Lecture 4
Hydrostatics and Time Scales

Glatzmaier and Krumholz 3 and 4
Prialnik 2
Pols 2

Assumptions – most of the time

- *Spherical symmetry*
  Broken by e.g., convection, rotation, magnetic fields, explosion, instabilities
  Makes equations a lot easier. Also facilitates the use of Lagrangian (mass shell) coordinates
- *Homogeneous composition at birth*
- *Isolation frequently assumed*
- *Hydrostatic equilibrium*
  When not forming or exploding

Uniqueness

One of the basic tenets of stellar evolution is the *Russell-Vogt Theorem*, which states that the mass and chemical composition of a star, and in particular how the chemical composition varies within the star, uniquely determine its radius, luminosity, and internal structure, as well as its subsequent evolution.

A consequence of the theorem is that it is possible to uniquely describe all of the parameters for a star simply from its location in the *Hertzsprung-Russell Diagram*. There is no proof for the theorem, and in fact, it fails in some instances. For example if the star has rotation or if small changes in initial conditions cause large variations in outcome (chaos).

Hydrostatic Equilibrium

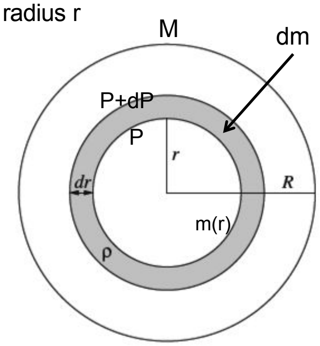
Consider the forces acting upon a spherical mass shell

$$dm = 4\pi r^2 dr \ \ \rho$$

The shell is attracted to the center of the star by a force *per unit area*

$$F_{grav} = \frac{-Gm(r)dm}{\left(4\pi r^2\right)r^2} = \frac{-Gm(r)\rho}{r^2} dr$$

where m(r) is the mass interior to the radius r
It is supported by the pressure gradient. The pressure on its bottom is smaller than on its top. dP is negative



$$F_P = P(r) \bullet area - P(r+dr) \bullet area$$

or

$$\frac{F_P}{area} = \frac{dP}{dr} dr$$

Hydrostatic Equilibrium

If these forces are unbalanced there will be an acceleration per unit area

$$\frac{F}{Area} = \frac{F_{grav} - F_P}{Area} = \frac{dm}{Area}\ddot{r} = \frac{4\pi r^2 \rho dr}{4\pi r^2}\ddot{r}$$

$$(\rho\, dr)\ddot{r} = -\left(\frac{Gm(r)\rho}{r^2} + \frac{dP}{dr}\right)dr$$

$$\ddot{r} = \frac{-Gm(r)}{r^2} - \frac{1}{\rho}\frac{dP}{dr}$$

$\ddot{r}$ will be non-zero in the case of stellar explosions or dynamical collpase, but in general it is very small in stars compared with the right hand side, so

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}$$

where $m(r)$ is the mass interior to the radius r. This is called the *equation of hydrostatic equilibrium*. It is the most basic of the stellar structure equations.

Examples of hydrostatic equilibrium:

1)Consider, for example, an isothermal atmosphere composed of ideal gas, P = nkT (T= constant; n is the number density). Let the atmosphere rest on the surface of a spherical mass M with radius R. R and M are both constant. h is the height above the surface:

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \qquad r = R + h \qquad h << r$$

$$dr = dh$$

$$\frac{N_A kT}{\mu}\frac{d\rho}{dh} = -\frac{GM\rho}{R^2} = -g\rho \qquad P = nkT \quad n = \frac{\rho N_A}{\mu} \quad T\,const.$$

$$\int_{\rho_0}^{\rho}\frac{d\rho}{\rho} = -\frac{\mu g}{N_A kT}\int_0^h dh \quad \Rightarrow \ln\rho - \ln\rho_0 = -\frac{\mu g h}{N_A kT}$$

$$\ln\left(\frac{\rho}{\rho_0}\right) = -\frac{h}{H} \qquad H = \frac{N_A kT}{\mu g} = \text{ density scale height}$$

$$\rho = \rho_0\, e^{-\frac{h}{H}} \qquad \text{equivalently } n = n_0 e^{-\frac{h}{H}} \qquad H = \frac{kT}{g}$$

Examples:

2) Or water in the ocean (density constant; incompressible)

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \qquad r = R - h \qquad h << r$$

$dr = -dh$ h = depth below surface

$$-\frac{dP}{dh} = -\frac{GM\rho}{R^2} = -g\rho$$

$$P = g\rho h \qquad \textit{e.g.}, \text{ at one km depth}$$

$$= (980)(1)(10^5) = 9.8 \times 10^7 \text{ dyne cm}^{-2}$$

$$= 98\, bars$$

Examples:

3) Stellar photospheres

As we will disuss next week, a beam of light going through a medium with opacity, κ ($cm^2\, g^{-1}$), suffers attenuation

$$\frac{dI}{dr} = -\kappa\rho I \quad \Rightarrow I = I_0\, e^{-\kappa\rho r} \equiv I_0 e^{-\tau}$$

where τ is the "optical depth". The photosphere is defined by place where, integrating inwards, $\tau \approx 1$ (a value of 2/3 is also sometimes used).

