Lecture 1

Overview
Time Scales, Temperature-density Scalings, Critical Masses

The Virial Theorm implies that if a star is neither too degenerate nor too relativistic (radiation or pair dominated)

$$\frac{GM^2}{R} \sim MN_A kT \quad \text{(for an ideal gas)}$$

$$T \sim \frac{GM}{N_A kR}$$

$$R \sim \left(\frac{3M}{4\pi\rho}\right)^{1/3} \quad \text{for constant density}$$

So $$\boxed{T \sim \frac{GM^{2/3}}{N_A k}\rho^{1/3}}$$

That is as stars of ideal gas contract, they get hotter and since a given fuel (H, He, C etc) burns at about the same temperature, more massive stars will burn their fuels at lower density, i.e., higher entropy.

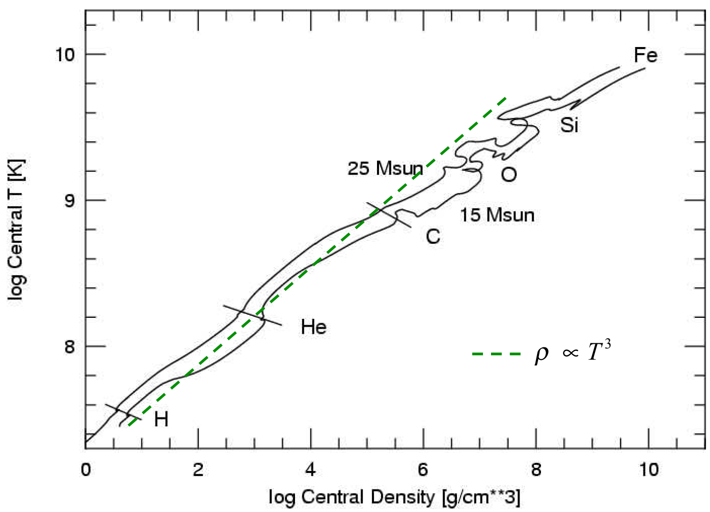
I. Preliminaries

The life of any star is a continual struggle between the force of gravity, seeking to reduce the star to a point, and pressure, which holds it up. *Stars are gravitationally confined thermonuclear reactors*.

So long as they remain non-degenerate and have not encountered the pair instability, overheating leads to expansion and cooling. Cooling, on the other hand, leads to contraction and heating. Hence stars are generally stable. The Virial Theorem works.

But, since ideal gas pressure depends on temperature, stars must remain hot. By being hot, they are compelled to radiate. In order to replenish the energy lost to radiation, stars must either contract or obtain energy from nuclear reactions. Since nuclear reactions change their composition, stars must evolve.

Central Conditions



at death the iron cores of massive stars are somewhat degenerate

Four important time scales for a (non-rotating) star to adjust its structure can be noted. The shortest by far is the time required to approach and maintain hydrostatic equilibrium. Stars not in a state of dynamical implosion or explosion maintain a balance between pressure and gravity on a few sound crossing times. The sound crossing time is typically comparable to the free fall time scale.

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$
$$P_{cent} \sim \frac{GM\rho}{R}$$

and

$$c_s = \gamma\left(\frac{P}{\rho}\right)^{1/2} \sim \left(\frac{GM}{R}\right)^{1/2}$$
$$v_{esc} \sim \left(\frac{GM}{R}\right)^{1/2}$$

So the escape speed and the sound speed in the deep interior are comparable

The free fall time scale is ~R/v_{esc} so

$$\tau_{esc} \sim \frac{R}{v_{esc}} = \left(\frac{R^3}{2GM}\right)^{1/2}$$
$$\sim \left(\frac{3}{8\pi G\rho}\right)^{1/2}$$

The number out front depends upon how the time scale is evaluated. The e-folding time for the density is 1/3 of this

$$\boxed{\tau_{HD} \sim \left(\frac{1}{24\pi G\rho}\right)^{1/2} \sim \frac{446}{\sqrt{\rho}}\ \text{sec}}$$

which is often used by the nuclear astrophysicist

The second relevant adjustment time, moving up in scale, is the thermal time scale given either by radiative diffusion, appropriately modified for convection. This is the time it takes for a star to come into and maintain thermal steady state, e.g., for the energy generated in the interior to balance that emitted in the form of radiation at the surface.

