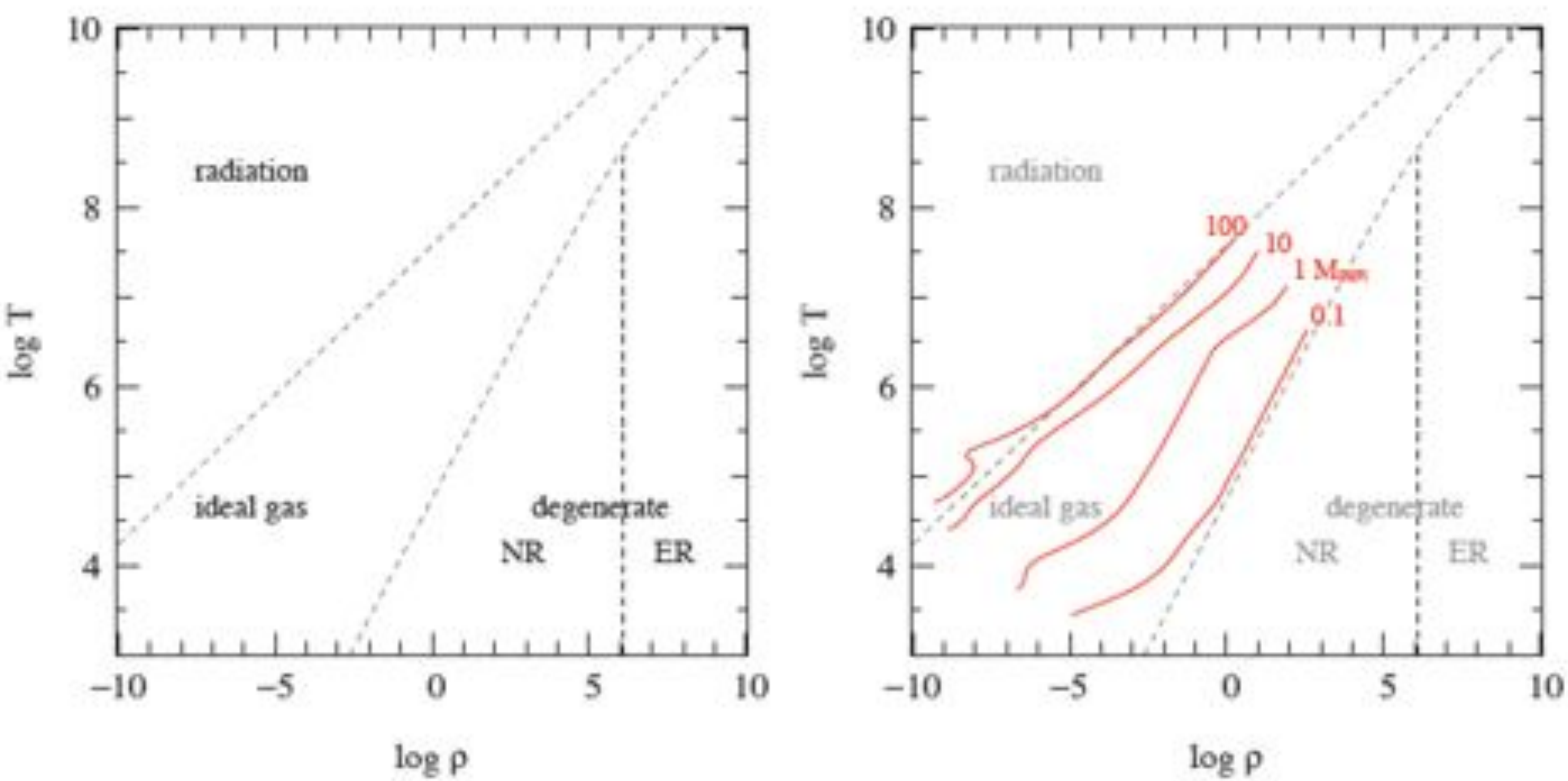
Lecture 6

Radiation Transport

Prialnik Chapter 5
Krumholtz and Glatzmaier 6
Pols Chapter 5

Summary figure EOS regimes – Pols Fig 3.4

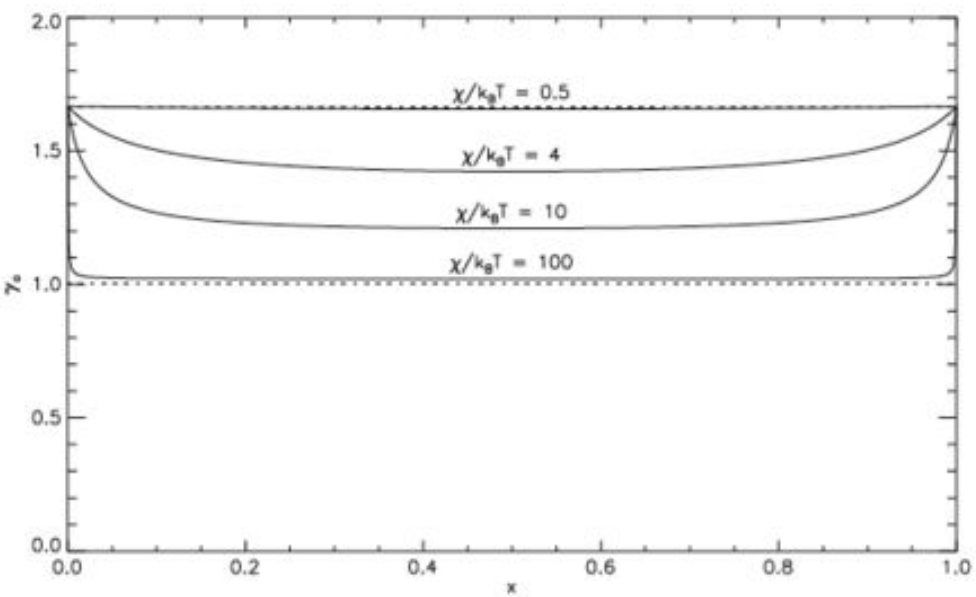


See Pols p 31 for discussion

For an adiabatic expansion or compression regardless of equation of state (ideal, degenerate, etc.) the first law of thermodynamics gives

$$P = K\rho^{\gamma_{ad}}$$

The value of γ_{ad} varies with the EOS.



For an ideal gas, fully ionized,

$$u = \phi\frac{P}{\rho} = \frac{3}{2}\frac{P}{\rho}$$

$$\gamma_{ad} = \frac{\phi+1}{\phi} = 5/3$$

For a relativistic gas $\gamma_{ad} = 4/3$. In regions of partial ionization (or pair production or photo disintegration) γ_{ad} can be suppressed.

Very roughly (do not use this for anything quantitative):

$$\frac{dP}{dm} = -\frac{Gm(r)}{4\pi r^4} \quad \Rightarrow P \propto m^2 r^{-4}$$

$$\rho \propto m / r^3 \quad r^{-4} \propto \rho^{4/3} m^{-4/3}$$

so for hydrostatic equilibrium, crudely, P $\propto m^{2/3}\rho^{4/3}$.

If the global density changes due to expansion or contraction, leading to new pressure and density, P' and ρ', hydrostatic equilibrum suggests

$$\left(\frac{P'}{P}\right) = \left(\frac{\rho'}{\rho}\right)^{4/3}$$

If the pressure increases more than this,

i.e., $\left(\frac{P'}{P}\right) > \left(\frac{\rho'}{\rho}\right)^{4/3}$ there will be a restoring force that will lead to expansion. If it is less, the contraction will continue and perhaps accelerate.

Now consider an adiabatic compression:

$$P = K\rho^{\gamma}$$

So $$\left(\frac{P'}{P}\right) = \left(\frac{\rho'}{\rho}\right)^{\gamma} \qquad \rho' > \rho$$

If, for a given mass, $\left(\frac{\rho'}{\rho}\right)^{\gamma} > \left(\frac{\rho'}{\rho}\right)^{4/3}$ one has stability,

hydrostatic equilibrium is satisfied. If not, things are unstable.

Thus if $\gamma > \frac{4}{3}$ the star is stable and if $\gamma < \frac{4}{3}$ it is not.

This is a global analysis and doesn't necessarily apply to small regions of the star but illustrates the importance of $\gamma = 4/3$. We will return to this later.

Radiation Transport

So far we have descriptions of hydrostatic equilibrium and the equation of state. Still missing:

- How the temperature must vary in order to transport the energy that is generated (by both radiative diffusion and convection).
- A way of calculating the opacity of stellar matter
- An explicit description of the relevant nuclear and particle physics
- A technique for solving the resulting differential equations

Transport by radiative diffusion in a star

Because of the high density and long time scales, hydrodynamic equilibrium is an excellent description. Locally, and frequently globally, thermal equilibrium is also a valid assumption.

The radiation field in the optically thick stellar interior is thus to high accuracy, given by the Planck function.

Spherical symmetry and isotropy can be assumed (except for rapid rotators). In “radiative” (as opposed to “convective”) regions, the heat is transported because a small excess of diffusion proceeds in one direction owing to the decreasing temperature in that direction, but to good accuracy heat is diffusing both ways nearly equally.