Consider hydrostatic equilibrium in a stellar atmosphere where M and R can assumed to be constant

$$\frac{dP}{dr} = -\frac{GM}{R^2}\rho = -g\rho$$

Multiplying by κ and dividing by ρ

$$\frac{\kappa}{\kappa\rho}\frac{dP}{dr} = \kappa\frac{dP}{d\tau} = -g \qquad \text{or} \qquad \frac{dP}{d\tau} = -\frac{g}{\kappa}$$

This was first noted by K. Schwartzschild in 1906

Solar photosphere

Integrating inwards and assuming g and κ are constant (the latter is not such a good approximation)

$$P_{photosphere} \approx \frac{g}{\kappa}$$

For the sun g = $\frac{GM_{\odot}}{R_{\odot}^2} = 2.7\times10^4$

So if $\kappa \sim 1$, $P_{photosphere} \approx 0.027$ *bars*.

κ is due to the H^- ion and varies rapidly with temperature but is of order unity (between 0.1 and 1 in the region of interest).

Actually P ~ 10^5 dyne $cm^{-2} = 0.1$ bar

Solar Photosphere as a Function of Depth

Depth (km)	% Light from this Depth	Temperature (K)	Pressure (bars)
0	99.5	4465	6.8×10^{-3}
100	97	4780	1.7×10^{-2}
200	89	5180	3.9×10^{-2}
250	80	5455	5.8×10^{-2}
300	64	5840	8.3×10^{-2}
350	37	6420	1.2×10^{-1}
375	18	6910	1.4×10^{-1}
400	4	7610	1.6×10^{-1}

Source: Fraknoi, Morrison, and Wolf, *Voyages through the Universe*

As we learned last time, the solar photosphere is largely neutral. H and He are not ionized much at all (~10^{-4} for H). The electrons come mostly from elements like Na, Ca, K, etc. that are very rare. Consequently the ion pressure overwhelmingly dominates at the solar photosphere (radiation pressure is negligible). This is not the case if one goes deeper in the star.

The electron pressure at the photosphere, though small, gives the electron density which enters into the Saha equation and determines the spectrum. This is in fact how it is determined.

Holweger and Muller
Solar Physics, 1974

Table 2-1: The Holweger-Müller Model Atmosphere[7]

Optical Depth (τ_{5000})	Temperature (°K)	Pressure (dynes cm^{-2})	Electron Pressure (dynes cm^{-2})	Density (g cm^{-3})	Opacity (κ_{5000})
5.0×10^{-5}	4306	5.20×10^{2}	5.14×10^{-2}	1.90×10^{-9}	0.0033
1.0×10^{-4}	4368	8.54×10^{2}	8.31×10^{-2}	3.07×10^{-9}	0.0048
3.2×10^{-4}	4475	1.75×10^{3}	1.68×10^{-1}	6.13×10^{-9}	0.0084
6.3×10^{-4}	4530	2.61×10^{3}	2.48×10^{-1}	9.04×10^{-9}	0.012
0.0013	4592	3.86×10^{3}	3.64×10^{-1}	1.32×10^{-8}	0.016
0.0040	4682	7.35×10^{3}	6.76×10^{-1}	2.47×10^{-8}	0.027
0.010	4782	1.23×10^{4}	1.12	4.03×10^{-8}	0.040
0.025	4917	2.04×10^{4}	1.92	6.52×10^{-8}	0.061
0.040	5005	2.63×10^{4}	2.54	8.26×10^{-8}	0.075
0.063	5113	3.39×10^{4}	3.42	1.04×10^{-7}	0.092
0.10	5236	4.37×10^{4}	4.68	1.31×10^{-7}	0.11
0.16	5357	5.61×10^{4}	6.43	1.64×10^{-7}	0.14
0.25	5527	7.16×10^{4}	9.38	2.03×10^{-7}	0.19
0.50	5963	9.88×10^{4}	22.7	2.60×10^{-7}	0.34
1.0	6533	1.25×10^{5}	73.3	3.00×10^{-7}	0.80
3.2	7672	1.59×e+005	551	3.24×10^{-7}	3.7
16	8700	2.00×e+005	2.37×10^{3}	3.57×10^{-7}	12

Lagrangian coordinates

The hydrostatic equilibrium equation

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}$$

can also be expressed with the mass as the independent variable by the substitution $dm = 4\pi r^2 \rho\, dr$

$$\frac{dP}{dr}\frac{dr}{dm} = -\frac{Gm(r)\rho}{r^2}\frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dm} = -\frac{Gm(r)}{4\pi r^4}$$

This is the "Lagrangian" form of the equation. We will find that all of our stellar structure equations can have either an Eulerian form or a Lagrangian form.

Merits of using Lagrangian coordinates

- Material interfaces are preserved if part or all of the star expands or contracts. Avoids "advection".
- Avoids artificial mixing of of composition and transport of energy
- In a stellar code can place the zones "where the action is", e.g., at high density in the center of the star
- Handles large expansion and contraction (e.g., to a red giant without regridding.

- In fact the merit of Lagrangian coordinates is so great for spherically symmetric problems that all 1D stellar evolution codes are writen in Lagrangian coordinates
- On the other hand almost all multi-D stellar codes are written in "Eulerian" coordinates.