$$\boxed{\tau_{therm} = \frac{R^2}{D_{therm}}}$$

nb. These two time scales, τ_{HD} and τ_{therm} sometimes place limits on the accuracy with which jobs run in one code can be remapped into another (e.g., 1D -> 3D). τ_{KH} can also be important

The thermal diffusion coefficient (also called the thermal diffusivity) is defined as

$$D = \left(\frac{conductivity}{heat\ capacity}\right) = \frac{K}{C_P\rho}$$

where K appears in Fourier's equation

heat flow = $-K\ \nabla T$

For radiative diffusion the "conductivity", K, is given by

$$K = \frac{4acT^3}{3\kappa\rho} \qquad \text{(see Clayton 3-12)}$$

where κ is the opacity (cm^2 g^{-1}), thus

$$D = \boxed{\left(\frac{4a\,cT^3}{3\kappa C_P\rho^2}\right)}$$

where C_P is the heat capacity (erg g^{-1} K^{-1})

D here thus has units cm^2 s^{-1}

Note that

$$D \approx \left(\frac{c}{\kappa\rho}\right)\left(\frac{aT^4}{\rho C_P T}\right) = \left(\frac{c}{\kappa\rho}\right)\left(\frac{radiation\ energy\ content}{total\ heat\ content}\right)$$

If radiation energy density is a substantial fraction of the internal energy (not true in the usual case), $D \sim c/\kappa\rho$ with κ the opacity, and taking advantage of the fact that in massive stars electron scattering dominates so that $\kappa = 0.2$ to $0.4\ \mathrm{cm}^2\ \mathrm{g}^{-1}$, the thermal time scale then scales like

$$\boxed{\tau_{rad} \approx \frac{R^2\kappa\rho}{c} \sim \left(\frac{0.2R^2}{c}\right)\left(\frac{3M}{4\pi R^3}\right) \propto \frac{M}{R}}$$

however, massive stars have convective cores so the thermal time is generally governed by the diffusion time in their outer layers. Since the dimensions are still (several) solar radii while the densities are less and the opacity about the same, the radiative time scales are somewhat less than the sun ($\sim 10^5$ yr; Mitalas and Sills, ApJ, 401, 759 (1992)).

Possible point of confusion. The radiative diffusion time vs the thermal time. The former is the time for a photon to diffuse out

$$\tau_{rad} \sim \frac{R^2}{\ell_{mfp}c} \sim \frac{R^2\kappa\rho}{c}$$

The latter is the time for an appreciable fraction of the heat content to diffuse out. It is longer if most of the heat is in the ions, not the radiation

$$\tau_{thermal} \sim \tau_{rad}\left(\frac{\textit{total thermal energy in star}}{\textit{radiation content of star}}\right)$$

In the sun these differ by a factor of about 100 and the thermal time is of order the Kelvin Helmholtz time

$$\tau_{thermal} \sim \tau_{rad}\left(\frac{E_{matter}}{E_{rad}}\right) \sim \left(\frac{E_{matter}}{L_{rad}}\right) \sim \left(\frac{\alpha GM^2}{2RL_{rad}}\right) = \tau_{KH}$$

http://link.springer.com/article/10.1023/A%3A1022952621810

In general a diffusion coefficient is given by

$$D \sim \frac{1}{3}v\,l$$

where v is a typical speed (light or a convective element) and l is a characteristic length scale (mean free path or pressure scale height)

A closely related time scale is the Kelvin Helmholtz time scale

$$\boxed{\tau_{KH} \approx \frac{GM^2}{RL} \propto \frac{M^{5/3}\rho^{1/3}}{L}} \qquad i.e.\ R \sim (M/\rho)^{1/3}$$

Except for very massive stars, L on the main sequence is proportional to M to roughly the power 2 to 4, and ρ decreases with M so the Kelvin Helmholtz time scale is shorter for more massive stars. Note that there are numerous Kelvin Helmholtz time scales for massive stars since they typically go through six stages of nuclear burning. During the stages after helium burning, L in the heavy element core is given by pair neutrino emission and the Kelvin Helmhotz time scale becomes quite short - e.g. a protoneutron star evolves in a few seconds.