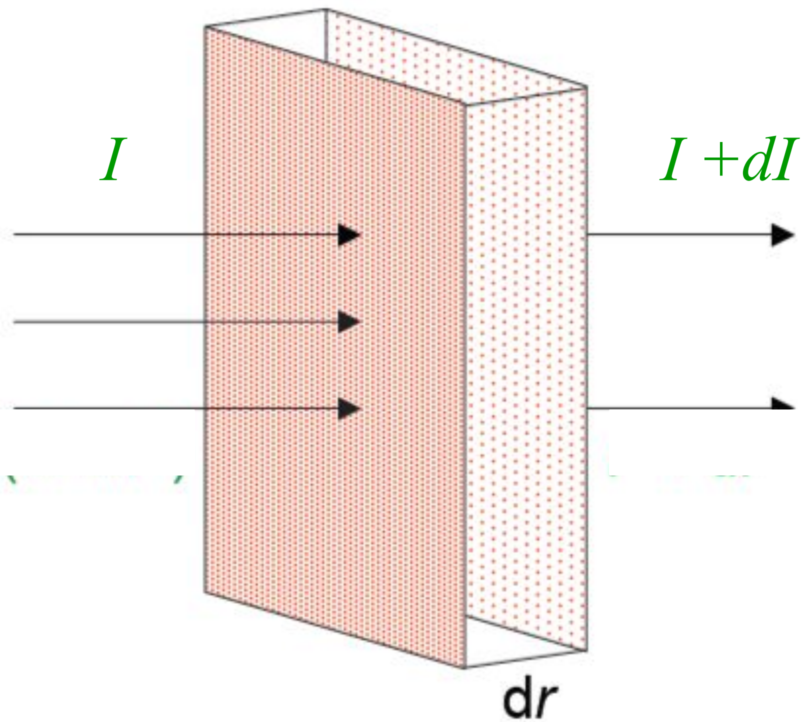
From our prior discussions, for blackbody radiation, we have the Planck function which tells us everything about the equilibrium radiation

$$B_\nu(T,\Omega)\,d\nu = \frac{2h\nu^3}{c^2}\frac{d\nu}{e^{h\nu/(k_B T)}-1} \text{ erg cm}^{-2}\text{ s}^{-1}\text{ Hz}^{-1}\text{ Ster}^{-1}$$

The total flux leaving a cm^2 comes from integrating this over solid angle and frequency

$$\text{F} = \int\int B_\nu\, d\nu \;\, d\Omega = \sigma T^4 \;\text{ erg cm}^{-2}\text{ s}^{-1}$$

It will also be useful to define a monochromatic intensity, $I_\nu(\Omega)$ with units like B_ν (erg cm^{-2} s^{-1} Hz^{-1} Ster^{-1}), but taken to describe the energy flux of a beam of radiation with a single frequency moving into a given solid angle.



As radiation passes through matter it can be absorbed or scattered.

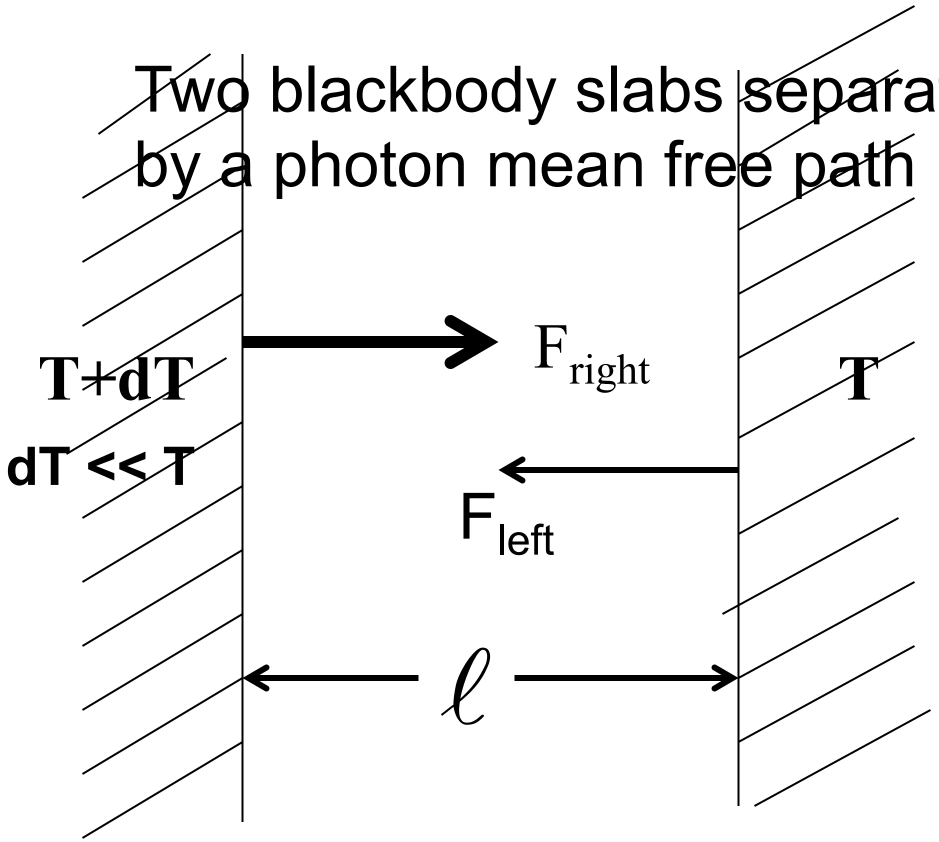
One way of quantifying this is by a typical "cross section" σ for an absorber or scatterer.

The mean free path of a given photon is $1/(n\sigma)$ where n is the number of absorbers/scatterers per cm^{-3}.

A related quantity is the "opacity" κ defined by $dI = - I\kappa\rho\, dr$ so that $I = I_0 \exp(-\kappa\rho r)$. The mean free path, ℓ, is thus $1/\kappa\rho$ since the radative flux declines by one e-fold in that distance. Thus $n\sigma = \kappa\rho$. In fact, σ and κ are both functions of the composition, temperature, density and frequency.

Simplified derivation of temperature gradient equation:

Two blackbody slabs separated by a photon mean free path



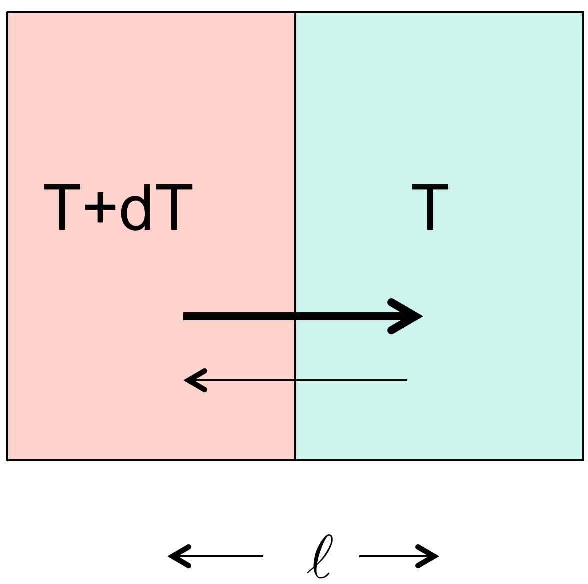
Clayton 3-2 page 171

$$\ell = \frac{1}{\kappa\rho}$$

net heat flux $F = F_{right} - F_{left} = \sigma\left[\left(T + dT\right)^4 - T^4\right] = \sigma T^4\left[\left(1 + \frac{dT}{T}\right)^4 - 1\right]$

$$\approx \sigma T^4\left[\left(1 + 4\frac{dT}{T}\right) - 1\right] = 4\sigma T^3\, dT$$

It is perhaps better to think of the gas as continuous, but with a small temperature jump. Radiation emitted from the hotter region typically goes a distance *l* before being absorbed. Similarly radiation moves from the cooler region the same distance back into the hotter region



One obtains the same equations as on the previous page

The temperature difference between two layers is given by the temperature gradient times the mean free path of a photon,

$\ell = 1/(\kappa\rho)$ so in 1D $\quad dT \approx \frac{dT}{dx}\ell$ and

$$F = 4\sigma T^3 dT = 4\sigma T^3 \ell \frac{dT}{dx} = \frac{4\sigma T^3}{\kappa\rho}\frac{dT}{dx} = \frac{acT^3}{\kappa\rho}\frac{dT}{dx}$$