This Lagrangian form of the hydrostatic equilibrium equation can be integrated to obtain the central pressure:

$$\int_{P_{cent}}^{P_{surf}} dP = P_{surf} - P_{cent} = -\int_0^M \frac{Gm(r)\,dm}{4\pi r^4}$$

and since $P_{surf} = 0$

$$P_{cent} = \int_0^M \frac{Gm(r)\,dm}{4\pi r^4}$$

To go further one would need a description of how m(r) actually varies with r. Soon we will attempt that . Some interesting limits can be obtained already though. For example, r is always less than R, the radius of the star, so $\frac{1}{r^4} > \frac{1}{R^4}$ and

$$P_{cent} > \frac{G}{4\pi R^4}\int_0^M m(r)\,dm = \frac{GM^2}{8\pi R^4}$$

A better but still very approximate result comes from assuming constant density (remember how bad this is for the sun!) $m(r) = 4/3\ \pi r^3 \bar{\rho} \sim 4/3\ \pi r^3 \rho_0$ (*off by 2 decades*!)

$$r^4 = \left(\frac{3m(r)}{4\pi\rho_0}\right)^{4/3} \Rightarrow \frac{m(r)}{r^4} = \left(\frac{4\pi\rho_0}{3m(r)}\right)^{4/3} m(r)$$

$$P_{cent} \approx \frac{-G}{4\pi}\left(\frac{4\pi\rho_0}{3}\right)^{4/3}\int_0^M \frac{dm}{m^{1/3}(r)} = \frac{G}{4\pi}\left(\frac{4\pi\rho_0}{3}\right)^{4/3}\frac{3M^{2/3}}{2}$$

$$but \quad \left(\frac{4\pi\rho_0}{3}\right)^{4/3} = \left(\frac{M}{R^3}\right)^{4/3} \quad so$$

$$P_{cent} = \frac{3GM^2}{8\pi R^4} \text{ or } \frac{GM\rho_0}{2R} \quad \text{since } \rho_0 \text{ is assumed } = \frac{3M}{4\pi R^3}$$

This is 3 times bigger but still too small

The sun's average density is 1.4 g cm^{-3} but its central density is about 160 g cm^{-3}.

Interestingly, most main sequence stars are supported primarily by ideal gas pressure (TBD), $P = \frac{\rho N_A kT}{\mu}$, so for the constant density (constant composition, ideal gas) case

$$\frac{\rho_0 N_A kT}{\mu} = \frac{GM\rho_0}{2R}$$

so the density cancels and one has

$$T_{cent} = \frac{GM\mu}{2N_A kR} \propto \frac{M}{R}$$

For a typical value of $\mu = 0.59$

$$T_{cent} = 6.8\times10^6 \left(\frac{M/M_\odot}{R/R_\odot}\right) K$$

This of course is still a lower bound. The actual present value for the sun is twice as large, but the calculation is really for a homogeneous (zero age) sun. A similar scaling (M/R) will be obtained for the *average* temperature using the Virial Theorem

Hydrodynamical Time Scale

We took $\ddot{r}=0$ to get hydrostatic equilibrium. It is also interesting to consider other limiting cases where $\ddot{r}$ is finite and the pressure gradient or gravity is negligible. The former case would correspond to gravitational collapse, e.g., a cloud collapsing to form a star or galaxy. The latter might characterize an explosion.

$$\ddot{r} = \frac{-Gm(r)}{r^2} - \frac{1}{\rho}\frac{dP}{dr}$$

If $\frac{dP}{dr} \to 0$, then $\ddot{r} = -Gm(r)/r^2 = g(r)$

which is the equation for free fall.

Free fall time scale

If g(r) is (unrealistically) taken to be a constant = g(R), the collapse time scale is given by

$\frac{1}{2} g\tau_{ff}^2 \approx R$ and for the whole star,

$$\frac{GM\tau_{ff}^2}{2R^2} \approx R \quad \text{but} \quad M = \frac{4\pi\bar{\rho}R^3}{3} \text{ (exact)}$$

$$\frac{4\pi G\bar{\rho}R\tau_{ff}^2}{6} \approx R$$

$$\tau_{ff}^2 = \frac{3}{2\pi G\bar{\rho}}$$

So

$$\tau_{ff} \approx \frac{1}{\sqrt{\frac{2\pi}{3}G\bar{\rho}}} = \frac{2680\ s}{\sqrt{\bar{\rho}}}$$

Clearly this is an overestimate since g(r) actually increases during the collapse.

Sometimes in the literature one sees instead

$$\tau_{ff} = \frac{R}{v_{esc}} = \frac{R}{\sqrt{2GM/R}} = \left(\frac{3R^3}{8\pi GR^3\bar{\rho}}\right)^{1/2}$$

$$= \left(\frac{3}{8\pi G\bar{\rho}}\right)^{1/2} = 1340/\sqrt{\bar{\rho}} \text{ sec}$$

And *for the density*, which changes logrithmically 3 times as fast as the density

$$\tau_{ff} = \frac{1}{3}\left(\frac{3}{8\pi G\bar{\rho}}\right)^{1/2} = 446/\sqrt{\bar{\rho}} \text{ sec}$$

In any case all go as $\bar{\rho}^{-1/2}$ and are about 1000 s for $\bar{\rho} = 1$

Explosion Time Scale

A related time scale is the explosive time scale.
Say g suddenly went to zero. An approximate expansion time scale for the resulting expansion would be

$$\ddot{r} = 4\pi r^2 \left(\frac{dP}{dm}\right) = \frac{1}{\rho}\left(\frac{dP}{dr}\right) \qquad R \sim 1/2\ \ddot{r}\,\tau_{\exp}^2$$

$$\frac{2R}{\tau_{\exp}^2} \sim \frac{P}{\rho R} \qquad \ddot{r} \sim \frac{2R}{\tau_{\exp}^2}$$

$$\tau_{\exp} \sim R\left(\frac{\rho}{P}\right)^{1/2} \approx R/c_{sound}$$

Usually the two terms in the "hydrostatic equilibrium" equation for $\ddot{r}$ are comparable, even in an explosion and $\tau_{ff} \approx \tau_{\exp}$ We shall just use $2680\ s/\sqrt{\rho}$ for both.