Finally, there is the nuclear time,

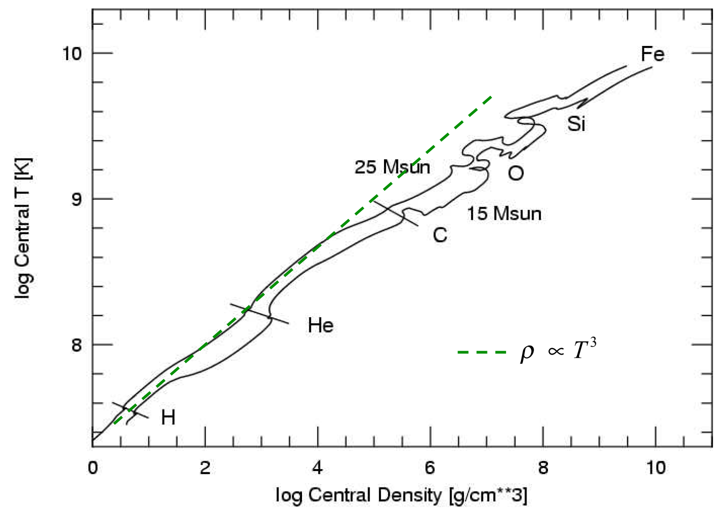
$$\boxed{\tau_{\rm nuc} = \left(\frac{1}{X}\frac{dX}{dt}\right)^{-1}},$$

where X is the mass fraction of the chief combustible fuel.

Usually, $\tau_{HD} < \tau_{\rm thermal} < \tau_{KH} < \tau_{\rm nuc}$. During the late stages of massive stellar evolution, however, the inequality $\tau_{\rm thermal} < \tau_{\rm nuc}$ actually begins to break down. During "explosive nucleosynthesis" in a supernova, there is near equality between $\tau_{\rm nuc}$ and τ_{HD}.

The life of a (non-degenerate) star is then typically a series of nuclear burning stages separated by periods of Kelvin-Helmholtz contraction. Hydrostatic equilibrium is maintained throughout the interior and thermal steady state is maintained if $\tau_{\rm therm}$ is short enough.

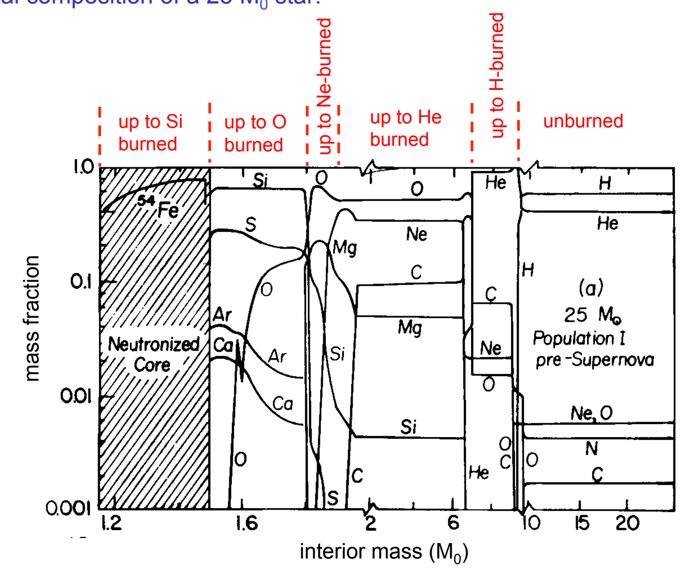
e.g., the sun is in thermal steady state. A presupernova star is not.



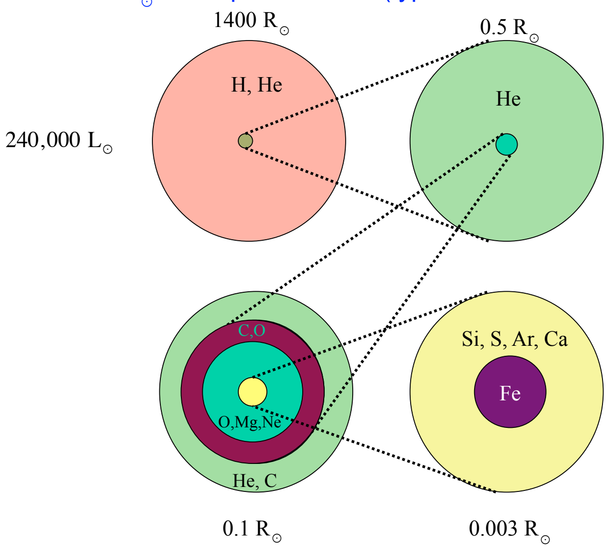
Advanced Nuclear Burning Stages
(e.g., 20 solar masses)