since $\sigma \equiv \frac{ac}{4}$

$a = 7.566 \times 10^{-15}$ erg cm^{-3} K^{-4}

$= 7.566 \times 10^{-15}$ dyne cm^{-2} K^{-4} *Pols p.iv*

$\sigma = 5.670 \times 10^{-5}$ erg cm^{-2} K^{-4}

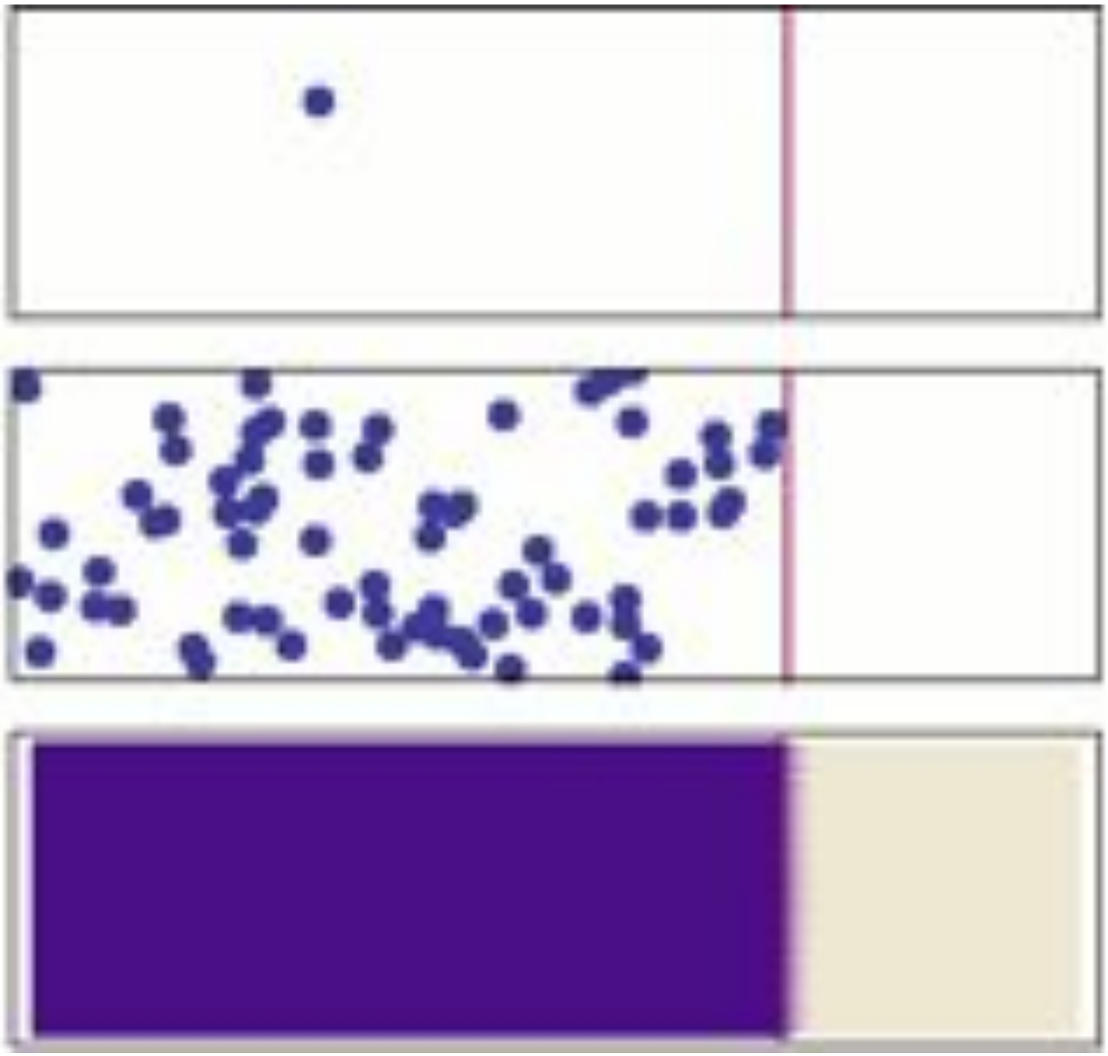
Multiplying F by the area, $4\pi r^2$, gives the luminosity

$$L(r) = \frac{4\pi r^2 acT^3}{\kappa\rho}\frac{dT}{dr} \quad \text{or}$$

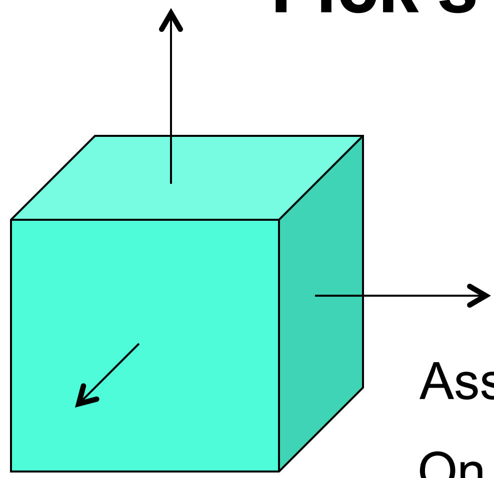
$$\frac{dT}{dr} \approx \frac{\kappa\rho}{4\pi r^2 acT^3} L(r)$$

The accurate result is ¾ times this

Fick's Law $J = -D \nabla n$



Fick's Law



In general Flux = nv

Assume an isotropic distribution, $\bar{v}_x = \bar{v}_y = \bar{v}_z$

On the average 1/3 of the particles at a boundary will be moving in the z direction and half of those up and half down

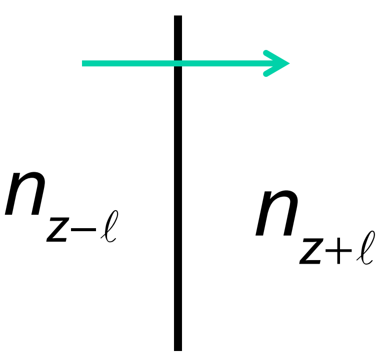
$$F_z \approx \frac{1}{6}\bar{v}_z\left[\bar{n}_{z-\ell} - \bar{n}_{z+\ell}\right] = \frac{1}{6}\bar{v}_z\left[n + \frac{dn}{dz}\ell - \left(n - \frac{dn}{dz}\ell\right)\right]$$

$$= \frac{1}{3}\bar{v}_z\frac{dn}{dz}\ell$$

$$= D\frac{dn}{dz}$$

where D is the "diffusion coefficient"

defined as $\frac{1}{3}\bar{v}\ell_{mfp} = \frac{1}{3}\bar{v}\frac{1}{\kappa\rho}$



$n_{z-\ell} > n_{z+\ell}$

Pols Chapter 5, Kippenhahn and Weigert

According to Fick's law the flux of something equals a constant times the rate at which something changes in space (the gradient)

$J = -D\nabla n$ with $D = \frac{1}{3}\bar{v}\ell$ the "diffusion coefficient"

and n, some quantity (like number density) that is diffusing. J is the flux of n across a surface. Since the only net gradiant in a spherical star is in the r direction $\nabla \rightarrow \frac{\partial}{\partial r}$

The unts of D are $\mathrm{cm}^2\ s^{-1}$ and so the units of J are $\mathrm{cm}^{-2}\ \mathrm{s}^{-1}$, the flux of particles per unit area per unit time.

In particular, an *energy* gradient ∇U in the internal energy per cm^3 gives rise to an energy flux (erg cm^{-2} s^{-1})

$$\text{Energy Flux} = F = -D\,\frac{\partial U}{\partial r}$$

*S*ince $\dfrac{\partial U}{\partial r} = \left(\dfrac{\partial U}{\partial T}\right)_V \dfrac{\partial T}{\partial r} = C_V \dfrac{\partial T}{\partial r}$ with C_V the "heat capacity" per unit volume (at constant volume)

$$F = -\mathrm{K}\frac{\partial T}{\partial r} \quad \text{with } \mathrm{K} \text{ the conductivity} = D\,C_V = \frac{1}{3}\bar{v}\ell C_V$$

D has units cm^2 s^{-1}; $C_V = c_V\rho$ has units erg cm^{-3} K^{-1}, so K, the conductivity, has units erg cm^{-1} s^{-1} K^{-1}.

The energy we are interested in here is blackbody radiation.

For photons, $\bar{v} = c$ and $U = aT^4$, so $\left(\frac{\partial U}{\partial T}\right) = C_V = 4aT^3$

The mean free path comes from the definition of opacity

$$\frac{dI_\nu}{ds} = -\kappa_\nu \rho \, I_\nu \Rightarrow I_\nu = I_0 \, e^{-s/\ell_{mfp}}$$

ℓ_{mfp} is the distance over which the intensity of a beam of radiation decreases by e, hence $\ell = \frac{1}{\kappa_\nu \rho}$. Ignore the frequency dependence for the moment.

In terms of the opacity, the conductivity is then

$$K_{rad} = \frac{1}{3}\bar{v}\,\ell\, C_V = \frac{4}{3}\frac{acT^3}{\kappa\rho}$$

and the radiative energy flux is

$$F_{rad} = -K_{rad}\frac{dT}{dr} = -\frac{4}{3}\frac{acT^3}{\kappa\rho}\frac{dT}{dr}$$

and since $\mathrm{F}_{rad} = \dfrac{L(r)}{4\pi r^2} \qquad (1D)$

$$\frac{L(r)}{4\pi r^2} = K_{rad}\nabla T = -\frac{4}{3}\frac{acT^3}{\kappa\rho}\frac{dT}{dr}$$

$$\frac{dT}{dr} = -\frac{3\kappa\rho}{16\pi acT^3}\frac{L(r)}{r^2}$$

¾ times what we got before

In Lagrangian coordinates this is

$$\frac{dT}{dm} = \frac{3\kappa}{64\pi^2 acT^3}\frac{L(r)}{r^4}$$

This is our second major equation for stellar structure and evolution. It describes heat transport so long as the material is not convecting.

Pols 5.17

The same approach can be used with a frequency-dependent flux , H_υ, as is necessary to correctly define the opacity

$$\mathrm{H}_\nu = -D_\nu \nabla U_\nu = -\frac{1}{3} v \ell_\nu \frac{dU_\nu}{dr} = -\frac{c}{3}\frac{1}{\kappa_\nu \rho}\frac{d}{dr}\left\{\frac{4\pi}{c} B_\nu \left(T, \Omega\right)\right\}$$

$$= -\frac{4\pi}{3\rho}\frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}\frac{dT}{dr}$$

$$F = \int_0^\infty H_\nu \, d\nu = -\frac{4\pi}{3\rho}\frac{dT}{dr}\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT} d\nu$$

and since

$$\int_0^\infty \frac{dB_\nu}{dT} d\nu = \frac{d}{dT}\int_0^\infty B_\nu \, d\nu \;\; = \frac{d}{dT}\frac{\sigma T^4}{\pi} = \frac{4\sigma}{\pi} T^3 = \frac{ac}{\pi} T^3$$

where we have used a result from lecture 2 that

$$\pi \int_0^\infty B_\nu d\nu = F = \sigma T^4$$

$$a \equiv \frac{4\sigma}{c}$$

the π comes from integration over solid angle

Defining a mean opacity $\bar{\kappa}$ (the Rosseland mean) by

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT} d\nu}{\int_0^\infty \frac{dB_\nu}{dT} d\nu} = \frac{\pi}{acT^3}\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT} d\nu$$

so $$\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT} d\nu \; = \frac{acT^3}{\pi\bar{\kappa}}$$

gives $F = -\frac{4\pi}{3\rho}\frac{dT}{dr}\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB_\nu}{dT}d\nu \quad = -\frac{4}{3}\frac{acT^3}{\bar{\kappa}\rho}\frac{dT}{dr}$ as before

but with a new definition of the opacity, $\bar{\kappa}$.

