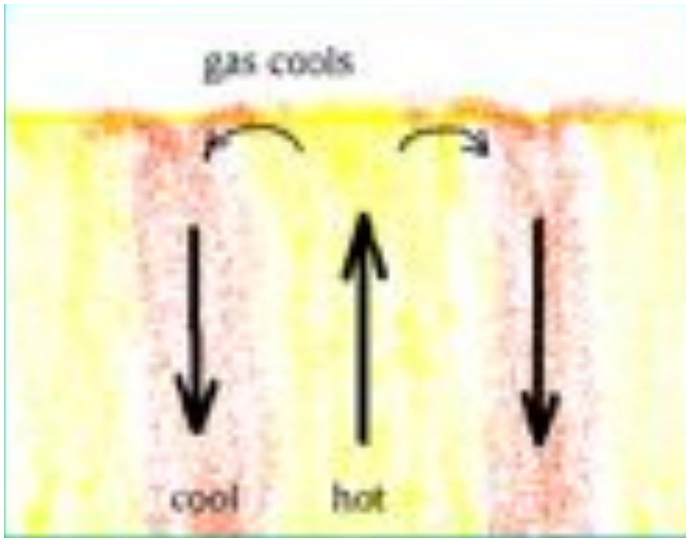
Convection and Other Instabilities

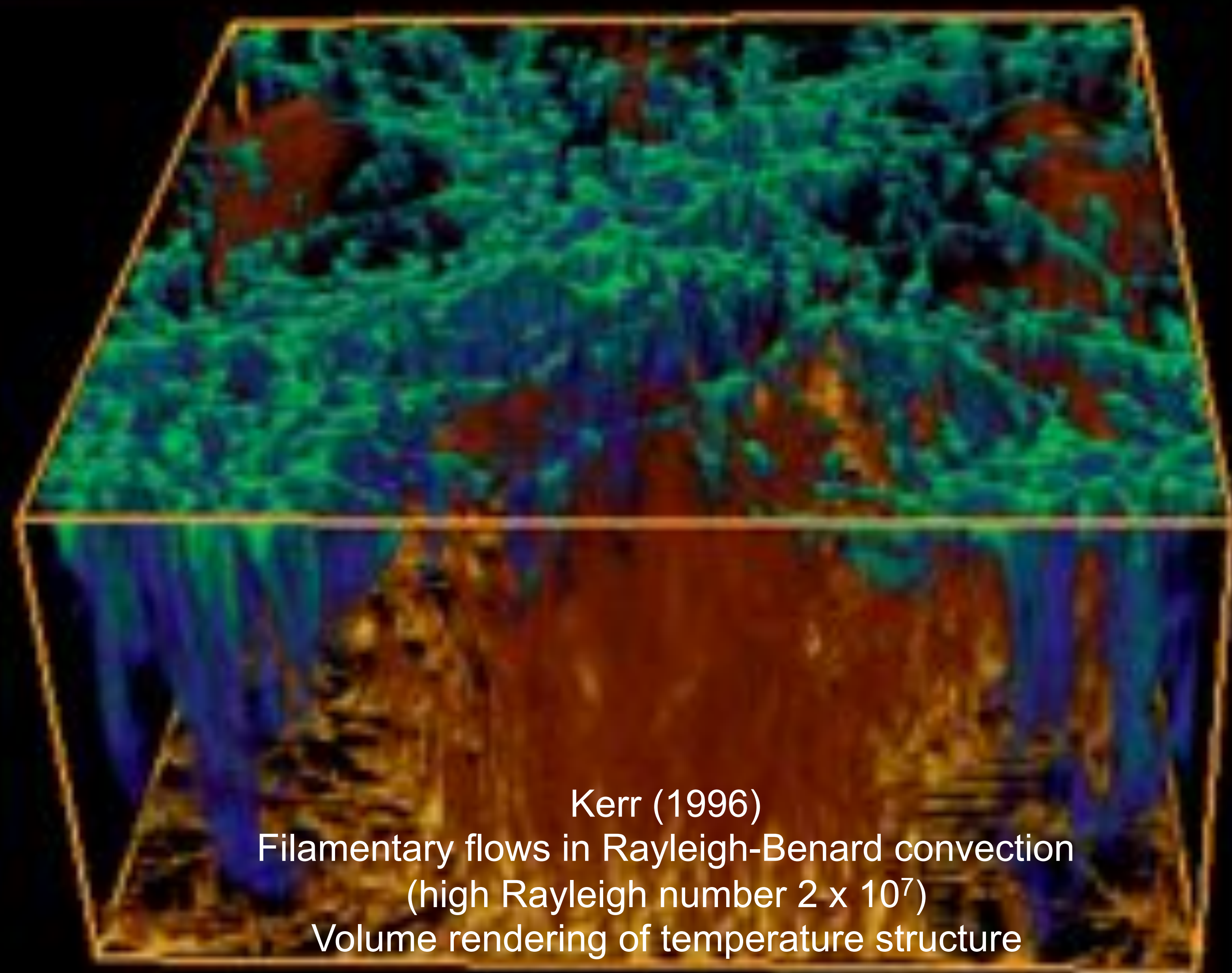
Prialnik 6
Glatzmair and Krumholz 11
Pols 3, 10

What is convection?



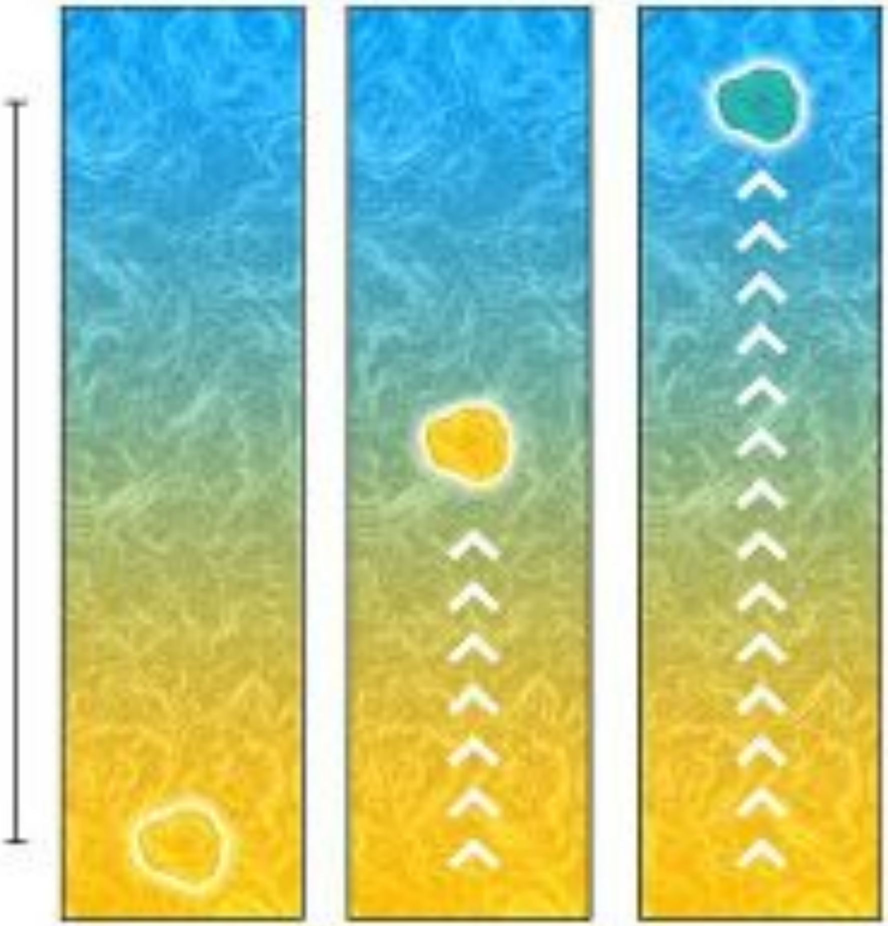
Highly idealized schematic. Real convection is not so ordered, but mass is conserved. Mass going up = mass going down.

- A highly efficient way of transporting energy in the stellar interior
- Heat carried by advection, not diffusion, Depends linearly on fluid speed, not on random walk
- Occurs when the heat gradient exceeds some critical value required for buoyancy



Kerr (1996)
Filamentary flows in Rayleigh-Benard convection
(high Rayleigh number 2×10^7)
Volume rendering of temperature structure


Solar surface



Which is denser after a a displacement? The blob or its surroundings?

Consider the radial displacement of a blob of gas. As it rises its internal density will decrease in accordance with the decrease in pressure in the surrounding medium. In the absence of heat exchange with its surroundings it will expand *adiabatically.*

The density in the surrounding medium will decrease too in accordance with hydrostatic equilibrium.

Adiabatic expansion

Recall from Lecture 5 (equation of state) that the first law of thermodynamics can be used to define an "adiabatic expansion" and an adiabatic exponent.

$$du + P\,d\left(\frac{1}{\rho}\right) = 0$$

which given that

$$u = \phi \frac{P}{\rho}$$

leads to

$$\frac{d \log P}{d \log \rho} = \frac{(\phi + 1)}{\phi} = \gamma_{ad}$$

and

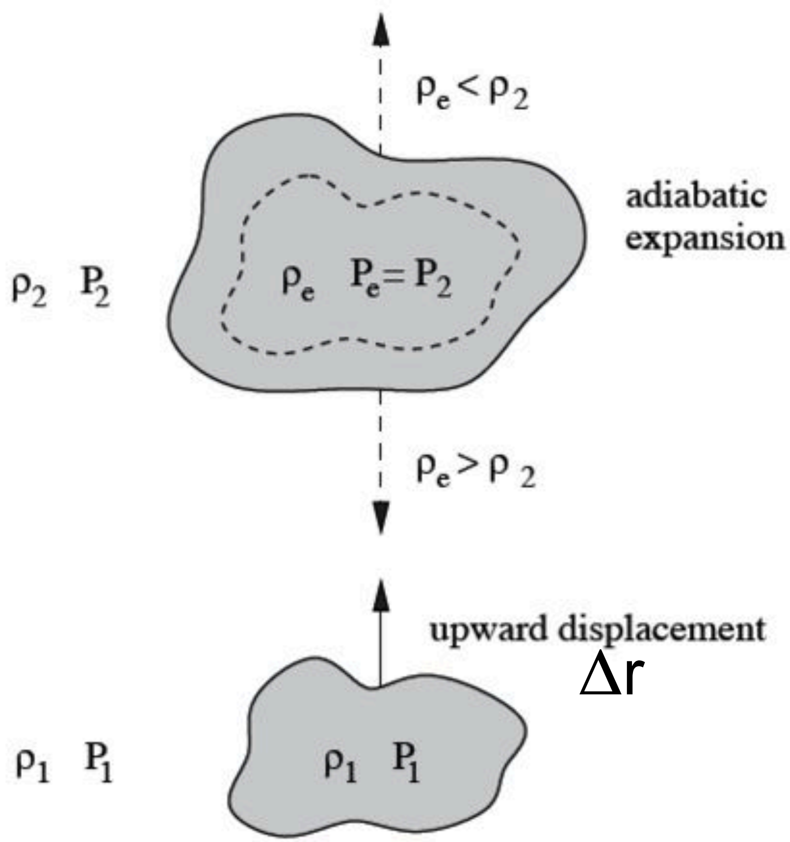
$$P \propto \rho^{\frac{\phi+1}{\phi}} \quad \text{or} \quad P = K \rho^{\frac{\phi+1}{\phi}} = K \rho^{\gamma_a}$$

Such a relation between pressure and density is called "adiabatic". If the gas expands due to decreased pressure on its extremities or is compressed by increased pressure without heat flow across its boundary, this expression gives the change in internal pressure relative to the change in density

From our previous discussion of ϕ, for ideal gas $\gamma_a = 5/3$; for radiation, $\gamma_a = 4/3$.