Examples

a) The sun $\bar{\rho} = 1.4$ g cm^{-3}

$$\tau_{HD} = 2680./\sqrt{1.4} = 1260\ s = 38\ \text{minutes}$$

If for some reason hydrostatic equilibrium were lost in the sun, this would be the time needed to restore it. If the pressure of the sun were abruptly increased by a substantial factor (~2), this would be the time for the sun to explode

b) A white dwarf $\rho \sim 10^6 - 10^8$ g cm^{-3}

$$\tau_{HD} = 2680./\sqrt{10^6} = 0.26 - 2.6\ \text{s}$$

~ Time scale for white dwarfs to vibrate (can't be pulsars)

~Time for the iron core of a massive star to collapse to a neutron star

Examples

c) A neutron star $\rho \sim 10^{15}$ g cm^{-3}

$$\tau_{HD} = 2680./\sqrt{10^{15}} = 0.084\ \text{ms}$$

~ Time for a neutron star to readjust its structure after core bounce

d) A red giant - solar mass, 10^{13} cm $\Rightarrow \rho \sim 5 \times 10^{-7}$ g cm^{-3}

$$\tau_{HD} = 2680./\sqrt{5 \times 10^{-7}} = 4 \times 10^5\ \text{s} \quad \approx 5\ \text{days}$$

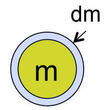
~Time for the shock to cross a supergiant star making a Type IIp supernova

~Typical Cepheid time scale

Generally the hydrodynamical, aka free fall, aka explosion time scale is the shortest of all the relevant time scales and stars, except in their earliest stages of formation and last explosive stages, are in tight hydrostatic equilibrium.

Some definitions

The total gravitational binding energy of a star of mass M, is



$$\Omega = -\int_0^M \frac{Gm}{r} dm = -\alpha \frac{GM^2}{R} \qquad \alpha \sim 1 \quad \text{(see next page)}$$

Similarly, the total internal energy is the integral over the star's mass of its internal energy per gram, u.

For an ideal gas $u = \frac{3}{2}\frac{N_A kT}{\mu} = \frac{3}{2}\frac{P}{\rho}$ (*TBD*)

$$U = \int_0^M u\, dm$$

$$= \frac{3}{2}\int_0^M \frac{P}{\rho} dm \quad \text{for an ideal gas}$$

Gravitational binding energy for a sphere of constant

If ρ = constant = ρ_0, $m(r) = \frac{4\pi r^3 \rho_0}{3}$ $dm = 4\pi r^2 \rho_0\, dr$

$$\Omega = -\int_0^M \frac{Gm(r)}{r} dm = -\int_0^R \frac{4\pi G r^2 \rho_0}{3} 4\pi r^2 \rho_0\, dr$$

$$= -\frac{16\pi^2 G \rho_0^2}{3}\int_0^R r^4\, dr = -\frac{16\pi^2 G \rho_0^2 R^5}{15}$$

$$= -\frac{3G}{5R}\left(\frac{4\pi R^3 \rho_0}{3}\right)^2 = -\frac{3GM^2}{5R}$$

i.e., α = 3/5 for the case of constant density. In the general case it will be larger.

Similarly

The total kinetic energy is

$$T = \int_0^M \frac{\dot{r}^2(m)}{2} dm \sim \frac{1}{2} M v^2 \text{ if the velocity is the same everywhere}$$

The total nuclear power is

$$L_{nuc} = \int_0^M \varepsilon(m)\, dm$$

where ε is the energy generation rate in erg g^{-1} s^{-1} from the nuclear reactions

ε is a function of T, ρ, and composition

and the luminosity of the star

$$L = \int_0^M F(m)\, dm$$

where F(m) is the energy flux entering each spherical shell whose inner boundary is at m

The Stellar Virial Theorem

The Lagrangian version of the hydrostatic equilibrium equation is

$$\frac{dP}{dm} = -\frac{Gm(r)}{4\pi r^4}$$

Multiply each side by the volume inside radius r, $V = \frac{4}{3}\pi r^3$,

and integrate

$$V\, dP(r) = \left(\frac{4}{3}\pi r^3\right)\left(-\frac{Gm(r)}{4\pi r^4}\right) dm = -\frac{1}{3}\frac{Gm(r)dm}{r}$$

$$\int_{P_{cent}}^{P(r)} V\, dP = -\frac{1}{3}\int_0^{m(r)} \frac{Gm'dm'}{r}$$

The integral on the right hand side is just the total gravitational potential energy in the star interior to r, $\Omega(r)$.

The left hand side can be integrated by parts out to radius r < R

$$\int_{P_{cent}}^{P(r)} V\, dP = PV\Big]_{cent}^{r} - \int_0^{V(r)} P\, dV = P(r)V(r) - \int_0^{V(r)} P\, dV$$

since V = 0 at the center

$$\int_{P_{cent}}^{P(r)} V\,dP = PV\Big]_{cent}^{r} - \int_0^{V(r)} P\,dV = P(r)V(r) - \int_0^{V(r)} P\,dV$$

Substituting $dV = 4\pi r^2 dr = \frac{4\pi r^2 \rho\, dr}{\rho} = \frac{dm}{\rho}$

$$\int_0^{V(r)} P\,dV = \int_0^{m(r)} \frac{P}{\rho} dm \qquad so$$

$$P(r)V(r) - \int_0^{m(r)} \frac{P}{\rho} dm = \frac{1}{3}\Omega(r)$$

This is true at any value of r, but pick r = R where $P(R) = 0$, then

$$-\int_0^{M} \frac{P}{\rho} dm = \frac{1}{3}\Omega_{tot}$$

(Prialnik 2.23) for any EOS

but for an ideal gas $\frac{P}{\rho} = \frac{2}{3}u$

The Virial Theorem for ideal gases

So for ideal gas $\int_0^{M} \frac{2}{3} u\, dm = -\frac{1}{3}\Omega$

The left hand side is 2/3 of the total internal energy of the star, hence

$$2U = -\Omega$$

(Prialnik 2.26) (Ω is defined < 0)

The internal energy is, in magnitude, 1/2 the binding energy of the star. We shall see later that similar though somewhat different expressions exist for radiation and relativistic degeneracy pressure. This is the Virial Theorem.