The Rosseland mean opacity is the average of $\frac{1}{\kappa_\nu}$ weighted by the function $\frac{dB_\nu}{dT}$.

$$B_\nu(T) = \frac{8\pi h}{c^3}\frac{\nu^3}{e^{h\nu/kT}-1} \quad \text{so}$$

$$\frac{dB_\upsilon}{dT} = \frac{2h^2\nu^4}{c^2}\frac{e^{h\nu/kT}}{kT^2}\left(\frac{1}{e^{h\nu/kT}-1}\right)^2$$

This weighting function has a maximum at $h\nu = 3.83\,kT$

A condition for the validity of this equation is that the background conditions vary only a tiny bit over a typical scattering length $\ell = \frac{1}{\kappa\rho}$.

In the sun for example the typical mean free path is 0.1 cm. The temperature of the sun varies from 10^7 K to essentially zero over a distance of 7×10^{10} cm so in a scattering length the temperature varies only about 10^{-4} K. Variations in the luminosity, opacity, and density are similarly small.

So the sun can be treated as a blackbody locally to high precision. The diffusion equation can be used and "local thermodynamic equilibrium" (LTE) prevails.

Sources of opacity : (Pols 59ff)

1) Electron scattering - frequency-independent

at low frequencies. Thomson cross section.

http://en.wikipedia.org/wiki/Thomson_scattering

$$\sigma_e = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right) = 6.652 \times 10^{-25} \ \ \text{cm}^2$$

which when multipled by the number of electrons in a given mass of gas, $\rho N_A Y_e$ and using n$\sigma = \kappa\rho$ gives

$$\kappa_{es} = N_A Y_e \sigma_e = 0.40 \ Y_e = 0.20\left(1 + X_H\right) \ \text{cm}^2 \ \ \text{g}^{-1}$$

$$\text{since } Y_e = \sum Z_i Y_i = X_H + \sum_{Z \geq 2} \frac{Z_i}{A_i} X_i = X_H + \frac{1}{2}\sum_{Z \geq 2} X_i$$

$$= X_H + \frac{1}{2}\left(1 - X_H\right) = \frac{1}{2}\left(1 + X_H\right)$$

Modifications to electron scattering:

- The peak of the radiation spectrum is at hν = 4.965 kT (Wien's law). When this becomes greater that about 0.1 $m_e c^2$ = 51 keV, so about 10^8 K, the Thomson cross section must be corrected for the recoil of the electron (Klein-Nishina corrections).

$$\varepsilon = h\nu / m_e c^2$$

$$\sigma_{KN} = \sigma_{Thom}\left(1 - 2\varepsilon + \frac{26}{5}\varepsilon^2 - \ldots\right) \qquad \varepsilon \ll 1$$

$$\sigma_{KN} = \pi r_e^2 \left(\frac{1 + 2\ln\varepsilon}{2\varepsilon}\right) \qquad \varepsilon \gg 1 \quad r_e = \frac{h}{m_e c}$$

http://physics.indiana.edu/~shylee/p672/EM_scattering_1.pdf

Modifications to electron scattering:

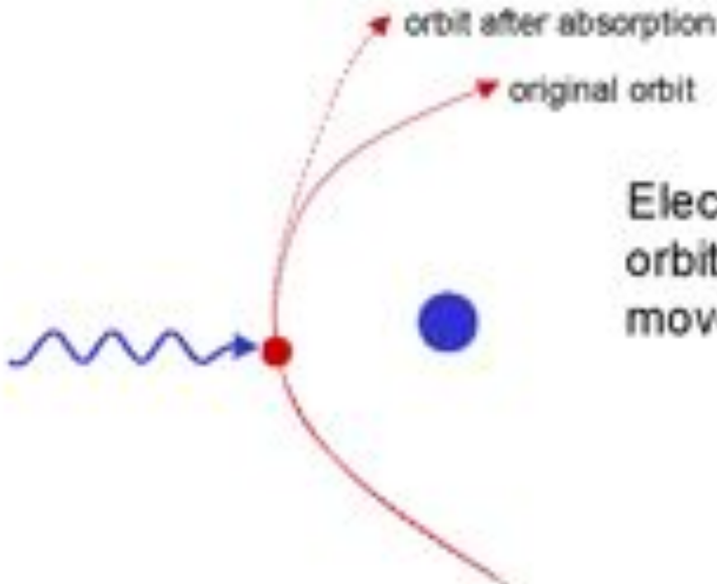
- Above about 10^9 K (kT = 86 keV) there are also corrections for the production of electron-positron pairs which increase the opacity.

- At high density, as in white dwarfs, the phase space for the scattered electron may be reduced by degeneracy. This reduces the electron scattering opacity

- At lower temperature the ionization must be explicitly computed using the Saha equation

- Below about 10^7 K other physics comes into play that involves bound states of atoms

2) Free-free absorption: photon absorbed by a free electron, in the presence of an ion, increasing electron energy. (Inverse process is *bremsstrahlung*)



Electron moving in an unbound (hyperbolic) orbit about an ion absorbs a photon and moves into a higher energy orbit

Similar to electron scattering but modified (increased) by the presence of the ion which absorbs some of the momentum and energy. Will depend on the abundances of both the electrons and ions.

The efficiency of such an absorption by a single electron is proportional to $Z_i^2 \nu^{-3}$ with Z_i the charge on the ion. The total efficiency depends on the number density of electrons and also the time the electron and ion stay sufficiently close for the interaction to occur. This depends on the average speed of the electron, $\Delta t \propto 1/\bar{v}$ with $\bar{v} = \left(3kT / m_e\right)^{1/2}$ so the cross section per ion

$$\sigma_{ff} \propto n_e T^{-1/2} Z_i^2 \nu^{-3}$$

See discussion of Kramer's opacity by Frank Shu on the class website.

Since $n\sigma = \kappa\rho$

$$\rho\kappa_{ff} = \sigma_{ff} \sum_i n_i \propto n_e T^{-1/2} Z_i^2 \nu^{-3} \sum_i n_i$$

Aside :

If $\kappa_\nu \propto \nu^{-n}$ the Rosseland mean integral gives $\bar{\kappa} \propto T^{-n}$

$$\frac{1}{\bar{\kappa}} \propto \frac{\dfrac{2h^2}{c^2kT^2}\displaystyle\int_0^\infty \nu^{n+4} \frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}\,d\nu}{\dfrac{2h^2}{c^2kT^2}\displaystyle\int_0^\infty \nu^{4} \frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}\,d\nu}$$

$$\frac{1}{\bar{\kappa}} \propto \left(\frac{kT}{h}\right)^n \frac{\displaystyle\int_0^\infty x^{n+4} \frac{e^{x}}{\left(e^{x}-1\right)^2}\,dx}{\displaystyle\int_0^\infty x^{4} \frac{e^{x}}{\left(e^{x}-1\right)^2}\,dx} \propto T^n$$

So, for the Rosseland mean opacity (leaving off $^{-}$)

$$\kappa_{ff} = \frac{n_e}{\rho}\sum_i n_i T^{-1/2} Z_i^2 \nu^{-3} \propto \frac{\rho N_A Y_e}{\rho} T^{-7/2} \sum_i \rho N_A Y_i Z_i^2$$

$$\propto \rho Y_e T^{-7/2} \sum_i \frac{X_i Z_i^2}{A_i} \propto \rho T^{-7/2} \qquad \sum_i \frac{X_i}{A_i} = \frac{1}{\bar{A}}$$

Any opacity of this form ($\propto \rho T^{-7/2}$) is referred to as a "Kramers opacity". Assuming full ionization and including the leading order composition dependence

$$\kappa_{ff} = 7.5\times 10^{22} Y_e \frac{\langle Z^2 \rangle}{\bar{A}} \rho T^{-7/2} \qquad \text{Clayton 3-170}$$

$$\approx 7.5\times 10^{22} \frac{(1+X)}{2} \frac{\langle Z^2 \rangle}{\bar{A}} \rho T^{-7/2} \ \ \text{cm}^2 \ \ \text{g}^{-1} \qquad \text{Close to Pols 5.32}$$

The interactions with more highly charged nuclei are stronger, but weaker if the mass of the nucleus is larger.