Note that the gamma here is a local gamma $\gamma_a(r)$, unlike the gamma that characterizes a polytrope

Consider a rising "blob" expanding adiabatically

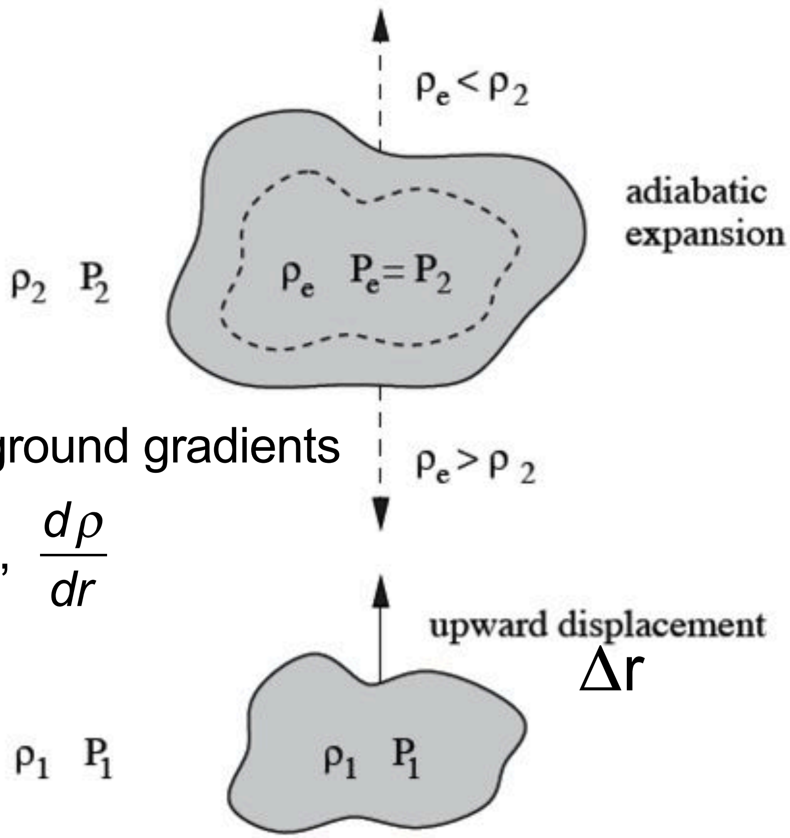


Because the hydrodynamic time is much shorter than the float time ($v << c_{sound}$) the pressure inside the floating blob at a given radius is always equal to the external pressure at the same radius. $P_e = P_2$, but ρ_e may not equal ρ_2

"e" stands here for "expanded"

$$\rho_e = \rho_1 + \delta\rho_e$$

$$\rho_2 = \rho_1 + \delta\rho$$



Δr

Background gradients

$$\frac{dP}{dr}, \frac{d\rho}{dr}$$

$$d \log P = \gamma_a \, d \log \rho \quad \text{(adiabatic)}$$

$$\Rightarrow \frac{\delta P_e}{P_e} = \gamma_a \frac{\delta\rho_e}{\rho_e}$$

where δP_e is determined by the existing pressure gradient in the star. So $\delta\rho_e = \frac{\rho_e}{P_e}\frac{1}{\gamma_a}\frac{dP}{dr}\Delta r$.

This will be stable against convection if

$$|\delta\rho_e| < |\delta\rho| = \rho_1 - \rho_2$$

but $\delta\rho_e$ and $\delta\rho$ are both negative numbers so stability $\Rightarrow \delta\rho_e > \delta\rho$

That is, if adiabatic expansion leads to a greater density (less density decrease) than the surroundings, there is no convection

$\delta\rho_e > \delta\rho$ for stability

$\delta\rho_e > \frac{d\rho}{dr}\Delta r$ or

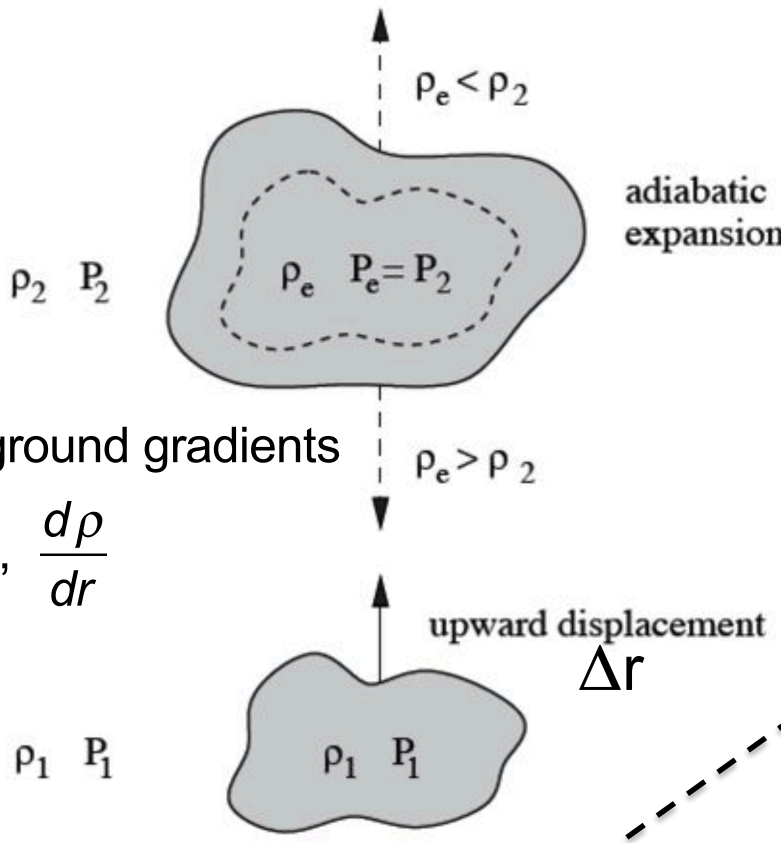
$$\frac{\rho_e}{P_e}\frac{1}{\gamma_a}\frac{dP_e}{dr}\Delta r > \frac{d\rho}{dr}\Delta r \qquad \frac{dP_e}{dr} = \frac{dP}{dr}$$

$$\frac{1}{\rho}\frac{d\rho}{dr} < \frac{1}{P}\frac{dP}{dr}\frac{1}{\gamma_a}$$

$d\rho$ and dP are both negative so

stable if
$$\frac{d\log\rho}{d\log P} > \frac{1}{\gamma_a}$$

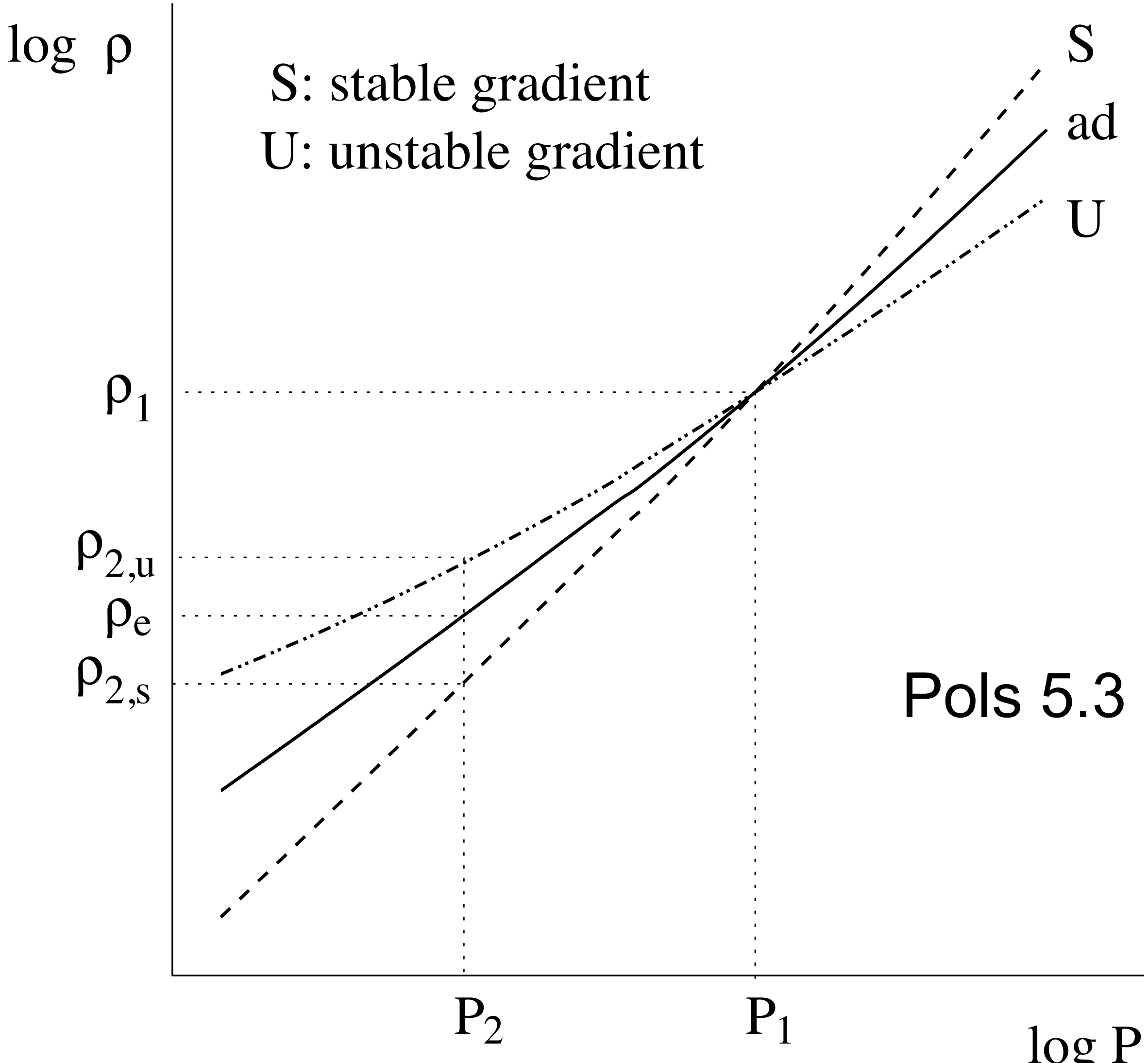
unstable if
$$\frac{d\log\rho}{d\log P} < \frac{1}{\gamma_a}$$



Background gradients

$\frac{dP}{dr}, \frac{d\rho}{dr}$

$P_e = P$ and $\rho_e \approx \rho$
if Δr is small $((\rho_e - \rho)/\rho$
might be $\sim 10^{-8}$)



stable if $\dfrac{d\log\rho}{d\log P} > \dfrac{1}{\gamma_a}$

so things stable if $\frac{d\log\rho}{d\log P} > \frac{1}{\gamma_a}$, but in general we don't know the density gradient. Instead we have information about $\frac{dT}{dr}$ and $\frac{dP}{dr}$

Expand dP in terms of partial derivatives. P is a function of T, ρ, and μ (for any EOS!)

$$dP = \left.\frac{\partial P}{\partial T}\right)_{\rho,\mu} dT + \left.\frac{\partial P}{\partial \rho}\right)_{T,\mu} d\rho + \left.\frac{\partial P}{\partial \mu}\right)_{\rho,T} d\mu$$

$$\frac{dP}{P} = \left(\frac{\partial \log P}{\partial \log T}\right)_{\rho,\mu} \frac{dT}{T} + \left(\frac{\partial \log P}{\partial \log \rho}\right)_{\mu,T} \frac{d\rho}{\rho} + \left(\frac{\partial \log P}{\partial \log \mu}\right)_{\rho,T} \frac{d\mu}{\mu}$$

$$= \chi_T \frac{dT}{T} + \chi_\rho \frac{d\rho}{\rho} + \chi_\mu \frac{d\mu}{\mu}$$

These values "χ" are dimensionless numbers which basically give the powers of the given quantities to which P is sensitive.

e.g., for ideal gas P $\propto \frac{\rho T}{\mu}$ so $\chi_T = 1; \chi_\rho = 1; \chi_\mu = -1$

$$d\log P = d\log T + d\log\rho + -d\log\mu$$

continuing:

$$d\log P = \chi_T d\log T + \chi_\rho d\log\rho + \chi_\mu d\log\mu$$

$$\chi_\rho \frac{d\log\rho}{d\log P} = 1 - \chi_T \frac{d\log T}{d\log P} - \chi_\mu \frac{d\log\mu}{d\log P}$$

$$\frac{d\log\rho}{d\log P} = \frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho}\nabla - \frac{\chi_\mu}{\chi_\rho}\nabla_\mu$$

where $\nabla \equiv \frac{d\log T}{d\log P}$; $\quad \nabla_\mu \equiv \frac{d\log\mu}{d\log P}$

∇ here (not to be confused with the "grad" operator) is a dimensionless temperature gradient describing how T behaves with pressure.