Fuel	Main Product	Secondary Products	Temp (10^9 K)	Time (yr)
H	He	^{14}N	0.02	10^7
He	C,O	^{18}O,^{22}Ne s- process	0.2	10^6
C	Ne, Mg	Na	0.8	10^3
Ne	O, Mg	Al, P	1.5	3
O	Si, S	Cl, Ar K, Ca	2.0	0.8
Si	Fe	Ti, V, Cr Mn, Co, Ni	3.5	1 week

Final composition of a 25 M_0 star:



$25\ M_\odot$ Presupernova Star (typical for $9 - 130\ M_\odot$)



Examples of Time Scales

- In stellar explosions, the relevant time scale is the hydrodynamic one.
- Explosive nucleosynthesis happens when $\tau_{\rm HD} \sim \ \tau_{\rm nuc}$
- During the first stages of stellar evolution $\tau_{\rm KH} < \tau_{\rm nuc}$. The evolution occurs on a Kelvin Helmholtz time
- In a massive presupernova star in its outer layers, $\tau_{\rm nuc} < \tau_{\rm thermal}$. The outer layers are not in thermal equilibrium with the interior. It also happens in these same stars that $\tau_{\rm nuc}\,(\text{core}) < \tau_{\rm KH}\,(\text{envelope})$. The core evolves like a separate star.
- Rotation and accretion can add additional time scales. E.g. Eddington Sweet vs nuclear. Accretion vs nuclear. Convection also has its own time scale.

Central Conditions for Polytropes

So long as a star is in hydrostatic equilibrium, it satisfies

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}.$$

If the density is assumed to be constant, and hence m(r) = const*r^3

$$\rho = \text{const} = \frac{3M}{4\pi R^3},$$

direct integration implies

$$P_c = \frac{1}{2}\frac{GM\rho_c}{R} \propto \frac{M^2}{R^4} \quad \propto GR^2\rho^2$$

where here $\rho_c = \bar{\rho} = \rho$.

It follows that

$$\frac{P_c}{\rho_c} = \frac{1}{2}GM\left(\frac{4\pi\rho_c}{3M}\right)^{1/3}$$

since

$$\frac{1}{R} = \left(\frac{4\pi\rho_c}{3M}\right)^{1/3}$$

and

$$\boxed{\frac{P_c^3}{\rho_c^4} = \frac{4\pi}{24}G^3M^2}$$

for constant density

More generally, for a polytrope of index n, $P \propto \rho^\gamma$; $\gamma = (n+1)/n$, see e.g., Clayton, Eq. 2-313

$$P_c = \frac{4\pi R^2 G}{(n+1)\zeta_1^2}\rho_c^2$$

$$\bar{\rho} = -\frac{3}{\zeta_1}\left(\frac{df}{d\zeta}\right)_{\zeta_1}\rho_c$$

$$= \frac{3M}{4\pi R^3}$$

where ζ_1 is the Emden constant given, e.g., in Table 2.5 of Clayton.

From this it follows that

$$\boxed{\frac{P_c^3}{\rho_c^4} = 4\pi G^3\left(\frac{M}{\phi}\right)^2}$$

where ϕ is a constant given by solution of the polytropic equation for index n,

$$\phi = (n+1)^{3/2}\zeta_1^2\left(\frac{df}{d\zeta}\right)_{\zeta_1}$$

$n = 0$ $\phi = 4.8988$ $= \sqrt{24}$
$n = 3$ $\phi = 16.145$
$n = 3/2$ $\phi = 10.73$

most stars have
$1.5 < n < 3$

Now, if P is P_{ideal} (NR, ND, ionized),

$$P_{\text{ideal}} = \frac{N_A k}{\mu} \rho T$$

where μ is the mean molecular weight

Aside: Abundance nomenclature

In general the mass fraction of a species "i" is X_i. The number density of i is then

$$n_i = \rho N_A \frac{X_i}{A_i}$$

with A_i the atomic mass number (integer) of isotope i and N_A, Avogadro's number, 6.02205×10^{23} particles/mole, or approximately the reciprocal mass of the nucleon in grams.

