a body to travel in its orbit from periapsis ($\theta = 0$) to any value of θ along the trajectory.

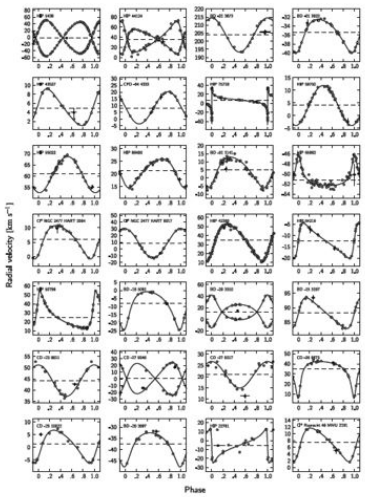


Fig. 2.6 Various velocity curves for several binary systems. Some are single-lined and some are double-lined. Figure taken from Matijevič, et al., *Astron. J.*, 141, 200 (2011). Reproduced by permission from the AAS

The general case can be solved but can be quite complicated.

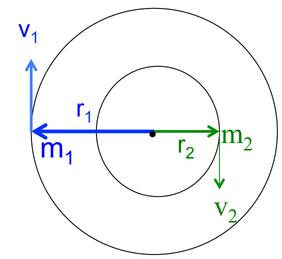
For this class we will restrict the examples to circular orbits.

One can extract "a" as well as (sometimes) information on inclination.

SPECTROSCOPIC BINARY MASSES

Assume circular orbit. Measure

- Period
- Velocity of each star
- v is constant for circular orbit



First get r_1 and r_2 from v_1 and v_2

$$r_i = \frac{v_i P}{2\pi}$$

$$P = \frac{2\pi r}{v}$$

Example:

$v_1 = 75 \text{ km s}^{-1}$ $\quad v_2 = 25 \text{ km s}^{-1}$

P= 17.5 days

Assume v is measured in plane of orbit, otherwise we just see the component of the velocity directed towards or away from us

$$a = r_1 + r_2 = \frac{P}{2\pi}(v_1 + v_2)$$

$$= \left(\frac{17\,days}{2\pi}\right)(100\ km\,s^{-1}) = 2.34\times10^{12}\ \text{cm}$$

$= 0.156$ AU

P = 17 d = 0.0465 *years*

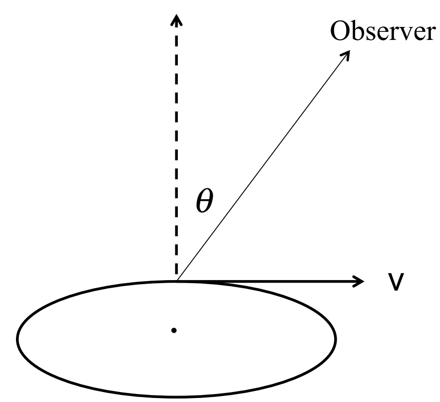
$$P^2(yr) = \left(\frac{M_1+M_2}{M_\odot}\right)^{-1}(0.0156)^3 \Rightarrow \left(\frac{M_1+M_2}{M_\odot}\right) = \frac{(0.156)^3}{(0.0465)^2}$$

$$= 1.76$$

Ratio of masses is $v_1 / v_2 = 3$

$4x = 1.76 \Rightarrow M_1 = 0.44 M_\odot \quad M_2 = 1.32 M_\odot$

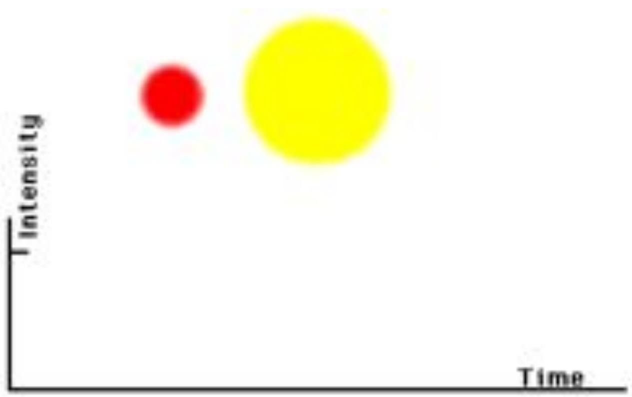
The larger mass has the slower speed.