∇ is usually a positive quantity. It is related to the radial temperature derivative by:

$$\nabla = \frac{d\log T}{d\log P} = \frac{P}{T}\frac{dT}{dP} = \frac{P}{T}\frac{dT}{dr}\left(\frac{dP}{dr}\right)^{-1} = -\frac{H_P}{T}\frac{dT}{dr} = \nabla$$

where $\mathrm{H}_P = \left(-\frac{1}{P}\frac{dP}{dr}\right)^{-1}$ is the pressure "scale helight",

the distance over which pressure declines by 1 e-fold

For ideal gas (only) in hydrostatic equilibrum

$$\mathrm{H}_P = -P / \frac{dP}{dr} = \frac{\rho N_A kT}{\mu}\frac{1}{\rho g} = \frac{N_A kT}{\mu g}; \quad \frac{H_P}{T} = \frac{N_A k}{\mu g}$$

Previously, $\dfrac{d\log\rho}{d\log P} = \dfrac{1}{\chi_\rho} - \dfrac{\chi_T}{\chi_\rho}\nabla - \dfrac{\chi_\mu}{\chi_\rho}\nabla_\mu$ and $\left(\dfrac{d\log\rho}{d\log P}\right)_{ad} = \dfrac{1}{\gamma_{ad}}$

For an adaibatic expansion at consant composition ($\nabla_\mu = 0$;

in the *displaced gas* the composition does not change)

$$\left(\frac{d\log\rho}{d\log P}\right)_{ad} = \frac{1}{\gamma_{ad}} = \frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho}\nabla_{ad} \Rightarrow \nabla_{ad} = \frac{1-\dfrac{\chi_\rho}{\gamma_{ad}}}{\chi_T} = \frac{\gamma_{ad}-\chi_\rho}{\gamma_{ad}\chi_T} = \nabla_{ad}$$

This is the "dimensionless adiabatic temperature gradient"

For ideal gas $\nabla_{ad} = \dfrac{5/3-1}{5/3} = 0.4$. For stability $\left(\dfrac{d\log\rho}{d\log P}\right)_{star} > \dfrac{1}{\gamma_{ad}}$, so

$$\frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho}\nabla - \frac{\chi_\mu}{\chi_\rho}\nabla_\mu > \frac{1}{\gamma_{ad}} \quad \text{but}$$

$$\frac{\gamma_{ad}-\chi_\rho}{\gamma_{ad}\chi_T} = \nabla_{ad} \Rightarrow \frac{1}{\chi_T} - \frac{1}{\gamma_{ad}}\left(\frac{\chi_\rho}{\chi_T}\right) = \nabla_{ad} \quad \text{and} \quad \left(\frac{\chi_T}{\chi_\rho}\right)\frac{1}{\chi_T} - \frac{1}{\gamma_{ad}} = \left(\frac{\chi_T}{\chi_\rho}\right)\nabla_{ad}$$

$$\frac{1}{\gamma_{ad}} = \frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho}\nabla_{ad}$$

continuing:

$$\frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho}\nabla - \frac{\chi_\mu}{\chi_\rho}\nabla_\mu > \frac{1}{\gamma_{ad}} = \frac{1}{\chi_\rho} - \frac{\chi_T}{\chi_\rho}\nabla_{ad}$$

$$-\frac{\chi_T}{\chi_\rho}\nabla - \frac{\chi_\mu}{\chi_\rho}\nabla_\mu > -\frac{\chi_T}{\chi_\rho}\nabla_{ad}$$

$$-\chi_T\nabla - \chi_\mu\nabla_\mu > -\chi_T\nabla_{ad} \qquad \nabla \equiv \frac{d\log T}{d\log P}$$

$$\chi_T\nabla + \chi_\mu\nabla_\mu < +\chi_T\nabla_{ad}$$

for stabiity $\nabla < \nabla_{ad} - \frac{\chi_\mu}{\chi_T}\nabla_\mu$ *Ledoux*

and if $\nabla_\mu = 0$ (or is ignored) $\nabla < \nabla_{ad}$ *Schwarzschild*

The dimensionless *radiative* temperature gradient is given by the chain rule and hydrostatic equilibrium and radiative diffusion

$$\frac{dT}{dm} = \frac{dP}{dm} \bullet \frac{dT}{dP} = \frac{-Gm}{4\pi r^4} \frac{T}{P} \frac{d\log T}{d\log P} = -\frac{3}{4ac} \frac{\kappa}{T^3} \left(\frac{L(r)}{\left(4\pi r^2\right)^2} \right)$$

$$\nabla_{rad} = \left.\frac{d\log T}{d\log P}\right)_{rad} = \frac{3}{16\pi acG} \frac{\kappa L(r) P}{mT^4}$$

The Schwarzschild condition for stability becomes, e.g.

$$\nabla_{rad} < \nabla_{ad}$$

For ideal gas for example $\gamma_a = 5/3, \chi_\rho = 1, \chi_T = 1$

$$\nabla_{ad} = \frac{\gamma_{ad} - \chi_\rho}{\gamma_{ad}\chi_T} = \frac{2/3}{5/3} = 0.4$$

For radiation $\gamma_a = 4/3, \chi_\rho = 0, \chi_T = 4$

$$\nabla_{ad} = \frac{\gamma_{ad} - \chi_\rho}{\gamma_{ad}\chi_T} = \frac{4/3}{16/3} = 0.25$$

Maximum luminosity for convective stability:

For stability: $\nabla_{rad} < \nabla_{ad}$

$$\frac{3}{16\pi acG}\frac{P}{T^4}\frac{\kappa L(r)}{m} < \nabla_{ad} \sim 0.4 \text{ (ideal gas)}$$

$$\frac{3}{16\pi acG}\frac{N_A k\rho}{\mu T^3}\frac{\kappa L(r)}{m} < 0.4$$

$$L(r) < 0.4\,\frac{16\pi acG\mu T^3 m(r)}{3N_A k\rho\kappa}$$

$$< 1.22\times10^{-18}\frac{\mu T^3}{\kappa\rho}m(r)\text{ erg s}^{-1}$$

$e.g.\ \mu = 0.61, \kappa\rho = 1, T = 10^6, M = M_\odot \qquad L < 1.5\times10^{33}\text{ erg s}^{-1}$

Where is convection important?

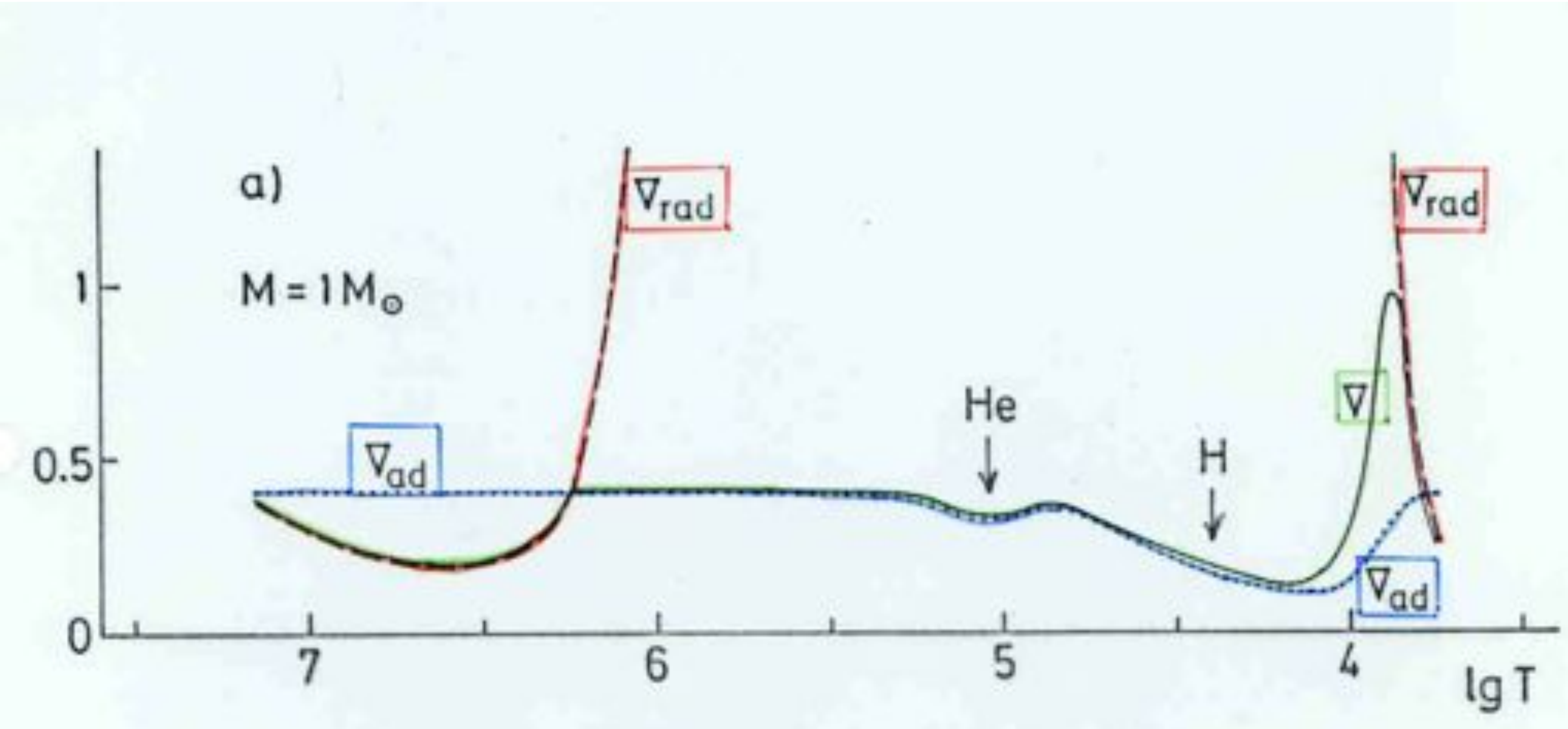
$$L(r) > 1.22 \times 10^{-18} \frac{\mu T^3}{\kappa \rho} m(r) \text{ erg s}^{-1}$$

Convection will be important in regions where the opacity is high or the energy generation is concentrated in a small mass giving a large ratio of L(r)/m(r). Examples:

- The cores of massive stars powered by the CNO cycle which is very temperature sensitive.

- Regions of high opacity especially in ionization zones and near stellar surfaces. Kramers opacity increases at low T.