In this class we will extensively use the notation

$$Y_i = \frac{X_i}{A_i}$$

where Y_i is like a dimensionless number density

$$Y_i = \frac{n_i}{\rho N_A}$$

Similarly we can define an electron abundance variable

$$Y_e = \frac{n_e}{\rho N_A}$$

Actually the dimensions of Y are Mole/gm and N_A has dimensions particles per Mole.

The total gas pressure for an ideal, non-relativistic, non-degenerate ionized gas is then

$$\boxed{P_{\text{ideal}} = \rho N_A k T \left[\Sigma\, Y_i + Y_e\right]}$$

$$P = \Sigma\, n_i\, kT = \frac{N_A k}{\mu} \rho T$$

which implies

$$\mu = \left[\Sigma\, Y_i + Y_e\right]^{-1}$$

Also the mean atomic weight, $\bar{A}$, is given by

$$\bar{A} = \frac{\Sigma\, n_i A_i}{\Sigma\, n_i} = \frac{\rho N_A \Sigma\, Y_i A_i}{\rho N_A \Sigma\, Y_i} = \frac{\Sigma\, X_i}{\Sigma\, Y_i}$$
$$= (\Sigma\, Y_i)^{-1}$$

X_i= mass fraction of species "i"

Similarly

$$\boxed{Y_e = \Sigma\, Z_i Y_i}$$

$$\rho N_A Y_e = n_e = \sum Z_i n_i$$
$$= \rho N_A \sum Z_i Y_i$$

and

$$\boxed{\mu = (\Sigma\, (1 + Z_i) Y_i)^{-1}}$$

$0.5 < \mu < 2$

Some examples:

a) Pure hydrogen:

$$Y_H = 1 \qquad \bar{A} = 1 \qquad Y_e = 1$$
$$\mu = (1+1)^{-1} = \frac{1}{2}$$
$$P_{\text{ideal}} = 2 \rho N_A k T$$

(The limit μ=2 is achieved as A goes to infinity and Z = A/2, i.e. electrons dominate)

b) 75% H, 25% He:

$$\bar{A} = \left(0.75 + \frac{0.25}{4}\right)^{-1}$$
$$= 1.23$$

$$Y_e = 0.75 + (2)(\frac{0.25}{4})$$
$$= 0.875$$

$$\mu = \left[(1+1)(0.75) \ + \ (1+2)(\frac{0.25}{4})\right]^{-1}$$
$$= 0.5926$$

$$P_{\text{ideal}} = 1.69\, \rho N_A k T$$

As an exercise to the reader, for pure helium, $\bar{A} = 4$, $Y_e = 0.50$, $\mu = 4/3$, and $P_{\text{ideal}} = 0.75 \rho N_A k T$. For a mixture of 50% ^{12}C and 50% ^{16}O, $\bar{A} = 13.71$, $Y_e = 0.50$ (as it always does for a gas of isotopes having neutron number = proton number), μ

$= 1.745$, and $P_{\rm ideal} = 0.573\rho N_A kT$, (0.50 from e$^-$; 0.073 for ions).

For advanced stages of evolution where $\bar{A} > 1$, most of the pressure is due to the electrons (and radiation)

Back to the main discussion:

$$\frac{P_c^3}{\rho_c^4} \propto M^2$$

thus implies for an ideal gas equation of state

$$P_c \propto \rho_c \frac{T_c}{\mu}$$

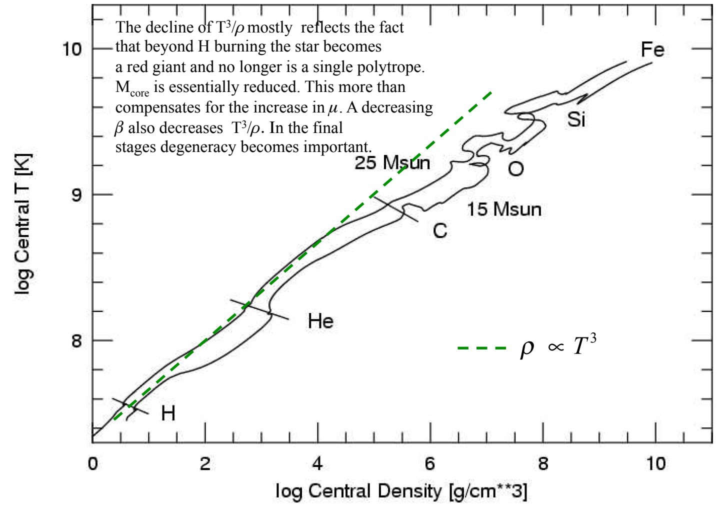
$$\boxed{\frac{T_c^3}{\rho_c} \propto M^2\mu^3}$$

This would suggest that the ratio would *increase* as the star evolved and μ became greater.