Note that we can also define a mass averaged temperature in the star

$$\bar{T} = \frac{1}{M}\int_0^{M} T(m)\,dm$$

The Virial temperature

For an ideal gas, $P = \frac{N_A k \rho T}{\mu}$ *(same as nkT but in terms of ρ)*

$$U = \frac{3}{2}\int_0^{M} \frac{P}{\rho} dm = \frac{3}{2}\int_0^{M} \frac{N_A k T}{\mu} dm \quad \text{but} \quad \int_0^{M} T\,dm = \bar{T} M$$

$$= \frac{3}{2}\frac{N_A k M \bar{T}}{\mu} = -\frac{1}{2}\Omega$$

$$\frac{3 M N_A k \bar{T}}{\mu} = \frac{\alpha G M^2}{R}$$

where e.g., $\alpha = 3/5$ for a sphere of constant density

$$\bar{T} = \frac{\alpha \mu G}{3 N_A k}\frac{M}{R}$$

Virial temperature

note again the inverse dependence of T on R

This is similar to the value obtained from hydrostatic equilibrium but $\frac{\alpha}{3}$ (about 0.2) instead of 0.5, so cooler, about 2.6×10^6 for the sun.

Note that the temperature of nearly homogeneous stars like the sun is set by their bulk properties, M and R. As the mass on the main sequence rises, R empirically rises only as about $M^{2/3}$ so the central temperature of massive stars rises gently with their mass. We shall see later that the density actually decreases.

Conservation of Energy
(aka First law of thermodynamics)

Consider the changes of energy that can be experienced by a small (spherical) mass element $\delta m = 4\pi r^2 \rho\, \delta r$. $\delta r \ll R$; $\delta m \ll M$. If the zone is sufficiently small, we can also think of δm as dm and it is customary to do so In stellar evolution codes. The zone's internal energy, u, can change as a consequence of:

- energy flowing in or out of its upper or lower boundary by radiation, conduction, or convection
- compression or expansion
- nuclear reactions generating or absorbing energy

The change in internal energy in a thin shell of mass, δm, during a small change of time, δt, is then

dm and δm used interchangably here

$$\delta u\, dm = \delta Q + \delta W$$

P is the pressure *in the zone*. ε is the energy generation in the zone.

F is the flux of energy at m or m+dm

where $\delta Q = \varepsilon\, \delta m \delta t + F(m)\delta t - F(m+\delta m)\, \delta t$ is the internal energy generation plus the net accumulation or loss of energy from fluxes at its upper and lower boundaries and

$$\delta W = -P\,\delta V = -P\delta\left(\frac{1}{\rho}\right)\delta m$$

is the energy lost to work because the zone expands and pushes on its boundaries or is compressed and gains energy. Note that $\delta V = 4\pi r^2\, \delta r$ and δm = constant = $4\pi r^2\, \rho\, \delta r$ so

$$\delta V = \delta m\ \delta\left(\frac{1}{\rho}\right)$$

$$\delta Q = \varepsilon\, \delta m \delta t + F(m)\delta t - F(m+\delta m)\, \delta t$$

F is the flux of energy at m or m+δm

and since

$$F(m+\delta m) = F(m) + \left(\frac{dF}{dm}\right)\delta m$$

$$\delta Q = \left(\varepsilon - \frac{dF}{dm}\right)\delta m\ \delta t$$

so $\quad \delta u \delta m + P\delta\left(\frac{1}{\rho}\right)\delta m = \left(\varepsilon - \frac{dF}{dm}\right)\delta m\ \delta t$

dividing by δm and δt and taking the limit as $\delta t \to 0$

$$\boxed{\frac{du}{dt} + P\frac{d}{dt}\left(\frac{1}{\rho}\right) = \varepsilon - \frac{dF}{dm}}$$

The energy conservation equation aka "the first law of thermodynamics

$$\frac{du}{dt} + P\frac{d}{dt}\left(\frac{1}{\rho}\right) = \varepsilon - \frac{\partial F}{\partial m}$$

$\frac{du}{dt}$ is the rate at which the internal energy in erg g^{-1} is changing, e.g., in a given zone δm of a (Lagrangian) stellar model

$P\frac{d}{dt}\left(\frac{1}{\rho}\right)$ is the PdV work being done on or by the zone as it contracts ($\rho \uparrow \Rightarrow PdV$ is negative) or expands ($\rho \downarrow \Rightarrow PdV$ is positive). Units are erg g^{-1} s^{-1}

ε is the nuclear energy generation rate minus neutrino losses in erg g^{-1} s^{-1}

$\frac{dF}{dm}dm$ is the difference in energy (per second) going out the top of the zone (by diffusion, conduction, or convection) minus the energy coming in at the bottom. It is positive if more energy is leaving than entering.