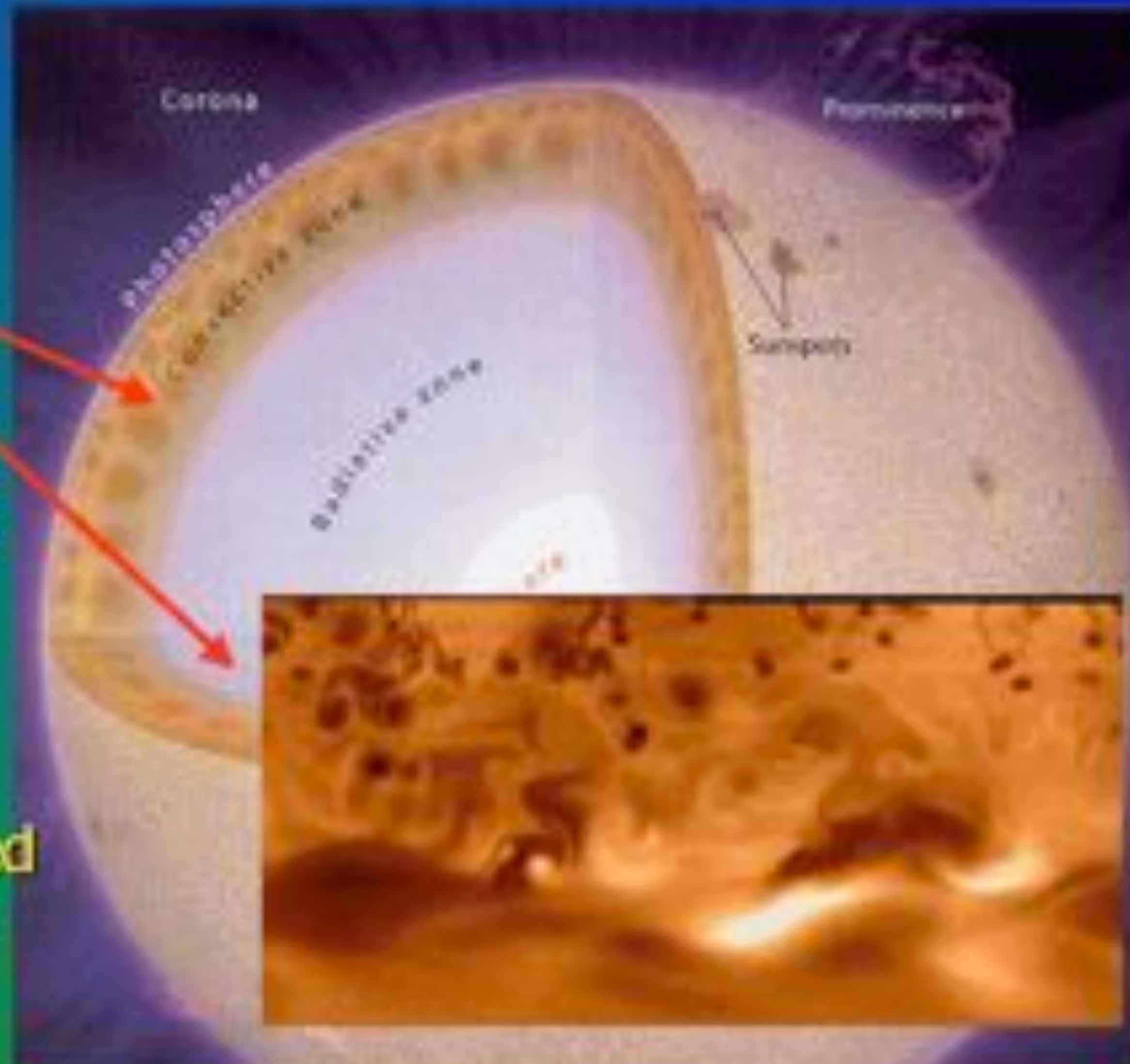
- The interiors of massive stars where L/M is large



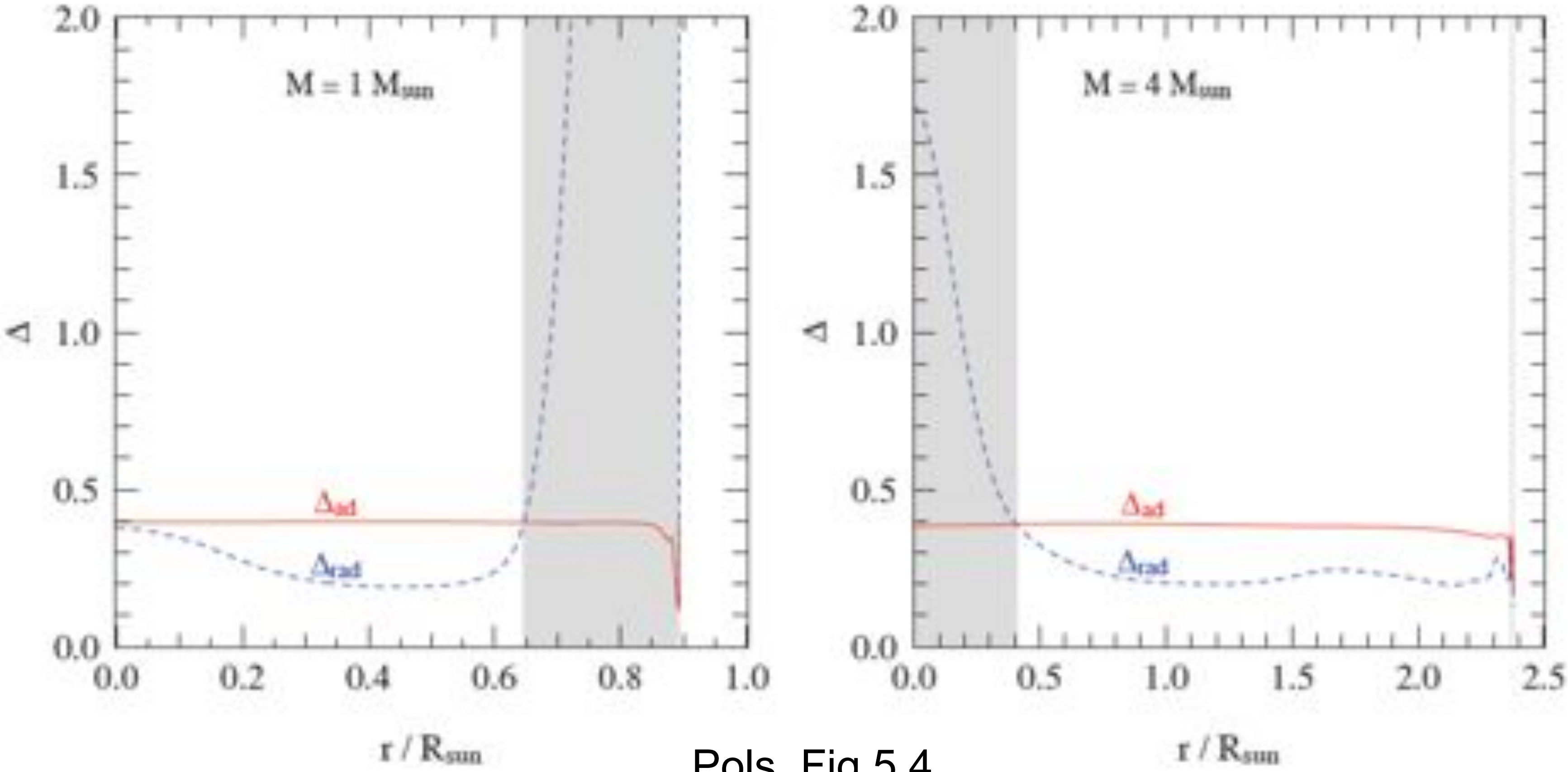
The sun – Manfred Schussler
Max Plank Insitut fur Aeronomie

The solar convection zone

- 200 Mm thick layer in turbulent motion
- Velocities range from 100 m/s (bottom) to 10 km/s (top)
- Energy flux nearly completely transported by convective motion

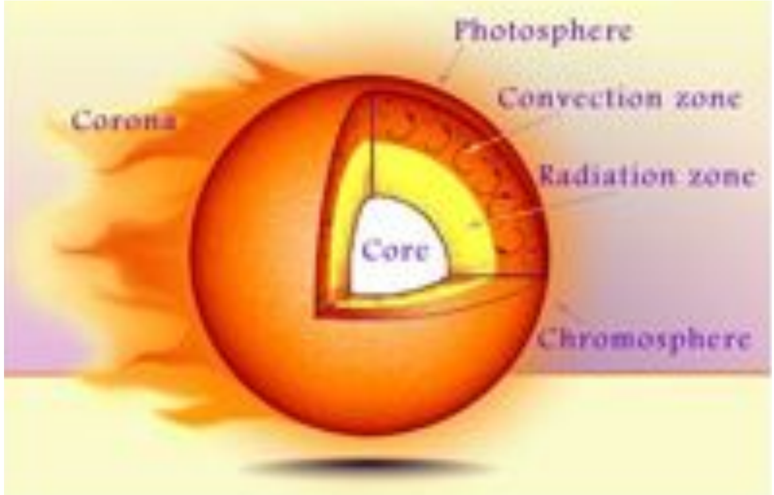


Convection in the sun and massive stars



Pols, Fig 5.4

Main sequence stars



The sun

Heat transport by convection and mixing length theory

In a region where the temperature gradient is even slightly superadiabatic, there will be efficient transport of energy by advection. Carrying entrained energy and dispersing it after some distance is much more efficient than radiative diffusion or conduction.

It turns out that the efficiency of the convection can be simply and fairly accurately (based upon 3D simulations) characterized by a representative length scale called the "mixing length". This is how far a relatively intact plume or bubble of hot gas will rise adiabatically before dispersing and depositing its energy by diffusive and turbulent processes.

To conserve mass, there will also be matter falling back down from large radii to smaller ones, but the down-flowing matter now carries less energy than when it went up, the difference having now been dissipated. The net heat being transported up (per gram) is approximately the heat capacity, $c_P = \left(\frac{\partial u}{\partial T}\right)_P$, times the temperature difference, ΔT, between the gas inside the convective plume and outside of it evaluated at $r + \ell_m$ where r is the radius where the transport starts and ℓ_m is the mixing length.

That is, ΔT is given by the difference between the temperature gradient in the external medium times ℓ_m, and the temperature change that happens to a rising adiabatic bubble after floating the same distance.

Heat transport = $C_P \Delta T$

$$\Delta T = \left[\left(\frac{dT}{dr}\right)_{ad} - \left(\frac{dT}{dr}\right)_{actual}\right]\ell_m = \Delta\left(\frac{dT}{dr}\right)\ell_m \quad > 0$$

$$\frac{dT}{dr} = T\frac{d\log T}{dr} = T\frac{d\log T}{d\log P}\frac{d\log P}{dr} = T\nabla\frac{d\log P}{dr}$$

$$= -\frac{T}{H_P}\nabla$$ as previously discussed (slide 14)

Note:

1) The "mixing length", ℓ_m, is a characteristic scale over which a given convection element dissipates. The total extent of a convective region can itself be greater (or even less) than the mixing length.

2) ∇ connects directy to the equation of state, pressure is a better variable here than radius. ∇ is dimensionless and the log's are base "e".

$$\Delta T = \left[\left(\frac{dT}{dr} \right)_{ad} - \left(\frac{dT}{dr} \right)_{actual} \right] \ell_m$$

$$\frac{dT}{dr} = T \frac{d \log T}{dr} = T \frac{d \log T}{d \log P} \frac{d \log P}{dr} = T \nabla \frac{d \log P}{dr}$$

3) We defined the reciprocal of $\frac{1}{P}\left(\frac{dP}{dr}\right) = \frac{d\log P}{dr}$ as the "pressure scale height" (the radial distance over which the pressure declines by a factor of *e*). Almost universally the mixing length ℓ_m is taken to be a (parametric) multiplier, α, times the pressure scale height.

$$\ell_m = -\alpha\left(\frac{d\log P}{dr}\right)^{-1} = \alpha H_p$$

$$0.3 < \alpha < 1.5$$

$\frac{d\log P}{dr}$ is < 0

$H_P = -\frac{d\log P}{dr}$ is > 0

Then

$$\ell_m = -\alpha\left(\frac{d\log P}{dr}\right)^{-1} = \alpha H_p \Rightarrow \frac{\ell_m}{H_P} = -\alpha$$

$$\Delta T = \left[\left(\frac{dT}{dr}\right)_{ad} - \left(\frac{dT}{dr}\right)_{actual}\right]\ell_m$$

$$= T\frac{\ell_m}{H_P}\left(\nabla_{ad} - \nabla\right) = \alpha T\left(\nabla - \nabla_{ad}\right)$$

Where

$$\nabla \equiv \frac{d\log T}{d\log P} \qquad \text{actual in the star}$$

$$\nabla_{ad} \equiv \frac{d\log T}{d\log P} \qquad \text{adiabatic from the EOS}$$

The heat being transported per unit area is then

$$F = v_c \rho \Delta u = v_c \rho c_P \Delta T$$

That is the flux is the mass flux in a given convective element, $v_c \rho$, times the excess heat content (erg/gm) in the element compared with its surroundings.

c_P, the heat capacity, can be obtained from the equation of state but is $\frac{5 N_A k}{2}$ for an ideal gas.

It remains to determine v_c, the typical convective speed. The acceleration, a, is given by

$$a = -\frac{\Delta\rho}{\rho} g$$

where $\Delta\rho$ is the density contrast between the floating plume and its surroundings, $\Delta\rho = \rho_{inside} - \rho_{outside} < 0$

As the plume floats ($v_c \ll$ sound speed), it remains in hydrostatic equilibrium like its surroundings and in pressure equilibrium with its surroundings. So $\Delta P/P = 0$.