Therefore, for a given temperature, as might be necessary to burn a given fuel, for example, the central density will be lower for a star of higher mass. And, in fact, for a given constant mass and composition, so long as the star closely resembles a single polytrope, and the pressure remains ideal, the central density will scale as

$$\boxed{\rho_c \propto \left(\frac{T_c}{\mu}\right)^3}$$

$$T_c \propto \mu \rho_c^{1/3}$$



Since μ increases as the fuel burns to heavier ashes, the relation $\rho \propto T^3$ works pretty well at the stars center but tends to be an overestimate. The onset of degeneracy or near relativistic motion of the electrons at high temperature can also cause deviations.

Now, especially for massive stars, the radiation pressure will not be negligible. One traditionally defines a quantity

$$\beta = \frac{P_{\rm ideal}}{P_{\rm ideal} + P_{\rm rad}} = \left(\frac{P_{ideal}}{P_{total}}\right)$$

$$P_{\rm tot} = P_{\rm ideal}/\beta$$

$$P_{tot} \propto \frac{P_{ideal}}{\beta} \propto \rho_c \frac{T_c}{\mu\beta}$$

Then $P_c^3/\rho_c^4 \propto M^2$, implies

$$\boxed{\frac{T_c^3}{\rho_c} \propto M^2\beta^3\mu^3},$$

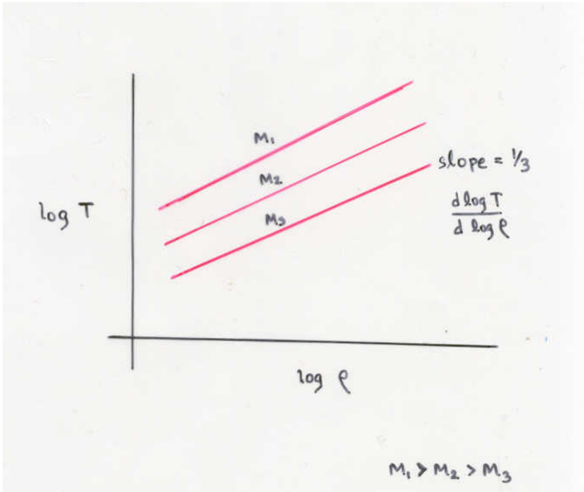
that is, so long as beta doesn't change much, one gets the same relation as before.

Decrease in β as star evolves acts to suppress T$^3/\rho$.

CRITICAL MASSES

Dropping, for now, the explicit dependence on μ and β, a contracting protostar of constant mass, or the contracting core of a massive star in between burning stages, so long as that core has an approximately constant polytropic index, will obey $T_c \propto \rho_c^{1/3} M^{2/3}$. Contraction leads to heating. The greater weight of the more compact configuration requires more pressure to hold it up and the pressure rises by increasing both T and ρ. A plot of log T$_c$ *vs.* log ρ_c gives a straight line with an upward slope of 1/3. Lines for larger mass will lie above those for lower mass. As the density grows ever higher, three possibilities emerge: a) collapse to a black hole; b) a dynamical event of some sort (e.g., neutron star formation) or c) the onset of degeneracy. For now, we are most interested in c).

Ideal gas plus radiation



A completely degenerate gas can be characterized by an equation of state of the form $P_{\rm deg} = K_\gamma(\rho Y_e)^\gamma$ with γ between 4/3 and 5/3.

The case $\gamma = 4/3$ has a well known singularity. For an n = 3 polytrope, which is appropriate here,

$$\frac{P_c^3}{\rho_c^4} = 4\pi G^3 \left(\frac{M}{16.14}\right)^2 = \frac{K_{4/3}^3 \rho_c^4 Y_e^4}{\rho_c^4}$$

$$M = \left(\frac{20.745 K_{4/3}^3}{G^3}\right)^{1/2} Y_e^2$$

$$K_{4/3} = 1.244 \times 10^{15} \ \text{dyne cm}^{-2}$$

note cancellation of ρ_c

$$\boxed{\frac{M}{M_\odot} = 5.80\ Y_e^2}$$

$= 1.45\, M_\odot$ if $Y_e = 0.50$ neglecting Coulomb corrections and relativistic corrections. 1.39 $M_\odot$ if they are included.