Thermal Equilibrium

If a steady state is reached where in a given zone the internal energy (u), density (ρ), and pressure (P) are not changing very much, then the left hand side of the 1st law becomes approximately zero and

$$\varepsilon = \frac{dF}{dm}$$

Energy flows into or out of the zone to accomodate what is released or absorbed by nuclear reactions (plus neutrinos).

If this condition exists through the entire star

$$\int_0^M \varepsilon \, dm = L_{nuc} = \int_0^M dF = L$$

then the star is said to be in thermal equilibrium. It is also possible for thermal equilibrium to exist in a subset of the star.

Examples:

- Main sequence stars are in thermal equilibrium
- Massive stars becoming red giants are not. More energy is being generated than is leaving the surface. The star's envelope is expanding
- White dwarfs are in a funny thermal equilibrium where ε_{nuc} = o and PdV is zero, but u is decreasing in order to provide L $\Rightarrow \dot{U} = L$
- Late stages of massive stellar evolution may also approach a funny equilibrium where neutrino losses balance nuclear energy generation, at least in those portions of the star where burning is going on at a rapid rate

Integrating the first law of thermodynamics

$$\frac{du}{dt} + P\frac{d}{dt}\left(\frac{1}{\rho}\right) = \varepsilon - \frac{dF}{dm}$$

Integrate over mass:

$$\int_0^M \frac{du}{dt} dm + \int_0^M P\frac{d}{dt}\left(\frac{1}{\rho}\right) dm = \int_0^M \varepsilon \, dm - F(M) + \cancel{F(0)}$$

$$= L_{nuc} - L$$

Since m does not depend on t the leftmost term can be rewritten

$$\int_0^M \frac{du}{dt} dm = \frac{d}{dt}\int_0^M u \, dm = \dot{U}$$

The second term takes some work, one can rewrite $1/\rho$ (since dm = dV / ρ)

$$\frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{d}{dt}\left(\frac{dV}{dm}\right) = \frac{d}{dm}\left(\frac{dV}{dt}\right)$$

Continuing:

$$\frac{d}{dt}\left(\frac{1}{\rho}\right) = \frac{d}{dt}\left(\frac{dV}{dm}\right) = \frac{d}{dm}\left(\frac{dV}{dt}\right)$$

$$= \frac{d}{dm}\left(4\pi r^2 \frac{dr}{dt}\right)$$

Integrate by parts:

(2nd term) $$\int_0^M P\frac{d}{dt}\left(\frac{1}{\rho}\right) dm = \int_0^M P d\left(4\pi r^2 \frac{dr}{dt}\right)$$

$$= 4\pi r^2 P \frac{dr}{dt}\Big]_0^M - \int_0^M 4\pi r^2 \frac{dr}{dt}\frac{dP}{dm} dm$$

The first term is zero at r = 0 and M (where P = 0) so

$$\int_0^M P\frac{d}{dt}\left(\frac{1}{\rho}\right) dm = -\int_0^M 4\pi r^2 \frac{dr}{dt}\frac{dP}{dm} dm$$

Continuing:

$$\int_0^M P\frac{d}{dt}\left(\frac{1}{\rho}\right)dm = -\int_0^M 4\pi r^2 \frac{dr}{dt}\frac{dP}{dm}dm$$

But $\frac{dP}{dm} = \frac{-Gm}{4\pi r^4} - \frac{\ddot{r}}{4\pi r^2}$

$$\int_0^M P\frac{d}{dt}\left(\frac{1}{\rho}\right)dm = \int_0^M \frac{Gm}{r^2}\frac{dr}{dt}dm + \int_0^M \ddot{r}\frac{dr}{dt}dm$$

and $\ddot{r}\frac{dr}{dt} = \frac{d}{dt}\left(\frac{\dot{r}^2}{2}\right)$ $\qquad \frac{Gm}{r^2}\frac{dr}{dt} = -\frac{d}{dt}\left(\frac{Gm}{r}\right)$

So

$$\int_0^M P\frac{d}{dt}\left(\frac{1}{\rho}\right)dm = -\frac{d}{dt}\int_0^M \frac{Gm}{r}dm + \frac{1}{2}\frac{d}{dt}\int_0^M \dot{r}^2\,dm = \dot{\Omega} + \dot{T}$$

$$\frac{1}{2}\dot{\Omega} = L_{nuc} - L$$ remember Ω is negative

- If $L_{nuc} = L$ then the star is in a state of balanced power. Over long period of time it neither expands or contracts. It is in thermal and dynamic equilibrium
- If $L_{nuc} > L$, more energy is being generated by nuclear reactions than is being radiated. Ω becomes less negative. The star expands to absorb the excess
- If $L_{nuc} < L$, the star is radiating more energy than nuclear reactions are producing. If possible, the star makes up the deficiency by contracting to a more tightly bound state. Ω becomes more negative

So all together

$$\dot{U} + \dot{\Omega} + \dot{T} = L_{nuc} - L$$

which expresses succinctly (and obviously) the conservation of energy for the star. Power generated or lost on the right balances the change in internal, gravitational, and kinetic energies on the left.

Suppose the star is static so that $\dot{T}=0$, and the Virial Theorem also applies so $\dot{U} = -1/2\dot{\Omega}$ (for an ideal gas)

Then

$$\frac{1}{2}\dot{\Omega} = L_{nuc} - L$$

This is interesting.