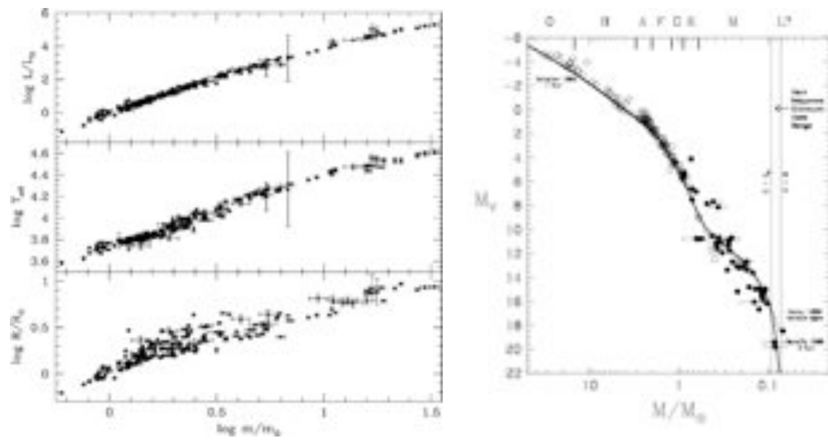
$\theta = 0$ face on, measure no velocity

$\theta = \frac{\pi}{2}$ edge on, measure full velocity

Eclipsing Binaries
(usually have circular orbits)



For an eclipsing binary you know you are viewing the system in the plane of the orbit. I.e., Sin i = 1

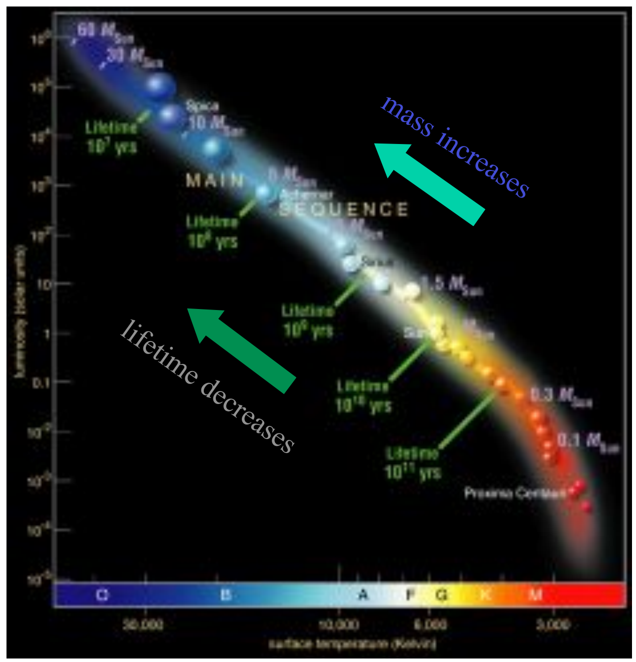


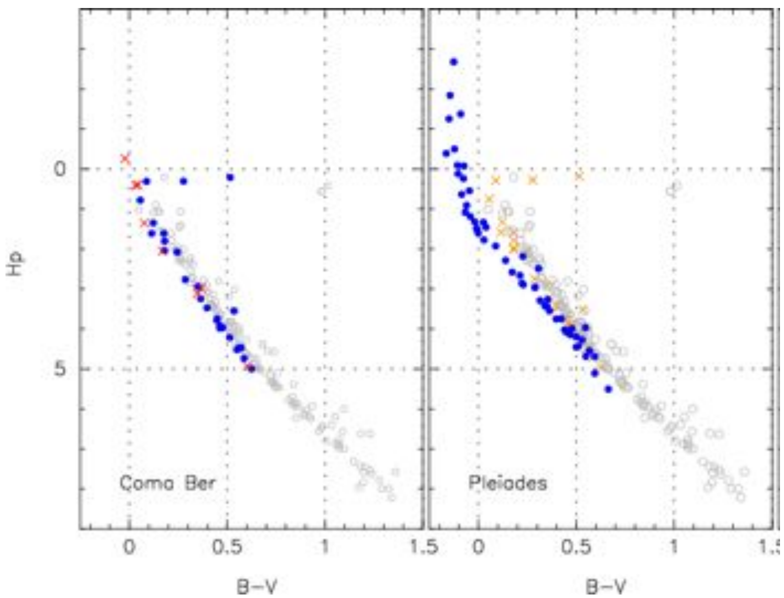
Two views of the mass-luminosity relation (Malkov 2007, left, and Henry 2004, right)

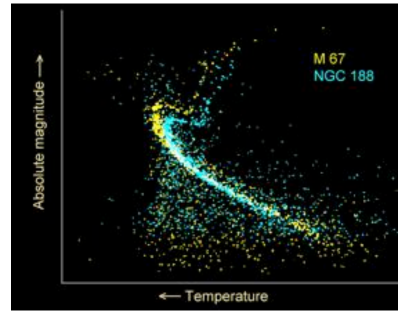
Recall

$$\tau_{MS} \approx \frac{f\,M\,q}{M^n} \qquad n \approx 3-4 \text{ for lower main sequence stars}$$

$$\propto M^{1-n} \quad \sim 10^{10}\ \text{yr}\left(\frac{M_\odot}{M}\right)^2$$