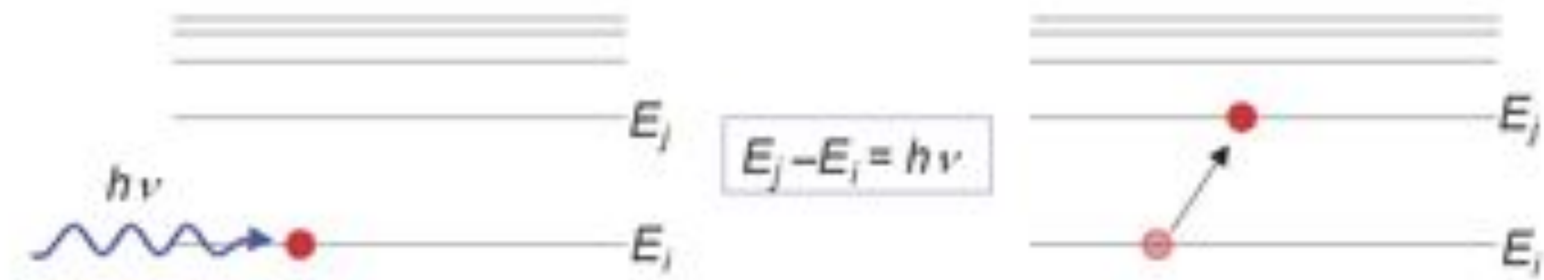
At very high temperatures the free-free opacity becomes negligible since the extra interaction between the ion and electron is negligible compared with the energy of the photon, $h\nu \sim 4\ kT$.

The free-free opacity then gradually becomes equal to the electron scattering opacity.

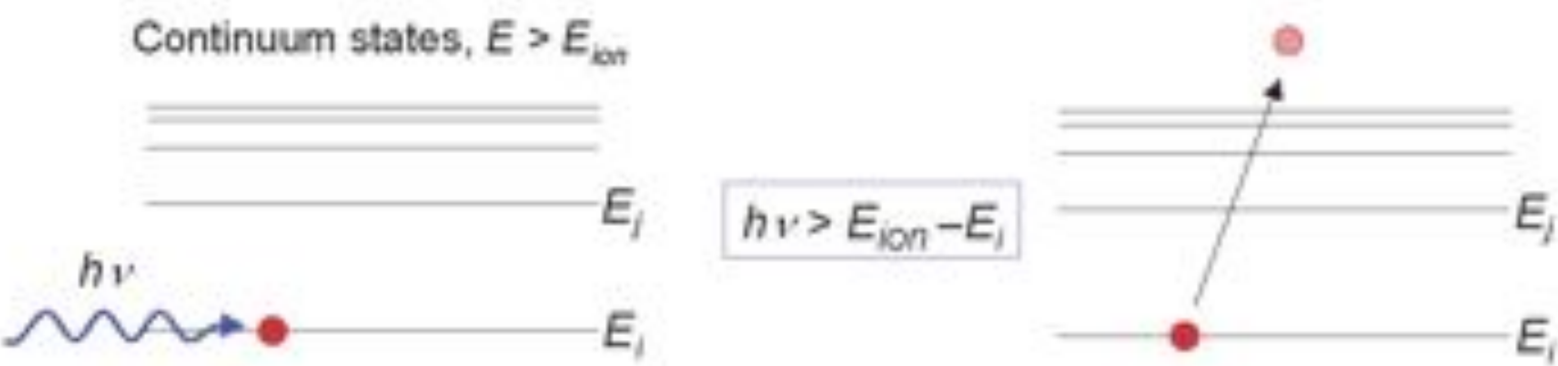
Similarly at low density the ion spacing is large and the electron ion interaction is diminished.

3) Bound-bound and bound-free opacity

Bound-bound absorption: photon is absorbed by a bound electron, exciting it to a higher energy state.



Bound-free absorption (or photo-ionisation): photon is absorbed by a bound electron, giving it energy above the ionisation potential.



Bound-free opacity

This is quite complicated to calculate since it depends on numerous transitions in many atoms. Classical considerations again give a cross section for bound-free absorption that again depends on ν^{-3} as long as hν > the ionization potential of the atom. For T >> 10^4 K, where bound-free is important,

$$\kappa_{bf} = 4.3 \times 10^{25} \left(1 + X\right) Z \rho T^{-7/2} \ \text{cm}^2 \ \text{g}^{-1}$$

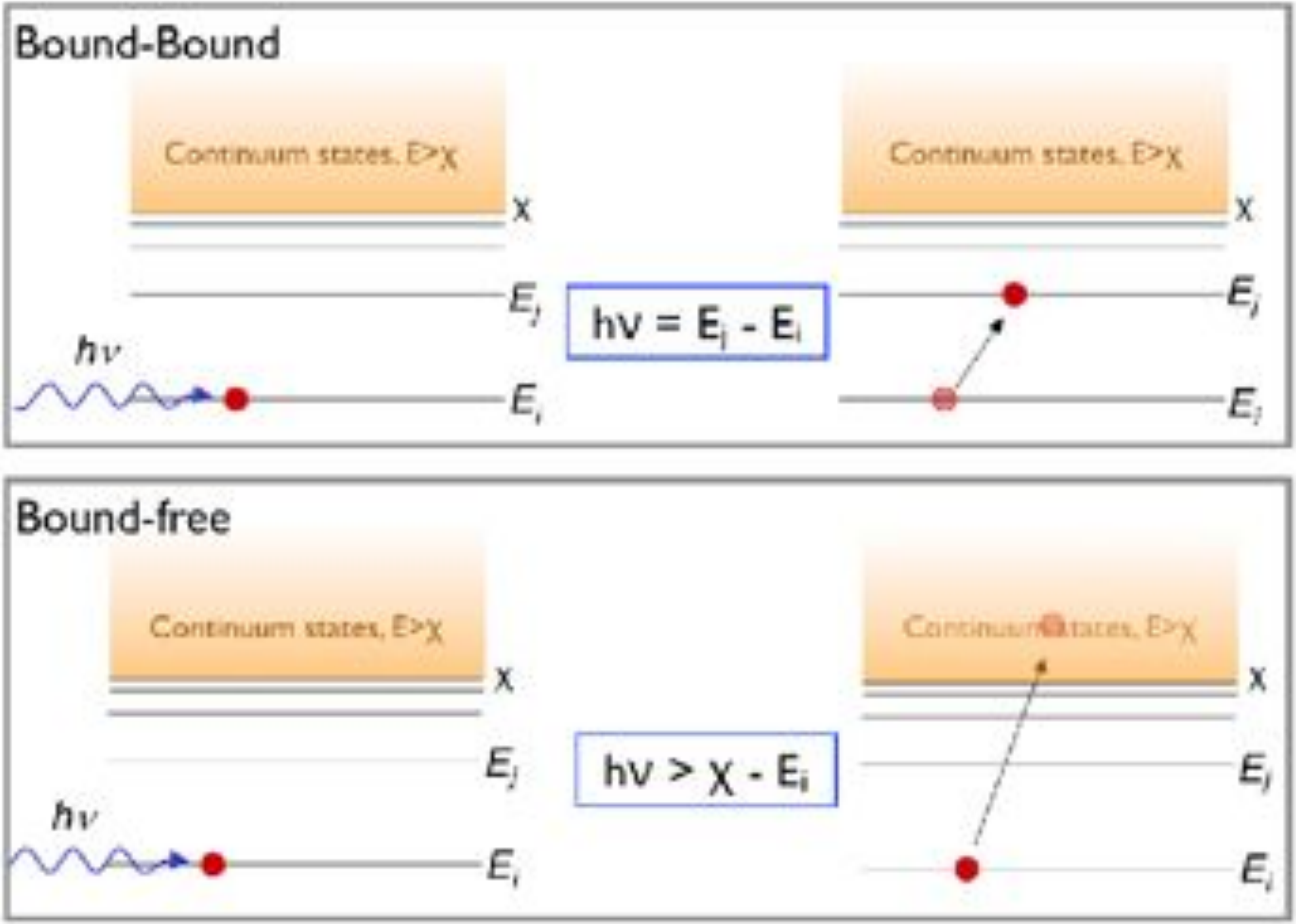
where X is the hydrogen mass fraction and Z the mass fraction of all elements heavier than helium (the metallicity). Unless Z is very small, this is much larger than the free-free opacity ($\kappa_{\mathrm{bf}} \sim 10^3 \, Z \, \kappa_{\mathrm{ff}}$). Note again the Kramer's-like dependence on T and ρ

Bound-bound opacity

Again the calculation is complex and composition dependent. In the general case one has to include millions of transitions in the states of many elements and many ionization stages of those elements. Especially important is the element iron because of its large abundance and many atomic levels.