For an ideal gas

$$\frac{dP}{P} = 0 = \frac{d\rho}{\rho} + \frac{dT}{T} - \frac{d\mu}{\mu}$$ so, if $\Delta\mu = 0$, $-\Delta\rho/\rho \approx \Delta T/T$.

$$a = -\frac{\Delta\rho}{\rho} g = \frac{\Delta T}{T} g \qquad where\ g = \frac{Gm(r)}{r^2}$$

Recall also $\Delta T = \alpha T\left(\nabla - \nabla_{ad}\right)$

The typical velocity while moving a distance $\ell_m = 1/2\ at^2$

$$v_c = \frac{\ell_m}{t} \approx \frac{\ell_m}{\sqrt{2\ell_m / a}} = \sqrt{\ell_m a / 2} = \sqrt{\frac{\ell_m g}{2}\left(\frac{\Delta T}{T}\right)}$$

$$v_c \approx \sqrt{\frac{\ell_m \alpha g}{2}\left(\nabla - \nabla_{ad}\right)} = \sqrt{\frac{\alpha^2 H_P g}{2}\left(\nabla - \nabla_{ad}\right)}$$

$$\ell_m = \alpha H_P$$

$$v_c \approx \sqrt{\frac{l_m \alpha g}{2}\left(\nabla - \nabla_{ad}\right)} = \sqrt{\frac{\alpha^2 H_P g}{2}\left(\nabla - \nabla_{ad}\right)}$$

and $F = \rho c_P \Delta T v_c$ $\qquad \Delta T = \alpha T\left(\nabla - \nabla_{ad}\right)$

$$= \rho c_P T\alpha\left(\nabla - \nabla_{ad}\right) v_c$$

$$= \rho c_P T\alpha^2 \sqrt{\frac{H_P g}{2}}\left(\nabla - \nabla_{ad}\right)^{3/2}$$

$$L(r) = 4\pi r^2 F = 4\pi r^2 \rho c_P T \alpha^2 \sqrt{\frac{H_P g}{2}}\left(\nabla - \nabla_{ad}\right)^{3/2}$$

Given the degree to which the actual temperature gradient is superadiabatic, $\left(\nabla - \nabla_{ad}\right)$, this gives the convective heat flux. Typically α = 1/3 to 1.5.

How superadiabatic is the temperature in the convection region?

$$L(r) = 4\pi r^2 \rho c_P T \alpha^2 \sqrt{\frac{H_P g}{2}} \left(\nabla - \nabla_{ad}\right)^{3/2}$$

Evaluate approximately for the solar convective zone, $r \approx R_\odot$,

$\rho \approx \bar{\rho}$ = 1.4 g cm^{-3}, $\bar{T}$ from Virial theorem about 3×10^6 K

For ideal gas and HE, $H_P = \left(\frac{1}{P}\frac{dP}{dr}\right)^{-1} = \frac{\rho N_A k T}{\mu g \rho} = \frac{N_A k T}{\mu g}$ $\qquad$ g cancels in $\sqrt{}$

$$L_\odot \approx 4\pi R_\odot^2 \bar{\rho} c_P \bar{T} \left(\frac{N_A k \bar{T}}{2\mu}\right)^{1/2} \left(\nabla - \nabla_{ad}\right)^{3/2}$$

$$\left(\nabla - \nabla_{ad}\right)^{3/2} = \frac{\left((2)(0.6)\right)^{1/2}\left(3.9\times 10^{33}\right)}{(4)(\pi)C_P(6.9\times 10^{10})^2(1.4)(3\times 10^6)[(6.02\times 10^{23})(1.38\times 10^{-16})(3\times 10^6)]^{1/2}}$$

$$= \frac{0.0011}{\frac{5}{2}N_A k} = 3.1\times 10^{-12}$$

$$\left(\nabla - \nabla_{ad}\right) \approx \frac{\Delta T}{T} \sim 10^{-8}$$

for α=1; $\quad \Delta T = \alpha T\left(\nabla - \nabla_{ad}\right)$

How superadiabatic is a convection region?

While one might quibble about factors of 2, the deviation from the adiabatic gradient is clearly very small. This is generally true in other places where convection happens as well. Exceptions: near stellar surfaces, in the outer layers of red supergiants, and in the final stages of massive star evolution.

Examining the various terms, ρ, T, R, L, we see that the excess scales as

$$\frac{\Delta T}{T} \propto L^{2/3}\left(\frac{R}{M}\right)^{5/3}$$

(Pols p 70)

At very high L, say 10^6 $L_\odot$ and large R the temperature gradient could start to become substantially superadiabatic. This rarely happens (SN progenitors), but the theory can also break down near stellar photospheres.

The small degree of super-adiabicity also implies that the required convective speeds are quite modest, very subsonic.

$$v_c = \sqrt{\frac{\alpha^2 H_P g}{2}\left(\nabla - \nabla_{ad}\right)}$$

$$\approx \sqrt{\frac{N_A kT}{2\mu}\left(\nabla - \nabla_{ad}\right)}$$

$$= 10^{-4}\left(\frac{N_A kT}{2\mu}\right)^{1/2} \sim 15 \text{ meters/s}$$

Note that this is also $\sim 10^{-4}\left(\frac{P}{\rho}\right)^{1/2} \approx 10^{-4}\ c_{sound}$

Stellar Stability

Glatzmaier and Krumholz 11 (p 127)

Stars may become unstable for a variety of reasons

- Insensitivity of the pressure to temperature as in a degenerate gas with a temperature sensitive energy generation rate

- Too great a component of radiation or relativistic degeneracy pressure

- Burning in too thin a shell

- Opacity and recombination driven pulsations

First, why are they usually stable in the first place?

Non-degenerate gas with some small amount of radiation

$$P = \frac{\rho N_A kT}{\mu} + \frac{1}{3} a T^4 \qquad u_{gas} = \frac{3}{2}\frac{P}{\rho} \qquad u_{rad} = 3\frac{P}{\rho}$$

The Virial theorem says that the pressure and binding energy are related by

$$\Omega = -3\int_0^M \frac{P}{\rho}\,dm$$

We can note immediately the neutral stability of a star supported entirely by radiation or relativistic particles. Then

$$E = U + \Omega$$

$$= 3\int_0^M \frac{P}{\rho}\,dm - 3\int_0^M \frac{P}{\rho}\,dm = 0$$

The star has no net energy and its radius is undefined. Such is the case with the Chandrasekhar mass white dwarf but also stars that are too massive and β too great.

But so long as ideal gas remains a significant contributor

$$\Omega = -3\int_0^M \left(\frac{2}{3} u_{gas} + \frac{1}{3} u_{rad} \right) dm = -\left(2U_{gas} + U_{rad} \right)$$

$$U_{gas} = -\frac{1}{2}(\Omega + U_{rad})$$

$$E = \Omega + U_{rad} + U_{gas} = \frac{1}{2}(\Omega + U_{rad}) = -U_{gas}$$

What does this say about the mass-averaged temperature?

$$\mathrm{U}_{gas} = \frac{3}{2}\frac{N_A k}{\mu}\int_0^M \frac{\rho T}{\rho} dm = \frac{3}{2}\frac{N_A k M \bar{T}}{\mu}$$

where the ρ in the denominator is because u is the internal energy *per gram* and

$$\bar{T} = \frac{1}{M}\int_0^M T\, dm$$

Conservation of energy then requires that

$$L_{nuc} - L = \frac{dE}{dt} = -\frac{3}{2} M \frac{N_A k}{\mu} \frac{d\bar{T}}{dt}$$

In thermal equilibrium the left hand side is zero, but assume a small positive imbalance $\delta L = L_{nuc} - L$

$$\frac{d\bar{T}}{dt} = -\frac{2}{3} \frac{\mu}{N_A k} \frac{\delta L}{M}$$

The negative sign is important here and relates again to the negative heat capacity of stars as imposed by the Virial theorem.

$(L_{nuc} - L) \uparrow \Rightarrow$ T goes down and vice versa

The star is stable.

Instability of a degenerate gas

Obviously if the pressure is insensitive to the temperature and the energy generation is very insensitive to the temperature, a very unstable situation exists that is prone to runaway. Temperature rises but there is no expansion and cooling. Energy generation rises and increases the temperature still more, etc.

To examine this in a bit more detail, consider the relation between central pressure and density we derived for polytropes.

$$P_c = C_n G M^{2/3} \rho_c^{4/3}$$

where C_n is a slowly varying function of polytropic index, but we might have n = 3 in mind.

Taking the time derivative and dividing by the original equation one has

$$\frac{dP}{dt} = \left[C_n G M^{2/3}\right] \frac{4}{3} \rho_c^{1/3} \frac{d\rho_c}{dt}$$

$$P = \left[C_n G M^{2/3}\right] \rho_c^{4/3}$$

dividing gives

$$\frac{dP}{P} = \frac{4}{3} \frac{d\rho_c}{\rho_c}$$

Now assume the pressure is of the form $\mathrm{P} = \mathrm{P}_0 \rho^a T^b$ where for ideal gas a = b = 1, for fully degenerate gas a = $\frac{4}{3}$ to $\frac{5}{3}$, $b = 0$, and so on.

A small perturbation in pressure thus perturbs the temperature and density according to

$$dP_c = P_0\left(a\rho_c^{a-1}T_c^b d\rho_c + b\rho_c^a T_c^{b-1} dT\right)$$

which when divided by $P_c = P_0\rho_c^a T_c^b$ gives

$$\frac{dP_c}{P_c} = a\frac{d\rho_c}{\rho_c} + b\frac{dT_c}{T_c}$$

Substituting the results for the polytrope, $\frac{dP_c}{P_c} = \frac{4}{3}\frac{d\rho_c}{\rho_c}$,

$$\frac{4}{3}\frac{d\rho_c}{\rho_c} = a\frac{d\rho_c}{\rho_c} + b\frac{dT_c}{T_c}$$

and

$$\frac{dT_c}{T_c} = \frac{1}{b}\left(\frac{4}{3} - a\right)\frac{d\rho_c}{\rho_c}$$

Cases $$\frac{dT_c}{T_c} = \frac{1}{b}\left(\frac{4}{3} - a\right)\frac{d\rho_c}{\rho_c}$$

1) Ideal gas - a = b = 1

$$\frac{dT_c}{T_c} = \frac{1}{3}\frac{d\rho_c}{\rho_c}$$

expansion ($d\rho_c < 0$)causes a decrease in T, stable

2) Degenerate gas - b << 1, a >4/3

$$\frac{dT_c}{T_c} = -\frac{1}{b}\left(a - \frac{4}{3}\right)\frac{d\rho_c}{\rho_c}$$

Expansion actully causes increased heating. a is always $\geq$ 4/3 for completely degenerate electrons. The temperature rises causing a runaway in energy generation that only stops when the gas has become sufficienty non-degenerate (a < 4/3).

Examples helium flash, Type Ia supernovae.

Thin shell instability

Let $r_{shell} = r_0 + dr \qquad dr \ll r_0 \qquad \delta m \ll M$

and $\delta m \ll m(r_{shell})$

The pressure in the shell is give by

$$P = -\int_{m(r_{shell})}^{M} \frac{Gm(r)}{4\pi r^4} dm = -\int_{r_{shell}}^{R} \frac{Gm(r)\rho}{r^2} dr$$

Change the pressure in the thin zone slightly, e.g., by burning.