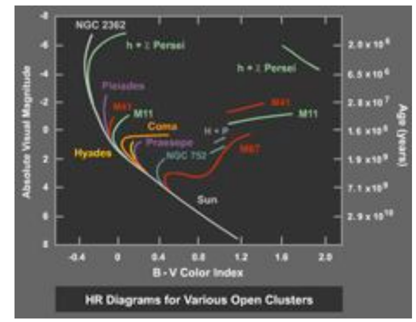


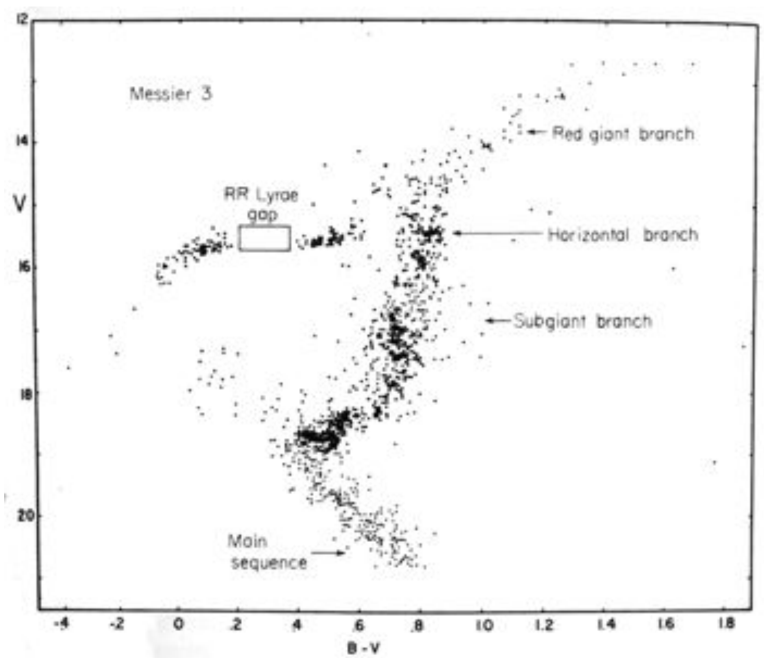


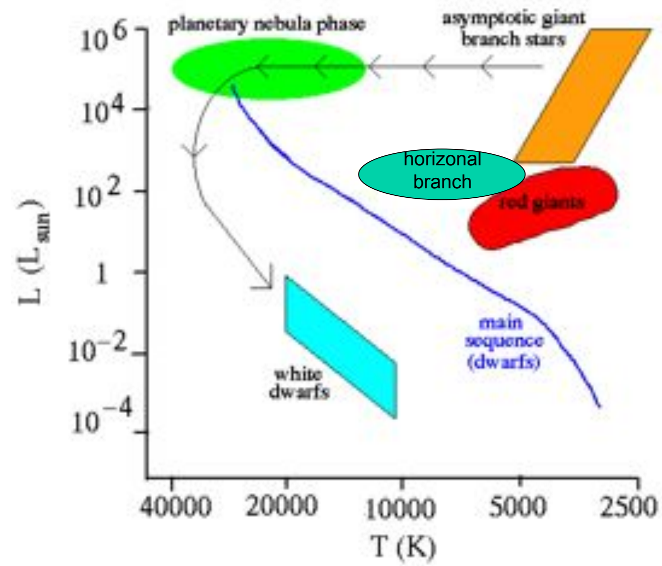


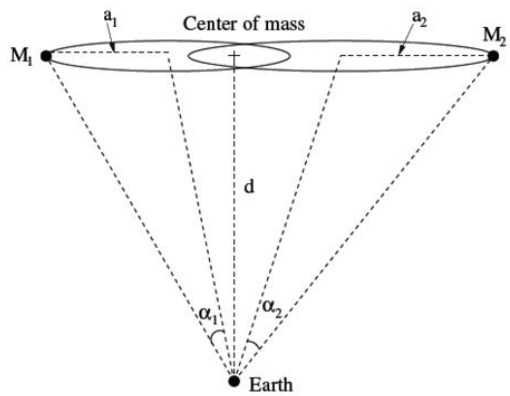
HR diagrams for open clusters M 67 and NGC 188

Schematic representation of HR diagrams and main sequence turn-offs observed for different open clusters









$$r_1 = \frac{\mu}{M_1} r = \frac{\mu}{M_1}\left(\frac{a(1-e^2)}{1+e\,Cos\theta}\right) \qquad r_2 = \frac{\mu}{M_2} r = \frac{\mu}{M_2}\left(\frac{a(1-e^2)}{1+e\,Cos\theta}\right)$$

$$a_1 = \frac{1}{2}\left(r_1(0) + r_1(\pi)\right) = \frac{\mu}{M_1} a \qquad a_2 = \frac{1}{2}\left(r_2(0) + r_2(\pi)\right) = \frac{\mu}{M_2} a$$

$$a_1 + a_2 = a$$