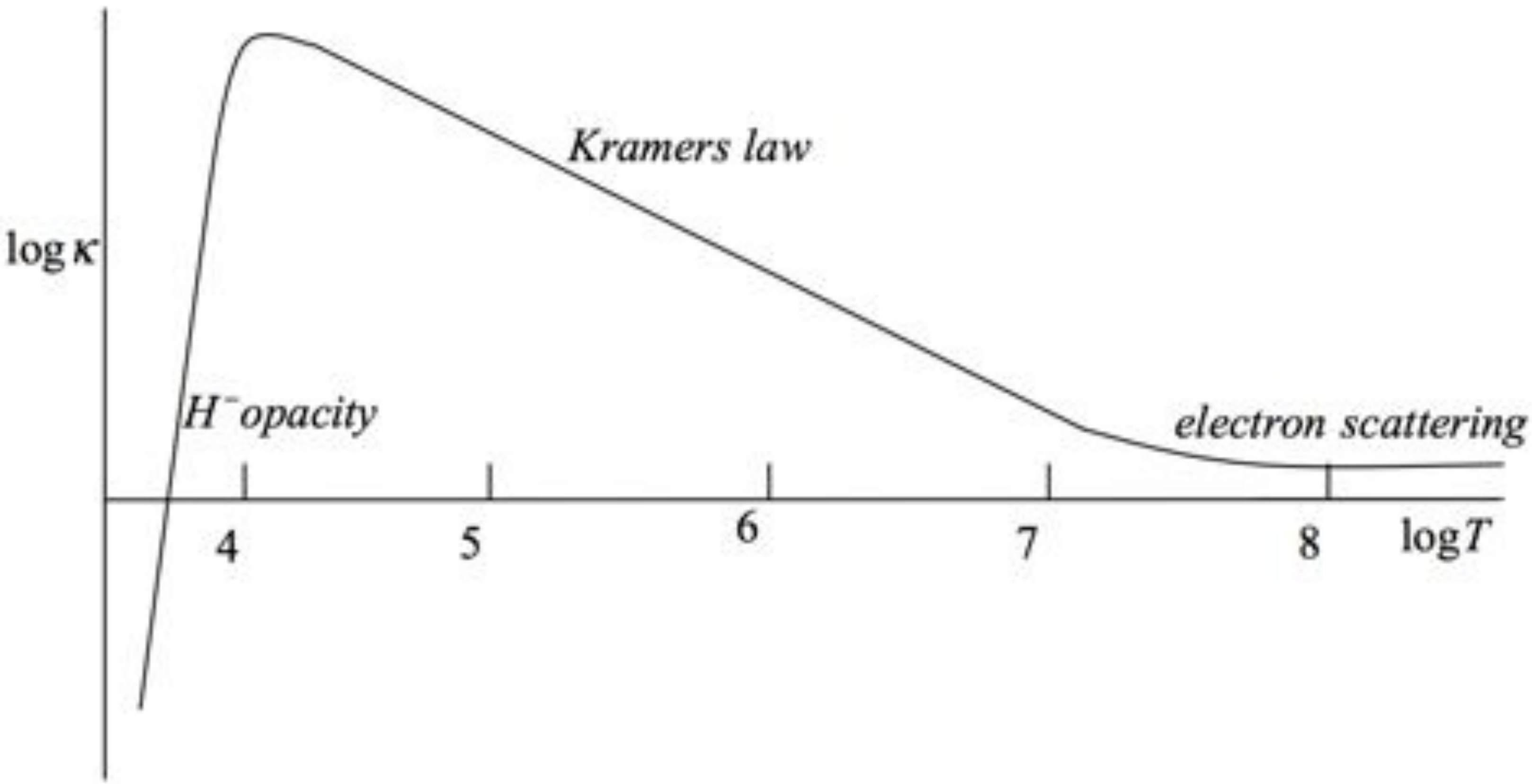
Each pair of levels can serve as a resonance for the absorption of radiation. The levels are broadened by collisions and thermal motion.

Bound-bound opacity is generally negligible above a few times 10^6 K



http://www.ast.cam.ac.uk/~bdavies/Stars2/Stars2_Lecture8.pdf

From Frank Shu, see also Prialnik p 51



The straight line n a log-log plot indicates a power law. For Kramers, the slope is -7/2.

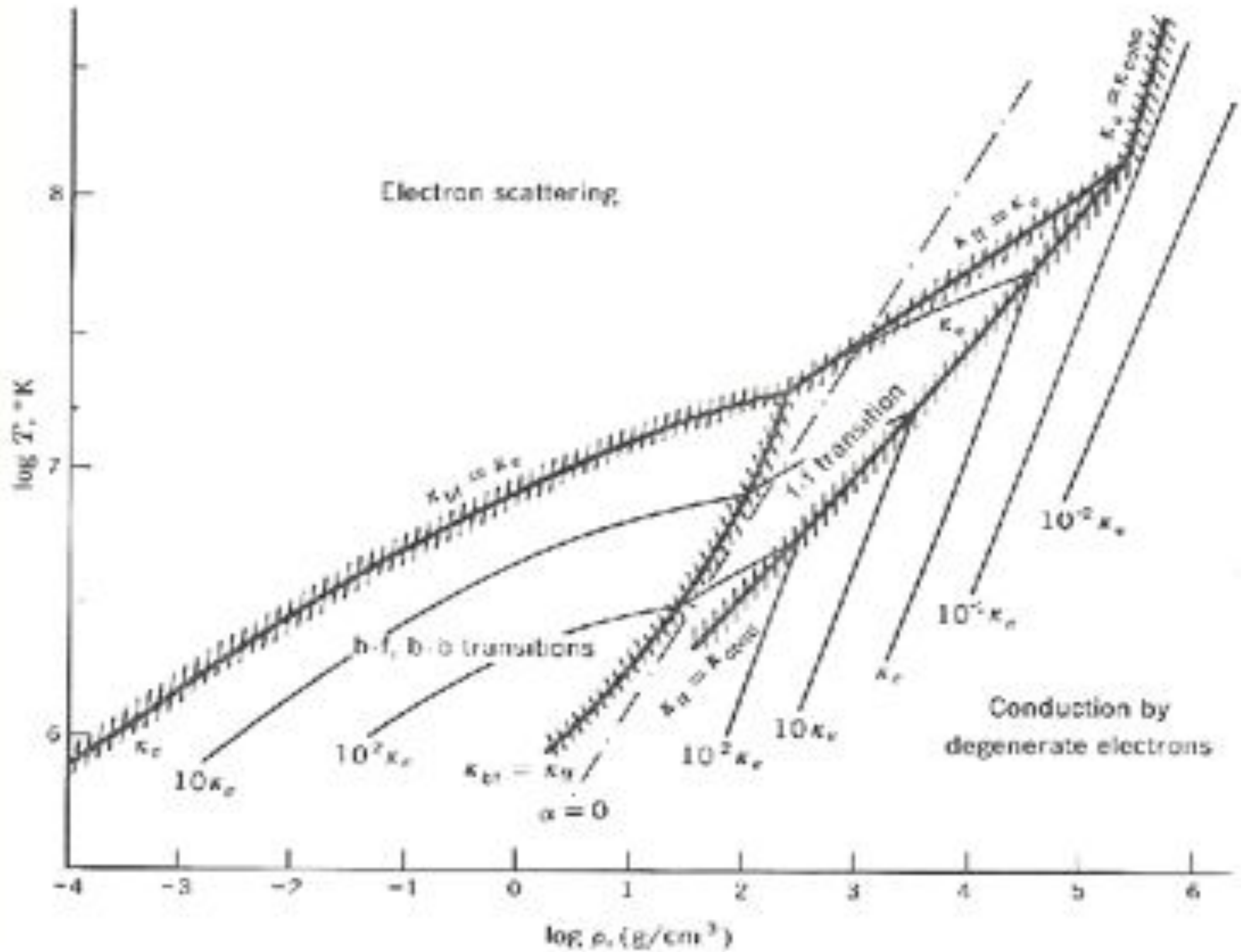


Figure 3 -15 from Clayton

4) Conduction

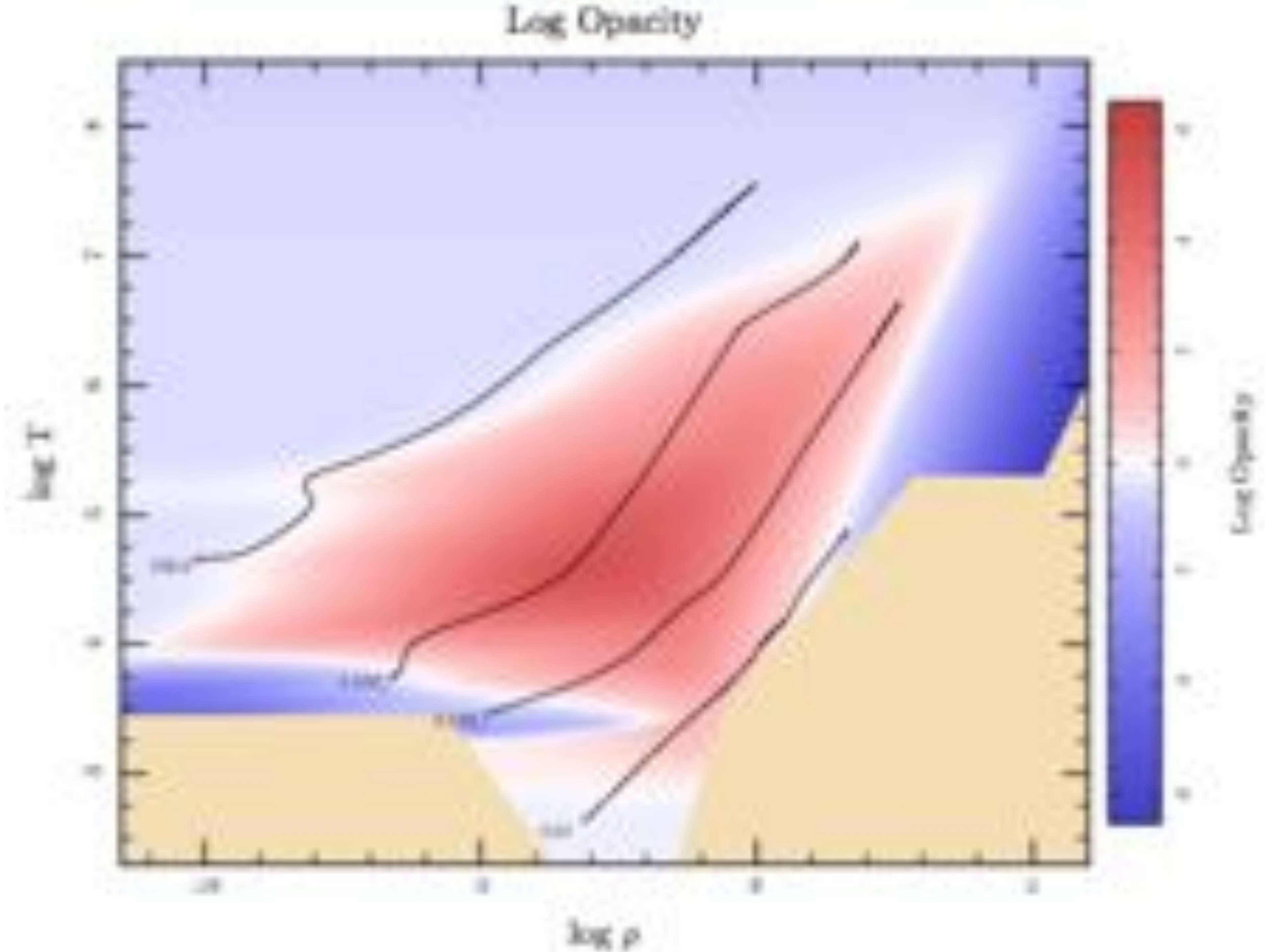
In a star heat may be transported by radiative diffusion, convection and conduction. We will treat convection separately.