This leads to a new outer radius $r_{shell} = r_0 + dr + \delta r$

$= \text{old } r_{shell} + \delta r \qquad (\delta r \text{ still } \ll r_{shell})$

$$P + \delta P = -\int_{m(r_{shell})}^{M} \frac{Gm(r)}{4\pi (r+\delta r)^4} dm \approx -\left(1 + \frac{\delta r}{r_{shell}}\right)^{-4} \int_{m(r_{shell})}^{M} \frac{Gm(r)}{4\pi r^4} dm$$

$$\approx \left(1 - 4\frac{\delta r}{r_{shell}}\right) P \qquad \text{(the integral is -P)}$$

$$1 + \frac{\delta P}{P} \approx 1 - 4\frac{\delta r}{r_{shell}} \Rightarrow \frac{\delta P}{P} = -4\frac{\delta r}{r_{shell}}$$

$$\frac{\delta P}{P} = -4\frac{\delta r}{r_{shell}}$$

The mass does not change when the zone expands

but $\delta m = 4\pi r_0^2\,\rho\, dr = 4\pi r_0^2(\rho+\delta\rho)(dr+\delta r)$

$$\rho+\delta\rho = \frac{\rho\, dr}{dr+\delta r} = \rho\left(1+\frac{\delta r}{dr}\right)^{-1} \approx \rho\left(1-\frac{\delta r}{dr}\right)$$

So $$\frac{\delta\rho}{\rho} \approx -\frac{\delta r}{dr} = -\frac{\delta r}{r_{shell}}\frac{r_{shell}}{dr} = \frac{1}{4}\frac{\delta P}{P}\frac{r_{shell}}{dr}$$

$$\frac{\delta P}{P} = 4\frac{\delta\rho}{\rho}\frac{dr}{r_{shell}}$$

$dr << r_{shell}$ so $\frac{\delta P}{P}$ is small even if $\frac{\delta\rho}{\rho}$ is large

$$\frac{\delta P}{P} = 4\frac{\delta\rho}{\rho}\frac{dr}{r_{shell}}$$

Using the same representation of the EOS as before,

$P = P_0 \rho^a \ T^b$ $$\frac{\delta P}{P} = a\frac{\delta\rho}{\rho} + b\frac{\delta T}{T} = 4\frac{\delta\rho}{\rho}\frac{dr}{r_{shell}}$$

So that

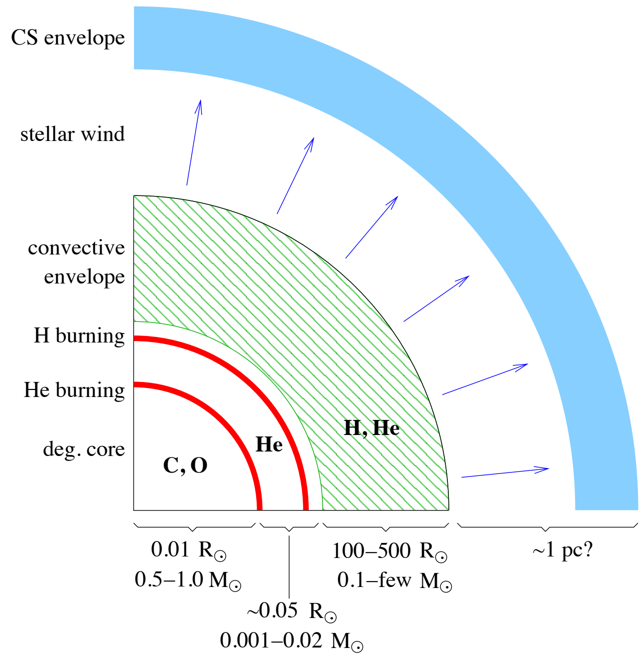
$$\frac{\delta\rho}{\rho}\left(a - 4\frac{dr}{r_{shell}}\right) = -b\frac{\delta T}{T}$$

$$\frac{\delta T}{T} = \frac{1}{b}\left(4\frac{dr}{r_{shell}} - a\right)$$

Since a and b are positive numbers and $dr \ll r_{shell}$ thin shells are always unstable. The instability is removed when dr becomes large or the fuel runs out.

Examples:

- Asymptotic Giant Branch Stars



Prialnik p 163 11.2

- Classical novae

- X-ray bursts on neutron stars

Dynamical instability

The instabilities discussed so far manifest over a long time – typically the nuclear time, but there are other more violent instabilities that affect hydrostatic equilibrium.

Quite basically, when a star, or a portion of a star contracts, its pressure goes up but so too does the force of gravity. If gravity goes up sufficiently faster than pressure, then an unbalanced state remains unbalanced and may collapse even faster. If on the other hand pressure goes up faster, the star when compressed will expand again and perhaps oscillate, but it is stable against collapse.

Very roughly:

$$\frac{dP}{dm} = -\frac{Gm(r)}{4\pi r^4} \quad \Rightarrow P \propto m^2 r^{-4}$$

Also $\rho \propto m / r^3$ so for hydrostatic equilibrium P $\propto m^{2/3}\rho^{4/3}$.

If the density changes due to expansion or contraction, leading to new pressure and density, P' and ρ', hydrostatic equilibrum demands

$$\left(\frac{P'}{P}\right) = \left(\frac{\rho'}{\rho}\right)^{4/3}$$

If the pressure increases more than this,

i.e., $\left(\frac{P'}{P}\right) > \left(\frac{\rho'}{\rho}\right)^{4/3}$ there will be a restoring force that will lead to expansion. If it is less, the contraction will continue and perhaps accelerate.

Now consider an adiabatic compression:

$$P = K\rho^{\gamma}$$

So $$\left(\frac{P'}{P}\right) = \left(\frac{\rho'}{\rho}\right)^{\gamma} \qquad \rho' > \rho$$

If, for a given mass, $\left(\frac{\rho'}{\rho}\right)^{\gamma} > \left(\frac{\rho'}{\rho}\right)^{4/3}$ one has stability, hydrostatic equilibrium is satisfied. If not, things are unstable.

Thus if $\gamma > \frac{4}{3}$ the star is stable and if $\gamma < \frac{4}{3}$ it is not.

This is a global analysis and doesn't necessarily apply to small regions of the star but illustrates the importance of $\gamma = 4/3$. We will return to this later.

Glatzmaier and Krumholz treat this more correctly using perturbation theory.

The force equation is

$$\ddot{r} = -\frac{Gm}{r^2} - \frac{1}{\rho}\frac{dP}{dr}$$

Now multiply by dm

$$dm\ddot{r} = -\frac{Gm}{r^2}dm - 4\pi r^2\ dP$$

Initially forces are balanced and

$$0 = -\frac{Gm}{r^2}dm - 4\pi r^2\ dP$$

But perturb and include $\ddot{r}$. A change in r by δr causes changes in P and ρ so that $r_0 \rightarrow r_0 + \delta r$, $P_0 \rightarrow P_0 + \delta P$, $\rho_0 \rightarrow \rho_0 + \delta\rho$, so

$$dm\frac{d^2}{dt^2}\left[r_0\left(1+\frac{\delta r}{r_0}\right)\right] = -\frac{Gm}{\left[r_0\left(1+\frac{\delta r}{r_0}\right)\right]^2}dm - 4\pi\left[r_0\left(1+\frac{\delta r}{r_0}\right)\right]^2 d\left[P_0\left(1+\frac{\delta P}{P_0}\right)\right]$$

Keep only first order terms $(1+\varepsilon)^n \approx 1 + n\varepsilon$

$$dm \frac{d^2}{dt^2}\left[r_0\left(1+\frac{\delta r}{r_0}\right)\right] = -\frac{Gm}{\left[r_0\left(1+\frac{\delta r}{r_0}\right)\right]^2} dm - 4\pi\left[r_0\left(1+\frac{\delta r}{r_0}\right)\right]^2 d\left[P_0\left(1+\frac{\delta P}{P_0}\right)\right]$$

$$dm\ \delta\ddot{r} = \left(1-2\frac{\delta r}{r_0}\right)\frac{Gm}{r_0^2} dm - \left(1+2\frac{\delta r}{r_0}+\frac{\delta P}{P_0}\right)4\pi r_0^2\, dP_0$$

but $0 = -\frac{Gm}{r_0^2} dm - 4\pi r_0^2\ dP_0$ so

$$dm\ \delta\ddot{r} = \frac{2Gm}{r_0^3}\delta r\ dm - \left(2\frac{\delta r}{r_0}+\frac{\delta P}{P_0}\right)4\pi r_0^2\, dP_0$$

To go further we must assume a behavior for P(ρ). Assume, because the time scale is too short for heat transport, that the compression is adiabatic, i.e., P = K ρ^γ.

Then since log P = log K+γ log ρ

$$d\log P = \frac{dP}{P} = \gamma\ d\ \log\rho = \gamma\ \frac{d\rho}{\rho}$$

$$\frac{dP}{P} = \gamma\ \frac{d\rho}{\rho}$$

Also linearizing

$$dm = 4\pi r^2 \rho\, dr$$

$$dm = 4\pi \left[r_0 \left(1 + \frac{\delta r}{r_0} \right) \right]^2 \rho_0 \left(1 + \frac{\delta\rho}{\rho_0} \right) dr_0 \left(1 + \frac{\delta r}{r_0} \right)$$

$$\approx 4\pi \left[r_0^2 \left(1 + \frac{2\delta r}{r_0} \right) \right] \rho_0 \left(1 + \frac{\delta\rho}{\rho_0} \right) dr_0 \left(1 + \frac{\delta r}{r_0} \right)$$

$$\approx 4\pi r_0^2 \rho_0 dr_0 \left(1 + \frac{2\delta r}{r_0} + \frac{\delta r}{r_0} + \frac{\delta\rho}{\rho_0} + ... \right) = 4\pi r_0^2 \rho_0 dr_0 \left(1 + \frac{3\delta r}{r_0} + \frac{\delta\rho}{\rho_0} \right)$$

But dm is precisely $4\pi r_0^2 \rho_0 dr_0$ so

$$\frac{\delta\rho}{\rho_0} = -\frac{3\delta r}{r_0} = \frac{1}{\gamma} \frac{\delta P}{P_0}$$

$$\frac{\delta P}{P_0} = -\frac{3\gamma\, \delta r}{r_0}$$

$$\frac{\delta P}{P_0} = -\frac{3\gamma\,\delta r}{r_0}$$

$$dm\,\delta\ddot{r} = \frac{2Gm}{r_0^3}\delta r\;dm - \left(2\frac{\delta r}{r_0} + \frac{\delta P}{P_0}\right)4\pi r_0^2\,dP_0$$

$$= \frac{2Gm}{r_0^3}\delta r\;dm - 4\pi r_0^2\left(2\frac{\delta r}{r_0} - \frac{3\gamma\,\delta r}{r_0}\right)dP_0$$

$$= \left[\frac{2Gm}{r_0^2}\,dm - 4\pi r_0^2\left(2-3\gamma\right)dP_0\right]\frac{\delta r}{r_0}$$

But for the unperturbed configuration we had

$$\frac{Gm}{r_0^2}\,dm = -4\pi r_0^2\,dP_0$$

So

$$dm\,\delta\ddot{r} = \left[-4\pi r_0^2\left(4-3\gamma\right)dP_0\right]\frac{\delta r}{r_0}$$

$$= \left[\frac{Gm}{r_0^2}\,dm\left(4-3\gamma\right)\right]\frac{\delta r}{r_0}$$

$$dm\,\delta\ddot{r} = \left[\frac{Gm}{r_0^2}\,dm\left(4-3\gamma\right)\right]\frac{\delta r}{r_0}$$