Also interesting is the solution for $\dot{U}$ in the same situation. Then

$$\dot{U} = -\frac{1}{2}\dot{\Omega} = L - L_{nuc}$$

In the absence of nuclear energy input, the star increases its internal energy (heat) as it radiates. The more a star, supported by ideal gas pressure, radiates, the hotter it gets. The heat capacity is defined by $C(T) = \Delta q/\Delta T$ where Δq is the energy flowing into or out of the matter. Here ΔT is positive when Δq is negative. We say that stars have a "negative heat capacity". The origin of the energy is gravity.

The Kelvin Helmholtz time scale

The time scale for adjusting to an imbalance in energy generation (historically L_{nuc} = 0) is the Kelvin Helmholtz timescale

$$\tau_{KH} = \frac{\Omega}{|\dot{\Omega}|} \approx \frac{\Omega}{2L}$$

$$\Omega = -\int_0^M \frac{Gm}{r} dm = -\int_0^R \frac{Gm(r)}{r} 4\pi r^2 \rho(r)\, dr$$

$$= -\frac{\alpha G M^2}{R} \quad \text{where } \alpha \text{ depends on } \rho(r)$$

I

This is (roughly) the time the sun could shine without nuclear reactions *at its present radius and luminosity*.

In fact, if the radius is allowed to change gravity could power the sun or any star much longer. Even keeping the luminosity the same the sun could, in principle, continue to contract until it became a white dwarf. Taking a white dwarf radius of ~5000 km, τ_{KH} could be increased to 1.3 Gy, but then the sun wouldnot look like it does (with the same luminosity it would radiate in the ultraviolet).

Taking this to the absurd limit (not allowed by quantum mechanics for a 1 $M_\odot$ star), Ω~Mc^2, for a black hole τ_{KH} ~10^{13} years. Gravity, natures weakest force, can, in the end provide more power than any other source including nuclear reactions.

Kelvin Helmholtz time scale for the sun

For the sun the constant density expression gives

$$\tau_{KH}(\odot) = \frac{3GM_\odot^2}{10 R_\odot L_\odot} = 2.95 \times 10^{14}\, s = 9.3 \text{ My}$$

but the actual sun is far from constant density and the actual value is closer to 30 My. We will obtain a better value when we study polytropes later.

One could in principle explain the modern sun without recourse to nuclear reactions by allowing it to become extremely centrally condensed (e.g., a black hole at the center). But this would be quite inconsistent with

- Stellar physics and known equations of state
- The solar neutrino experiments

Other places the Kelvin Helmholtz time scale enters:

- Star formation – it is the time required for a protostar to settle down on the main sequence and ignite nuclear burning
- The time scale between major nuclear burning stages where degeneracy does not enter in, e.g., between helium depletion and carbon ignition in a massive star
- The time a proto-neutron star requires to release its binding energy as neutrinos. During this time it can power a supernova

As the star contracts, so long as it remains approximately an ideal gas, the Virial Theorem (or hydrostatic equilibrium) implies:

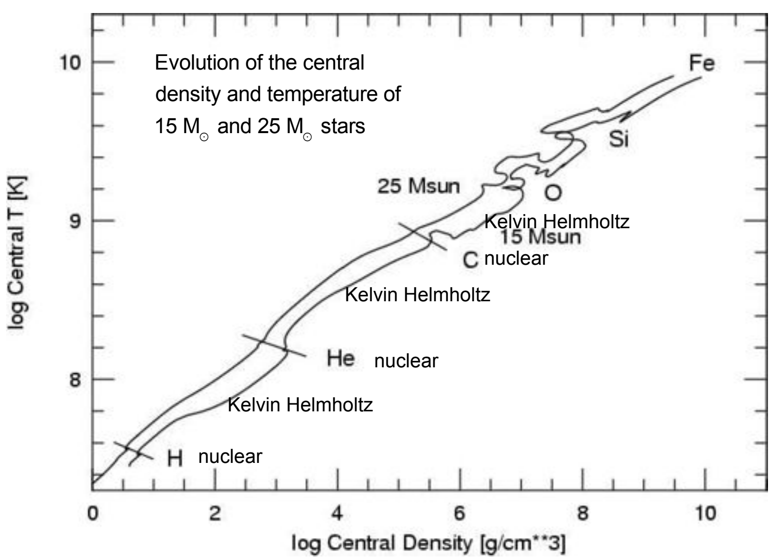
$$U=-\frac{\Omega}{2} \Rightarrow \frac{3}{2\mu} N_A k\bar{T} M = \alpha \frac{GM^2}{2R}; \quad or \quad P_c \propto \frac{GM\rho}{R}$$

$$\bar{T} \propto \frac{M}{R} \quad \text{but} \quad \rho \propto \frac{M}{R^3}$$

So $\bar{T} \propto M^{2/3} \rho^{1/3}$

As the the star contracts, the temperature rises as the cube root of the density and, for a given density, is less in stars of smaller mass.

We will make these arguments more quantitative later



The nuclear time scale

As we shall see later nuclear fusion – up to the element iron is capable of releasing large energies per gram of fuel (though well short of mc^2).

Roughly the energy release during each phase is the fraction of the star that burns times

Fuel	**fraction mc^2**
Hydrogen	7×10^{-3}
Helium	7×10^{-4}
Carbon	1×10^{-4}
Oxygen	3×10^{-4}
Silicon	1×10^{-4}

The nuclear time scale

One can also define a nuclear time scale

$$\tau_{nuc} = \frac{M q_{nuc}}{L}$$

but everywhere except the main sequence, one must be quite cautious as to what to use for L and M because more than one fuel may be burning at a give time and neutrinos can carry away appreciable energy. Also only a fraction of the star burns, just that part that is hot enough. For the sun if 10% burns

$$\tau_{nuc} \approx \frac{(0.1)(0.007\,c^2)(M_\odot)}{L_\odot}$$

$$= 10\,Gy$$

which is fortuitously quite close to correct.