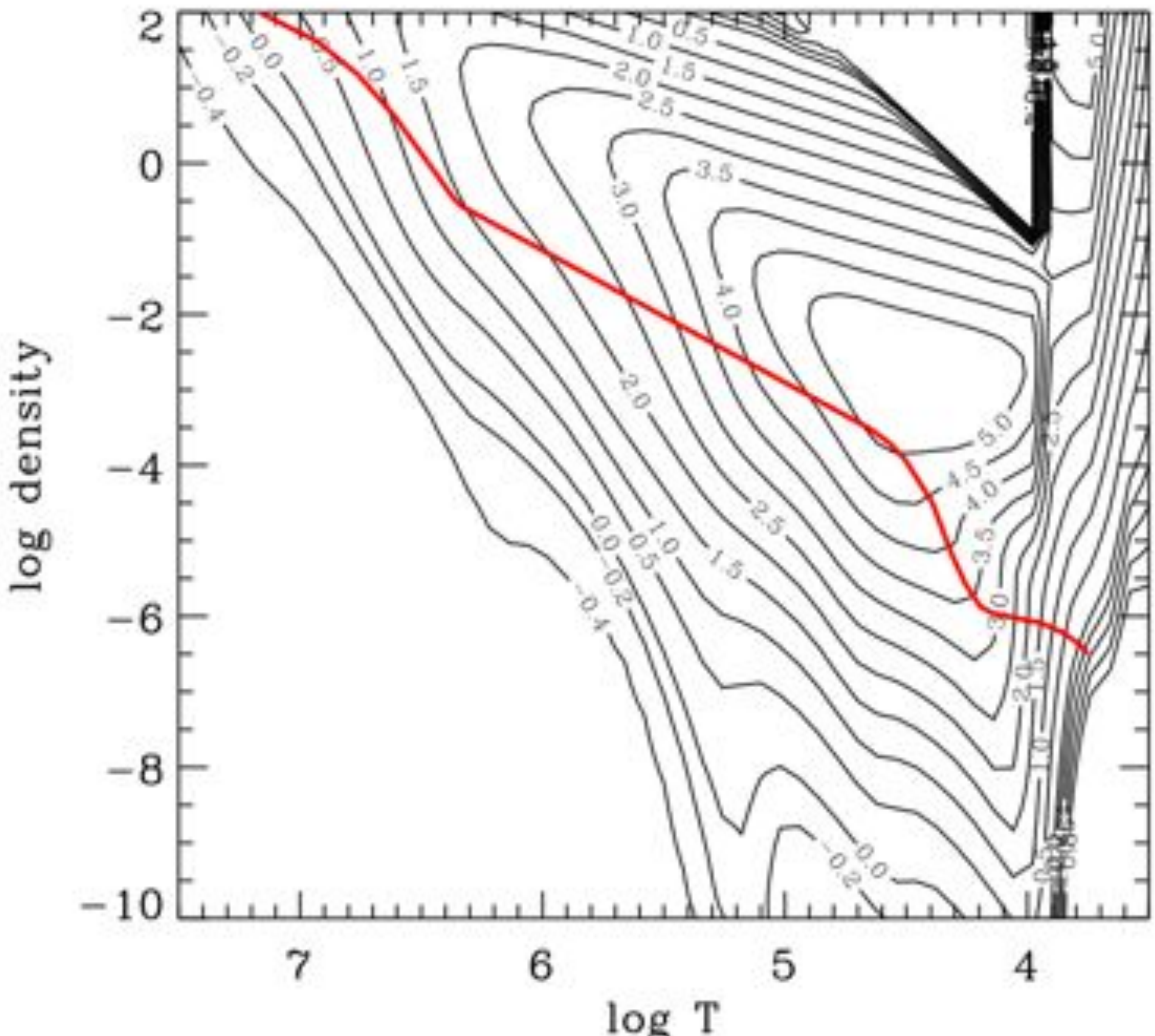
Conduction is the transport of heat by diffusing electrons. As a diffusion process it can be treated analogously to radiation, but it is electrons that are carrying the energy not photons. It is described by a conductive opacity.

Usually the mean free path of electrons is much shorter than photons and radiative diffusion dominates. At high density however a) the the path length of electrons is increased by the effects of degeneracy (full phase space in the outgoing channel) and b) there are more electrons.

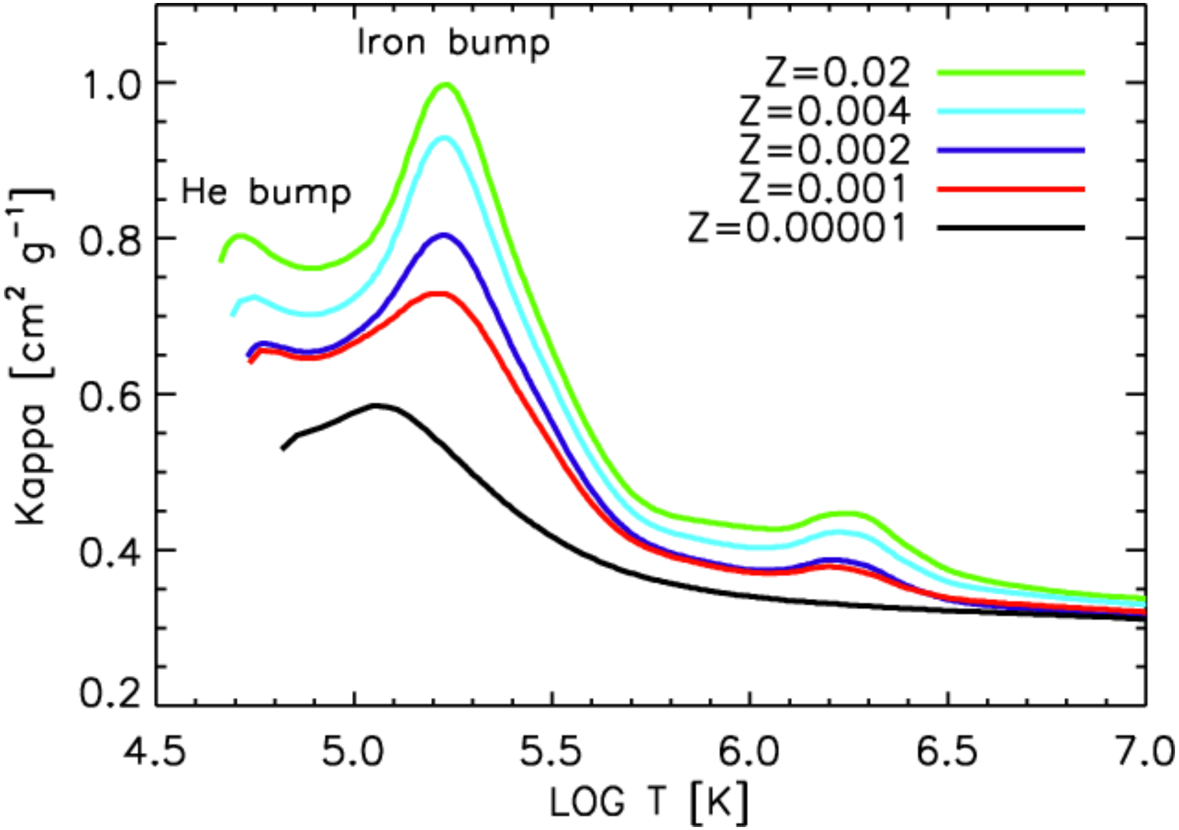
Conduction thus can dominate in situations where the electrons are degenerate, e.g. white dwarfs. It is enhanced by a)high density b) low temperature and c) lower charges on the nuclei that the electrons are scattering off of. An approximation to κ is given by Pols 5.35. Note that small κ implies long pathlength and thus high conductivity.

$$\kappa_{cond} \approx 4.4 \times 10^{-3} \frac{\sum_i Z_i^{5/3} X_i / A_i}{(1+X)^2} \frac{(T/10^7)^2}{(\rho/10^5)^2} \text{ cm}^2 \text{ g}^{-1}$$


Log Opacity
log T
log ρ
Log Opacity



Contours of constant opacity (OPAL); red line is the sun



Opacity vs T for 60 solar mass models of various metallicities (Cantiello et al 2009, A&A)

Metallicity affects opacity and this is one of the main ways it enters into stellar evolution.

Because of the differing opacity, energy generation, and mass loss, stars of low metallicity will be different from those of solar composition.

Nowadays stellar evolution codes use tables of opacities that have been generated off-line that include all the processes we have mentioned – electron scattering, free-free, bound-free, bound-bound. An exaple tablulation as a function of X, Y, and Z can be found at

http://cdsweb.u-strasbg.fr/topbase/OpacityTables.html

The Eddington Luminosity

There is an upper limit to the amount of radiation that can flow through a star by radiative diffusion without making the matter move.