$$\delta\ddot{r} = -\left(3\gamma-4\right)\frac{Gm}{r_0^3}\delta r$$

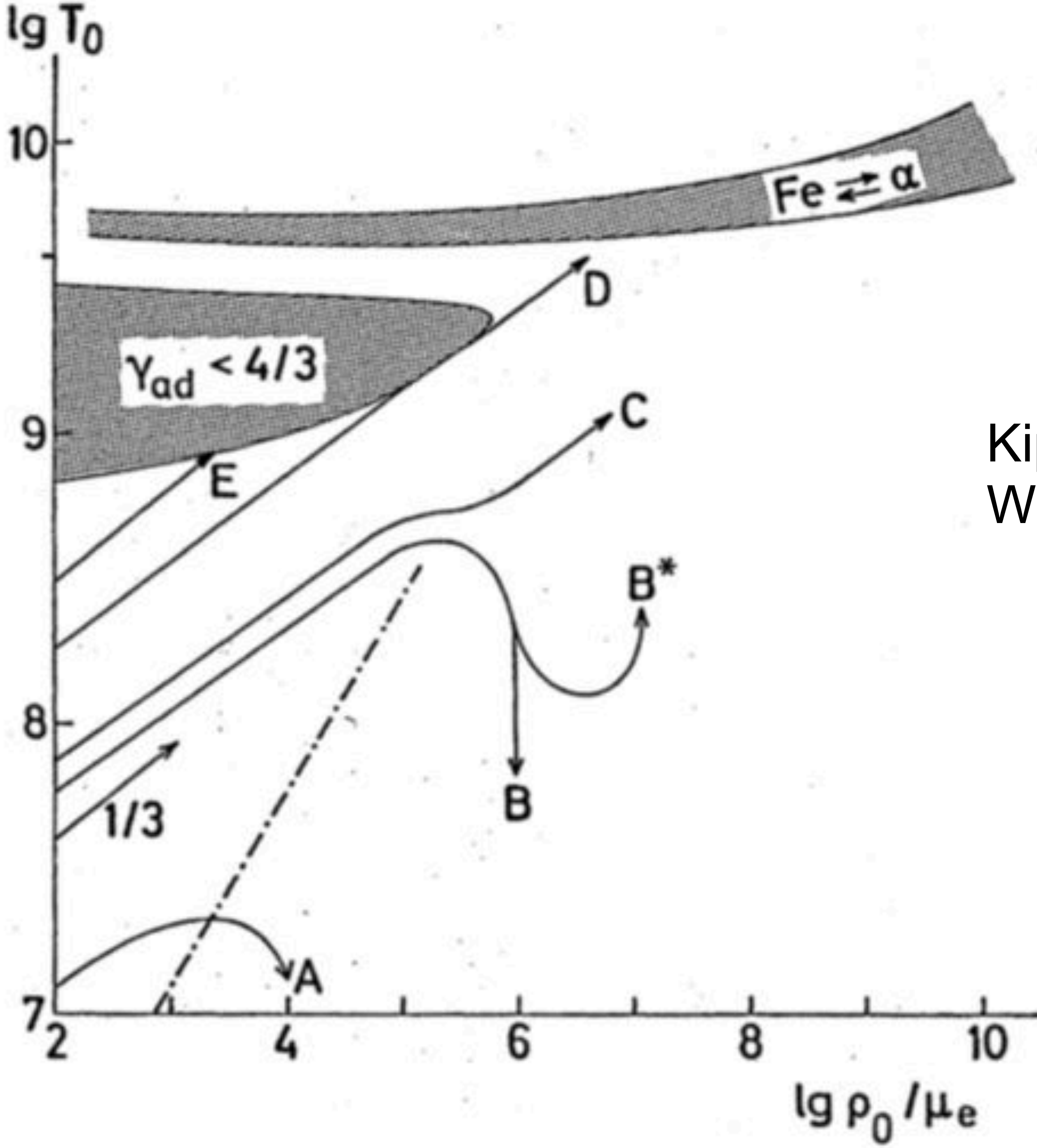
This is an equation for a harmonic oscillator. It has solution

$$\delta r = Ae^{i\omega t} \qquad \omega = \pm\sqrt{\left(3\gamma-4\right)\frac{Gm}{r_0^3}}$$

If ω is real, get oscillatory motion. If ω is imaginary ($\gamma < \frac{4}{3}$) get exponential growth or decay on a time scale

$$\tau \sim \frac{1}{i\omega} \sim \frac{1}{\sqrt{G\bar{\rho}}}$$

i.e., a hydrodynamic time scale



Kippenhahn and Weigert 34.1

Dynamic Instabilities

- Especially for stars with γ near 4/3, the nuclear energy generation may oscillate periodically. Compression leads to heating which leads to expansion and cooling but the net energy yield in a cycle is positive so the instability persists. This is called the *epsilon instability*

- Heat transport depends on the temperature gradient and opacity. If this combination behaves in such a way as to trap heat when the gas in the outer layers of a star is compressed and heated and release it when the gas expands and cools, then the star may be subject to the *kappa* instability

Pulsational instability by the κ mechanism requires that (Pols Chap 10, esp p 158)

$$\frac{d\log\kappa}{d\log P} > 0$$

or

$$\left.\frac{d\log\kappa}{d\log P}\right)_{ad} = \left.\frac{\partial\log\kappa}{\partial\log P}\right)_{T} + \left.\frac{\partial\log\kappa}{\partial\log T}\right)_{P} \frac{d\log T}{d\log P}$$

$$= \kappa_P + \kappa_T \nabla_{ad} > 0$$

For Kramer's opacity $\kappa \propto \rho \mathrm{T}^{-3.5}$ and if $\mathrm{P} \propto \rho \mathrm{T}$ $\kappa_P = 1$ and $\kappa_T = -4.5, \nabla_{ad} \approx 0.4,$ so the condition is not generally satisfied.

Two notable exceptions:

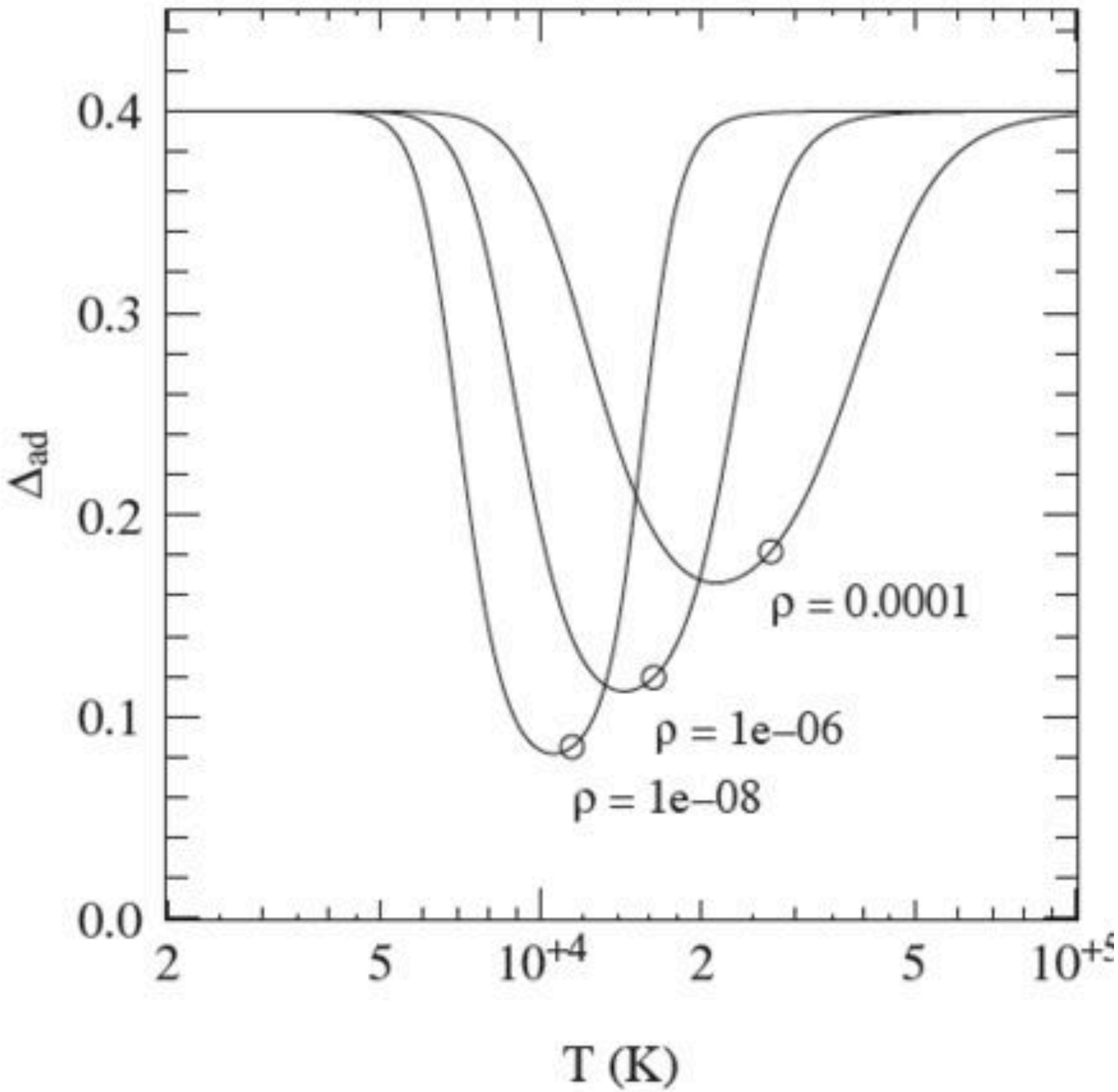
H^- opacity where the opacity depends on a positive power of the temperature (Mira variables)

Kramer's opacity but $\nabla_{ad} < 0.23$ as in ionization zones (Cepheid variables and RR-Lyrae stars)

There are two important ionization zones:

H I $\rightarrow$ H II and He I $\rightarrow$ He II both near 15,000 K

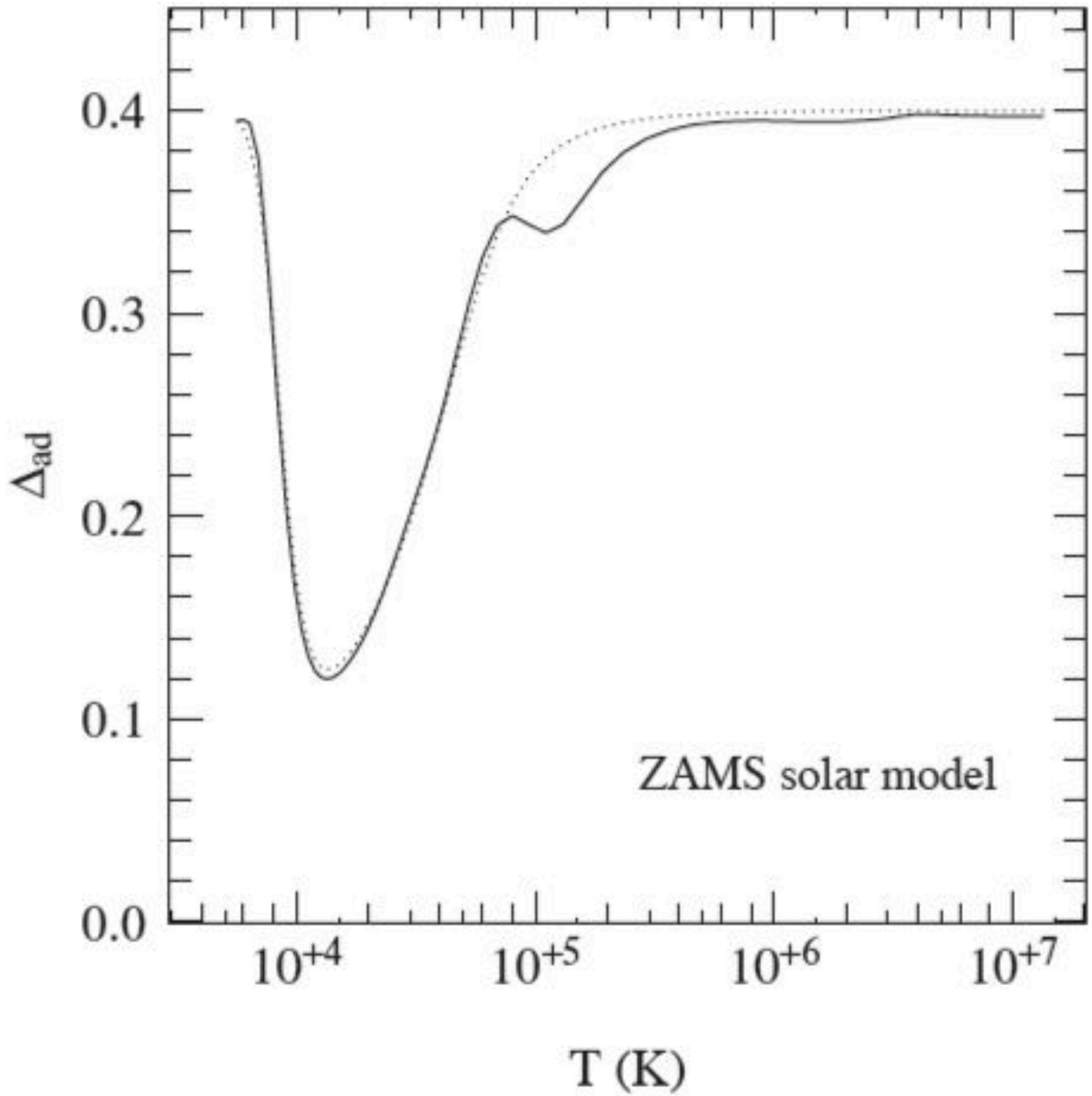
He II $\rightarrow$ He III (i.e., He^{++}) near 40,000 K