For lower densities and hence degenerate cores significantly less than the Chandrasekhar mass non-relativistic degeneracy pressure gives another solution (γ = 5/3)

$$\frac{P_c^3}{\rho_c^4} = 4\pi G^3 \left(\frac{M}{10.73}\right)^2$$

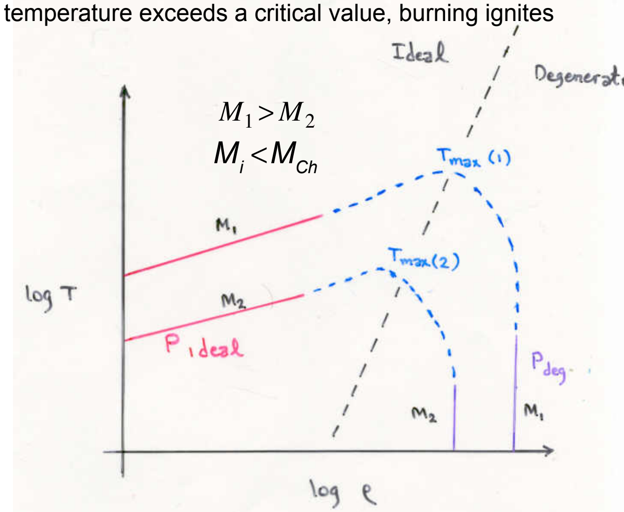
$$\rho_c = \frac{4\pi G^3 M^2}{K_{5/3}^3 Y_e^5 (10.73)^2}$$

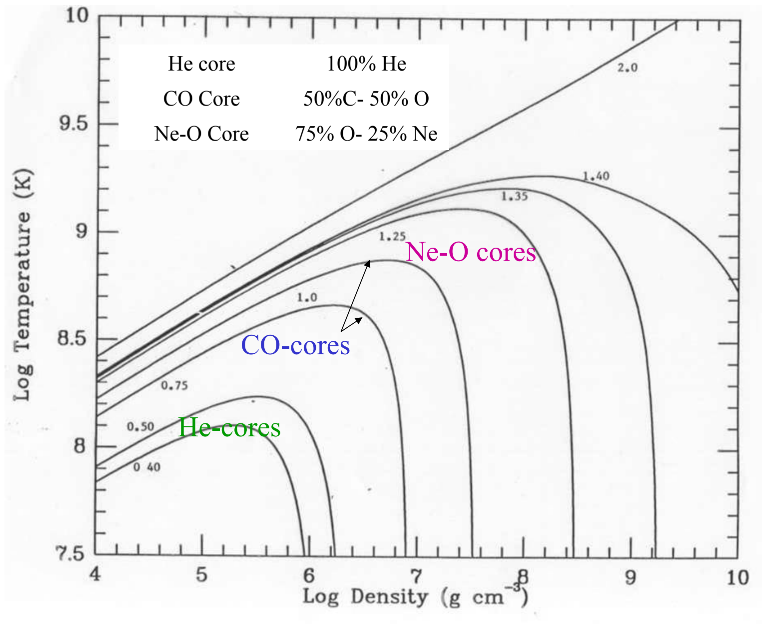
$$= 4.05 \times 10^6 \left(\frac{0.5}{Y_e}\right)^5 \left(\frac{M}{M_\odot}\right)^2 \ \text{g cm}^{-3}$$

This implies, for each mass, a stable permanent configuration of fixed ρ_c independent of T..

This is the well known central density mass relation for (non-relativistic) white dwarfs

Specifying a core mass gives a maximum temperature achieved before degeneracy supports the star. If that maximum temperature exceeds a critical value, burning ignites





Critical Temperatures

Fuel	Main Product	Secondary Products	Temp (10^9 K)	Time (yr)
H	He	^{14}N	0.02	10^7
He	C,O	^{18}O,^{22}Ne s- process	0.2	10^6
C	Ne, Mg	Na	0.8	10^3
Ne	O, Mg	Al, P	1.5	3
O	Si, S	Cl, Ar K, Ca	2.0	0.8
Si	Fe	Ti, V, Cr Mn, Co, Ni	3.5	1 week

These calculations give critical masses:

C	~1.0
Ne	~1.25
O	1.39
Si	1.39

More detailed and physical calculations exist in the literature, see especially Nomoto and Hashimoto (1986). The following should be regarded as standard

Fuel	Min. Mass
He	0.25
C	1.06
Ne	1.37
O	1.39
Si	1.39

These are core masses. The corresponding main sequence masses are larger.