Deep inside the star this excess power can result in convection, although there is an upper limit to what convection can carry too. Closer to the surface such high luminosities drive very rapid mass loss.

Since luminosity on the main sequence is generally sensitive to M^n with $n > 1$, there comes a limiting mass where the star cannot stay bound for long

In a region with appreciable radiation pressure there is a radiation presure gradient

$$\frac{dP_{rad}}{dr} = \frac{4}{3} a T^3 \frac{dT}{dr}$$

And if that region is transporting energy by radiative diffusion

$$\frac{dT}{dr} = \frac{3\kappa\rho}{16\pi a c T^3} \frac{L(r)}{r^2}$$

So

$$\frac{dP_{rad}}{dr} = -\frac{\kappa\rho}{4\pi c} \frac{L(r)}{r^2}$$

This cannot exceed the pressure gradient given by hydrostatic equilibrium

$$\left|\frac{dP_{rad}}{dr}\right| < \left|\frac{dP}{dr}\right| = \frac{GM\rho}{r^2}$$

Interestingly the ρ/r^2 term cancels and one has

$$\boxed{L < \frac{4\pi Gmc}{\kappa} = L_{Ed}} = 1.5\times 10^{38}\left(\frac{M}{M_\odot}\right)\left(\frac{0.34}{\kappa}\right)\text{ erg/sec}$$

$$= 3.8\times 10^{4}\left(\frac{M}{M_\odot}\right)\left(\frac{0.34}{\kappa}\right)L_\odot$$

where L_{Ed} is the Eddington luminosity.

Often, for situations where L approaches L_{Ed}, the temperature is so high that electron scattering opacity dominates. $\kappa = 0.34\ \text{cm}^2\ \text{g}^{-1}$ is appropriate for X = 0.7. For pure helium, carbon, oxygen, etc. $\kappa = 0.2\ \text{cm}^2\ \text{g}^{-1}$.

The Eddington Accretion Rate

Interestingly the Eddington luminosity also sets an upper bound to the rate at which stars can *accrete* mass. Accretion liberates the gravitational potential energy but can only occur at a rate smaller than that which gives an Eddington luminosity (at least in 1D)

$$\frac{GM\dot{M}}{R} < L_{Ed} = \frac{4\pi GMc}{\kappa}$$

$$\frac{dM}{dt} < \frac{4\pi Rc}{\kappa}$$

e.g., for a WD with R = 5000 km

$$\frac{dM}{dt} < 8.8 \times 10^{-6} \ \mathrm{M}_{\odot} \quad \mathrm{year}^{-1}$$

The Eddington Lifetime

And we shall see later these considerations also set a lower bound to the main sequence lifetime of a star.

Extremely massive stars have luminosities approaching L_{Ed}. They are almost fully convective and thus burn almost their entire mass on the main sequence

$$\tau_{Ed} = \frac{(6.8\times10^{18}\ \mathrm{erg/gm})(1.99\times10^{33}\ \mathrm{gm})M/M_{\odot}}{1.5\times10^{38}\,M/M_{\odot}\ \mathrm{erg/sec}}$$

$$= 9.0\times10^{13}\ \mathrm{sec} = 2.9\ \text{million years}$$

Appendix

GK Radiation transport

3) Or following Glatzmaier and Krumholz (6), use the equation of radiative transfer. (skipped in class)

$$\frac{dI}{ds} = -\kappa\rho I + j$$ where j is a source of radiation $\tau = \kappa\ \rho\ s$

http://www.cv.nrao.edu/course/astr534/Radxfer.html

So *I* is the original beam being absorbed and j is a source of new energy being added to the beam in a medium that is in thermal equilibrium

Define $S = \dfrac{j}{\kappa\rho}$

then for $\tau = \kappa\rho\,s$

$$\frac{dI}{d\tau} = -I + S$$

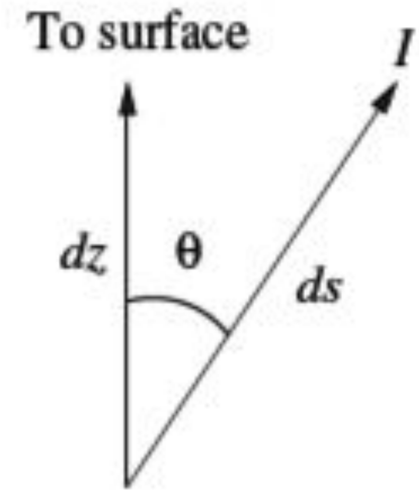
This is the equation of radiative transfer

In thermal equilibrium with no Tgradient
in a uniform, opaque medium the radiative flux
has to be given by the Planck distribution

$$\frac{dI}{d\tau} = 0 \qquad S = I = B_\nu(T) = \left(\frac{2h\nu^3}{c^2}\right)\frac{1}{e^{h\nu/kT} - 1}$$

This remains true for both terms to first order in
the presence of a small temperature gradient.
I is going to remain close in value to S and both
will be approxiately B

Introduce some explicit angle dependence and enforce spherical coordinates, consider a ray propagating at an arbitrary angle to a radial



z is in the radial direction.

The ray propagates in the s direction and has a component in the radial direction:

$$\text{Cos}\ \theta = \frac{dz}{ds}$$

$$\text{so ds} = \frac{dz}{Cos\,\theta}$$

Transfer equation:

$$\frac{dI(z,\theta)}{ds} = \frac{dI(z,\theta)}{dz}\,Cos(\theta) = \kappa\rho\left[S(T) - I(z,\theta)\right]$$

which can be rewritten

$$I(z,\theta) = S(T) - \frac{Cos\,\theta}{\kappa\rho}\,\frac{dI(z,\theta)}{dz}$$

Nothing new so far. Now recognize that the second term is very small (I is very nearly constant over short distances) and that S(T) is the Planck function, B(T), so write the beam flux as Plankian plus a small perturbation

$$\mathrm{I}(z,\theta)=B(T)+i(z,\theta) \qquad\qquad i(z,\theta)<<B(T)$$

but also from the previous page

$$\mathrm{I}(z,\theta)=S(T)-\frac{Cos\,\theta}{\kappa\rho}\;\frac{dI(z,\theta)}{dz}$$

The source function is $S(T)=B(T)\,so$

$$B(T)+i(z,\theta)=B(T)-\frac{Cos\,\theta}{\kappa\rho}\;\frac{dI(z,\theta)}{dz}$$

$$i(z,\theta)=-\frac{Cos\,\theta}{\kappa\rho}\;\frac{dI(z,\theta)}{dz}$$

$$=-\frac{Cos\,\theta}{\kappa\rho}\;\frac{d}{dz}\Big(B(T)+i(z,\theta)\Big)\approx-\frac{Cos\,\theta}{\kappa\rho}\;\frac{dB(T)}{dz}$$

neglecting the derivative of the small term

$$i(z,\theta) \approx -\frac{Cos\,\theta}{\kappa\rho}\frac{dB(T)}{dz}$$

By making the length scale small enough

$\approx$ becomes =

$$I(z,\theta) = B(T) - \frac{Cos\,\theta}{\kappa\rho}\frac{dB(T)}{dz}$$

To compute the flux integrate the intensity over angles and frequencies like we did for radiation pressure.

$$H_\nu(z) = \int I(z,\theta)\, Cos(\theta)\, d\Omega \qquad nb.\ z = r$$

$$= \int \left[B(T) - \frac{Cos\theta}{\kappa_\nu \rho} \frac{dB(T)}{dz} \right] Cos\theta\ d\Omega$$

$B(T)$ is isotropic

$$= -\int \left[\frac{Cos\theta}{\kappa_\upsilon \rho} \frac{dB(T)}{dz} \right] Cos\theta\ d\Omega$$

$$d\Omega = 2\pi\ Sin\theta\ d\theta$$

$$= -\frac{2\pi}{\kappa_\upsilon \rho} \frac{dB(T)}{dz} \int_0^\pi Cos^2\theta\, Sin\theta\, d\theta$$

$$= -\frac{2\pi}{\kappa_\upsilon \rho} \frac{dB(T)}{dz} \int_{-1}^{1} Cos^2\theta\, (dCos\theta)$$

$$\int \Rightarrow \left. \frac{x^3}{3} \right]_{-1}^{1} = \frac{2}{3}$$

$$= -\frac{4\pi}{3\kappa_\nu \rho} \frac{dB(T)}{dz} = -\frac{4\pi}{3\kappa_\nu \rho} \frac{dB(T)}{dT} \frac{dT}{dz}$$

$$F = \frac{-4\pi}{3\rho}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\kappa_\nu}\frac{dB}{dT}\,d\nu \qquad F = \int H_\nu\, d\nu$$

Define as before in 2) :

$$\frac{1}{\bar{\kappa}} = \frac{\int_0^\infty \kappa^{-1}\frac{dB}{dT}\,d\nu}{\int_0^\infty \frac{dB}{dT}\,d\nu} = \frac{\pi}{4\sigma T^3}\int_0^\infty \kappa_\upsilon^{-1}\frac{dB}{dT}\,d\nu$$

Rosseland Mean

Then:

$$F = -\frac{16\sigma T^3}{3\bar{\kappa}\rho}\frac{\partial T}{\partial z} = -\frac{16\sigma T^3}{3\bar{\kappa}\rho}\frac{dT}{dr}$$

$$L(r) = -4\pi r^2\,\frac{16\sigma T^3}{3\bar{\kappa}\rho}\frac{dT}{dr}$$

as before