Pols 3.5 – suppression of the adiabatic index

$$\Delta_{ad} = \frac{d \log T}{d \log P}$$

by partial ionization of a pure hydrogen gas. D returns to its standard ideal gas value when the hydrogen is either neutral or fully ionized

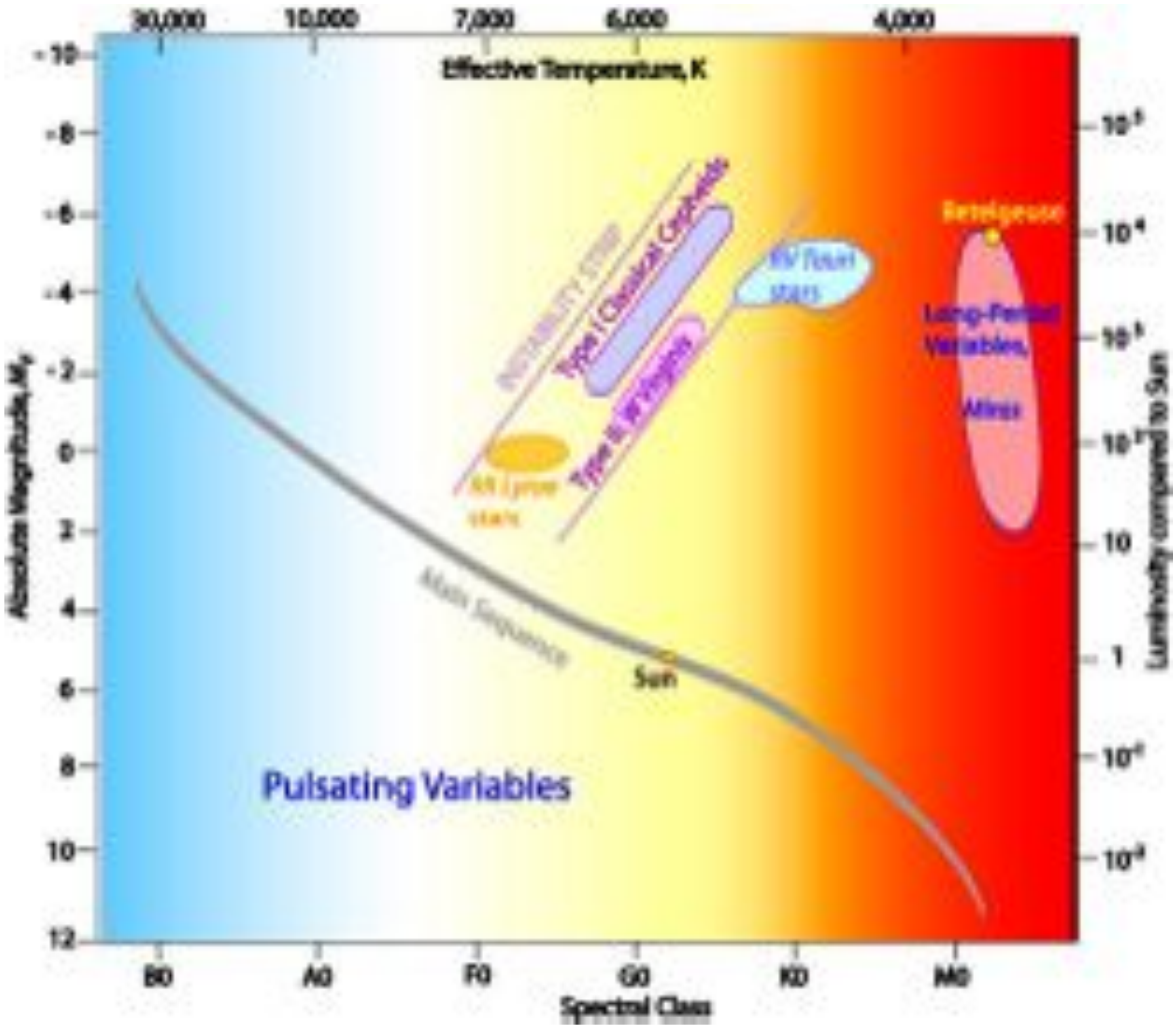


Ionization in the outer layers of the sun. The second shallow depression is He^{+} turning into He^{++}

If the ionization zone lies very close to the surface – and generally in a region that is already convective, it has little effect. Not enough mass (or heat capacity) participates. This is the case for T_{eff} > 7500 K

If the ionization zone is located too deep, at high density, the effect of partial ionization is diminished. The adiabatic gradient is not greatly supressed and the heat trapped is not enough to drive an oscillation of the heavy overlying material.
This happens if T_{eff} < 5500 K

Between 5500 and 7500 K, instability can ccur.



http://www.atnf.csiro.au/outreach/education/senior/astrophysics/variable_pulsating.html

Period-luminosity relation for Cepheids

As we showed previously when γ<4/3, the instability is dynamical

$$\tau = \left((3\gamma - 4)\frac{Gm}{r_0^3} \right)^{-1/2} \sim \frac{1}{\sqrt{G\bar{\rho}}}$$

For more massive Cepheids the radius is larger and the average density less, hence the period is longer

Also Cepeids obey a mass luminosity relation somewhat like main sequence stars. So brighter Cepheids have longer periods.

Obtaining a good match with the observations remains a challenge.

 2001 A&A

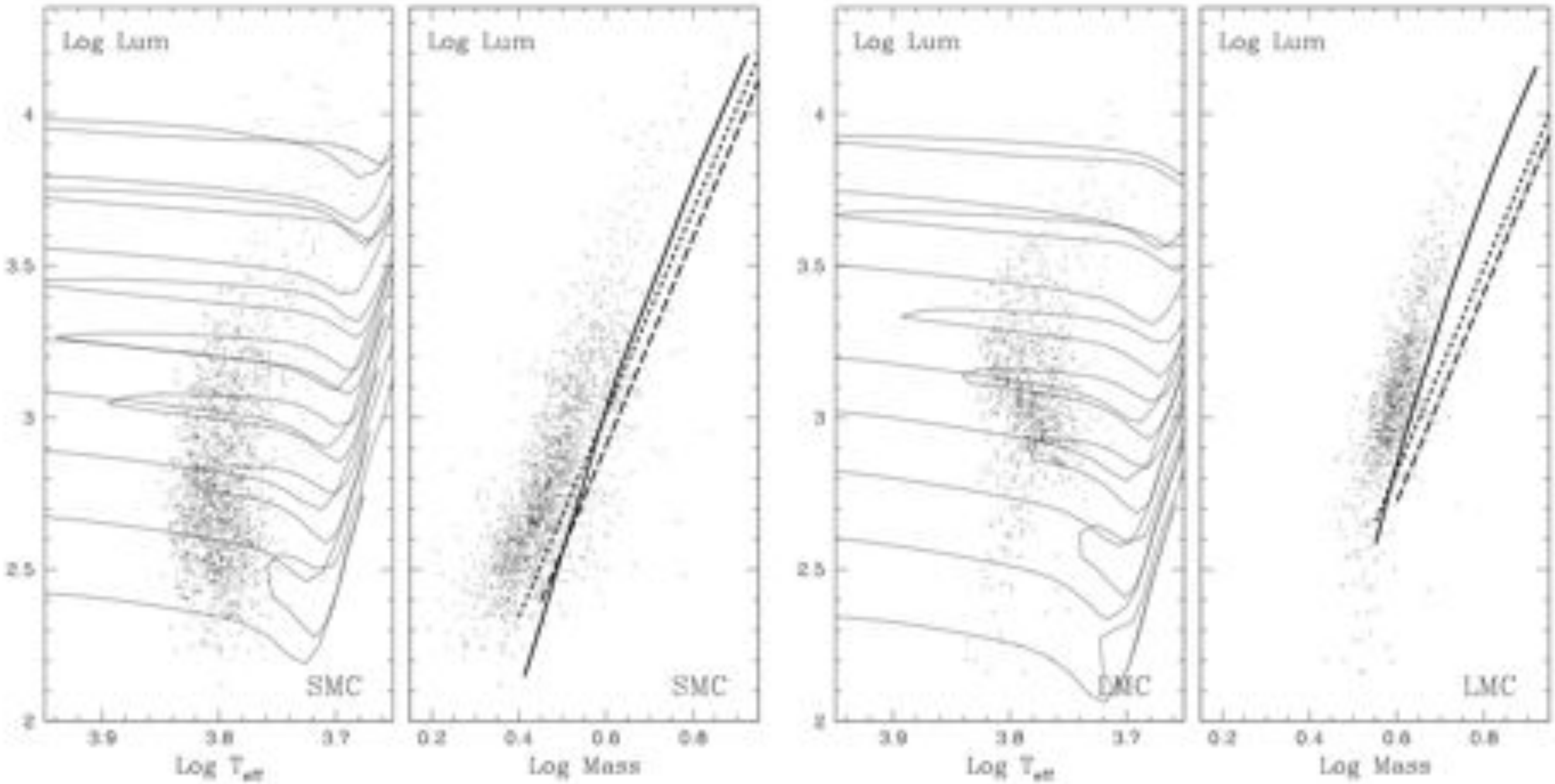


Fig. 5. Theoretical HR diagram and Mass–luminosity relations for SMC (the two left panels) and for LMC (the two right panels). As in Fig. 1, fundamental Cepheids are shown as solid and overtones as open circles calculations: Theoretical HR diagram with superposed evolutionary tracks from Girardi *et al.* (2000). M–L relations from evolutionary calculations; solid lines: Girardi *et al.* 2000, dotted lines: Alibert *et al.* 1999, dashed lines: Bono *et al.* 2000 .

Equivalence to Glatzmaier Chap 11

$$F=\rho c_P T\alpha^2\sqrt{1/2\,gH_p}\left(\nabla-\nabla_{ad}\right)^{3/2}$$

Substituting, for ideal gas and HE, $H_P=\left(\frac{1}{P}\frac{dP}{dr}\right)^{-1}=\frac{N_A kT}{\mu g}$

$$\nabla=\frac{H_P}{T}\left(\frac{dT}{dr}\right)\qquad\left(\nabla-\nabla_{ad}\right)=\frac{N_A k}{\mu g}\delta\left(\frac{dT}{dr}\right)$$

$$F=\rho c_P T\alpha^2\sqrt{1/2\,gH_p}\left(\nabla-\nabla_{ad}\right)^{3/2}$$

$$=\rho c_P T\alpha^2\left(\frac{N_A kT}{2\mu}\right)^{1/2}\left(\frac{N_A k}{\mu g}\delta\left(\frac{dT}{dr}\right)\right)^{3/2}$$

$$=\rho c_P\left(\frac{N_A k}{\mu}\right)^2\left(\frac{T}{g}\right)^{3/2}\frac{\alpha^2}{\sqrt{2}}\left[\delta\left(\frac{dT}{dr}\right)\right]^{3/2}$$

$$v_c \approx \sqrt{\frac{\alpha^2 H_P g}{2}\left(\nabla_{ad} - \nabla\right)}$$

$$\nabla - \nabla_{ad} = \frac{H_P}{T}\delta\left(\frac{dT}{dr}\right)$$

$$v_c = \sqrt{\frac{\alpha^2 H_P g}{2}\frac{H_P}{T}\delta\left(\frac{dT}{dr}\right)}$$

$$= \alpha H_P \sqrt{\frac{g}{2T}\delta\left(\frac{dT}{dr}\right)}$$

For an ideal gas and hydrostatic equilibrium

$$H_P = P / \left(\frac{dP}{dr}\right) = \frac{\rho N_A k T}{\mu g \rho} = \frac{N_A k T}{\mu g}$$

$$v_c = \sqrt{\frac{\alpha^2 H_P g}{2}\frac{H_P}{T}\delta\left(\frac{dT}{dr}\right)} = \alpha\frac{N_A k}{\mu}\sqrt{\frac{T}{2g}\delta\left(\frac{dT}{dr}\right)}$$

which is Glatzmaier if β=1/2