There are several things to notice here.

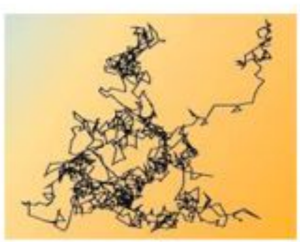
First the energy yield is a small fraction of the rest mass energy. It just comes from shuffling around the same neutrons and protons inside different nuclei. The number of neutrons plus protons is being conserved. No mass is being annihilated.

Less energy is produced once helium has been made. The binding energies of neutrons and protons in helium, carbon, oxygen and silicon are similar, all about 8 MeV/nucleon. More energy is released by oxygen burning simply because the oxygen abundance is about 5 times the carbon abundance.

We shall see later that the nuclear time scale can be greatly accelerated by neutrino losses

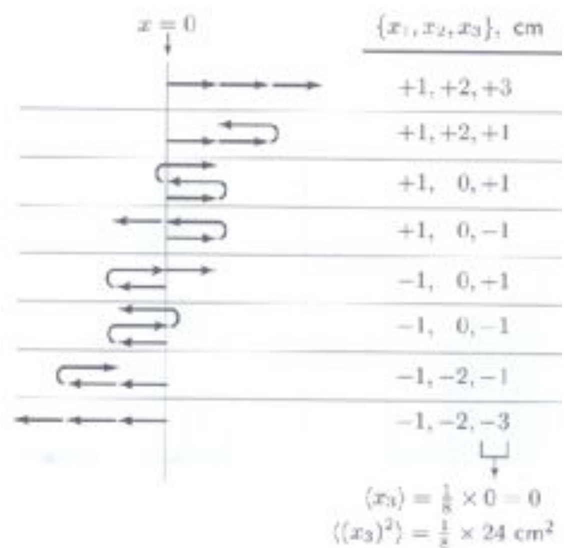
Radiation diffusion time scale

ng



In a random walk, how far are you from the origin in n steps, and how long did it take to get there?

Random walk:



$= 3\ \text{cm}^2$

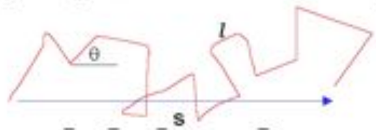
$\langle x_3 \rangle = \sqrt{3}$ cm

The thermal time scale

As it traverses the star, a photon is scattered by electrons and ions

We can describe this by the *scattering mean free path* – the average distance that a photon can travel in before changing direction.

Let the mean free path be $\bar{l}$. Total vector distance travelled by photon is $\bar{s}$. At each scattering its angle θ, to the direction of $\bar{s}$, changes



$$\bar{s} = \bar{l}_1 + \bar{l}_2 + \bar{l}_3 + ... + \bar{l}_N$$ The average of s itself is zero

so $\vec{s} \bullet \vec{s} = s^2 = \bar{l}_1.\bar{l}_1 + \bar{l}_2.\bar{l}_2 + \bar{l}_3.\bar{l}_3 .... + \bar{l}_N.\bar{l}_N + 2(\bar{l}_1.\bar{l}_2 + \bar{l}_1.\bar{l}_3 ...)$

$$= Nl^2 \qquad (77)$$

where there are N scatterings, and for random θ, $\bar{l}_i.\bar{l}_j$ vanishes on average.

Let the time between collisions be τ. Then

$$\tau = \frac{|\bar{l}|}{c} = \frac{l}{c}$$

The total time, t_{cross}, taken for a photon to cross distance s is then equal to the number of collisions in distance s, multiplied by the time between collisions.

$$t_{cross} = N\tau = \frac{s^2}{l^2}\frac{l}{c} = \frac{s^2}{lc}$$

An average value of the mean free path in the sun is about 1mm. Therefore, to escape the Sun it takes a photon a time

$$\tau_{dif}(\odot) \approx \frac{R_\odot^2}{\ell c} = \left(\frac{(6.9\times10^{10})^2}{(0.1)(3\times10^{10}}\right) = 1.6\times10^{12} \text{ sec}$$

or about 50,000 years.

A more accurate value is 170,000 years

http://adsabs.harvard.edu/full/1992ApJ...401..759M

Ordering of Time scales

To summarize we have discussed 4 time scales that characterize stellar evolution.

- Hydrodynamical time – $446/\rho^{1/2}$ to $2680/\rho^{1/2}$ sec – the time to adjust to and maintain hydrostatic equilibrium. Also the free fall or explosion time scale
- Thermal – R^2/lc – the time to establish thermal equilibrium if diffusion dominates (it may not)
- Kelvin-Helmholtz – $\alpha GM^2/(2RL)$ – the time to adjust structure when the luminosity changes.
- Nuclear – $q_{nuc}M/L$ – the time required to fuse a given fuel to the next heavier one

In general, and especially on the main sequence

$$\tau_{HD} << \tau_{therm} < \tau_{KH} < \tau_{nuc}$$

There are however interesting places where this ordering breaks down:

$\tau_{therm} \sim \tau_{KH} \sim \tau_{nuc}$ during the late stages of massive star evolution

$\tau_{HD} \sim \tau_{nuc}$ for SN Ia; for explosive nucleosynthesis in SN II

There are also other relevant time scales for e.g., rotational mixing, angular momentum transport, convective mixing, etc. Usually a phenomenon can be characterized and its importance judged by examining the relevant time scales.