All stars with main sequence mass above the Chandrasekhar mass could in principle go on to burn Si. In fact, that never happens below 9 solar masses. Stars develop a red giant structure with a low density surrounding a compact core. The convective envelope "dredges up" helium core material and causes it to shrink. Only for stars above about 7 or 8 solar masses does the He core stay greater than the Chandrasekhar mass after helium burning.

Main Sequence Critical Masses

0.08 $M_\odot$ Lower limit for hydrogen ignition

0.45 $M_\odot$ helium ignition

7.25 $M_\odot$ carbon ignition

9.00 $M_\odot$ neon, oxygen, silicon ignition (off center)

10.5 $M_\odot$ ignite all stages at the stellar center

~ 80 $M_\odot$ First encounter the pair instability (neglecting mass loss)

~35 $M_\odot$ Lose envelope if solar metallicity star

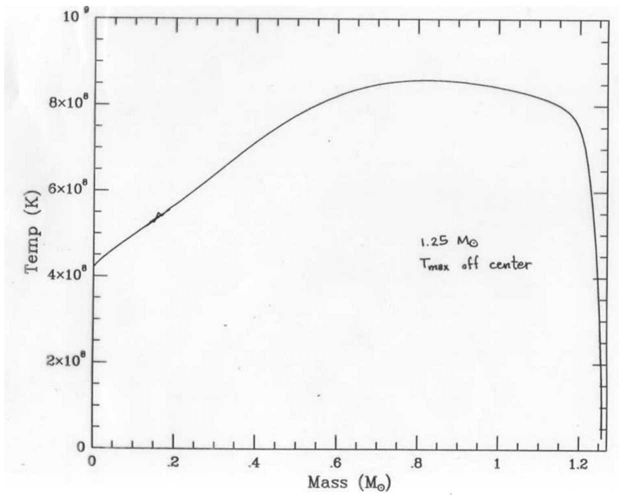
These are for models calculated with the KELER code including semiconvection and convective overshoot mixing but ignoring rotation. With rotation the numbers may be shifted to slightly lower values. Low metallicity may raise the numbers slightly since less initial He means a smaller helium core. Other codes give different results typically to within 1 solar mass.

Between 7.25 and 10.5 solar masses the evolution can be quite complicated and code dependent owing to the combined effects of degeneracy and neutrino losses. Off-center ignition is the norm for the post-carbon burning stages.

Mass loss introduces additional uncertainty, especially with regard to final outcome. Does a 8.5 solar mass main sequence star produce a NeO white dwarf or an electron-capture supernova (TBD).

Above 9 solar masses an iron core eventually forms – on up to the pair instability limit – around 150 solar masses.

Off-Center Ignition

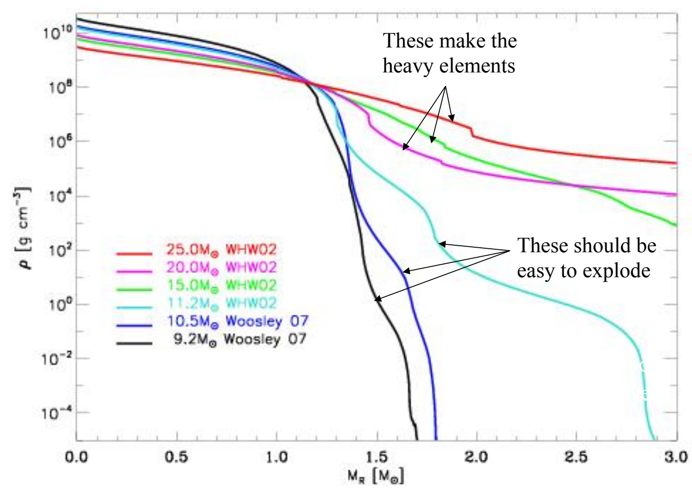


The death of a star and how it may potentially explode is also very sensitive to:

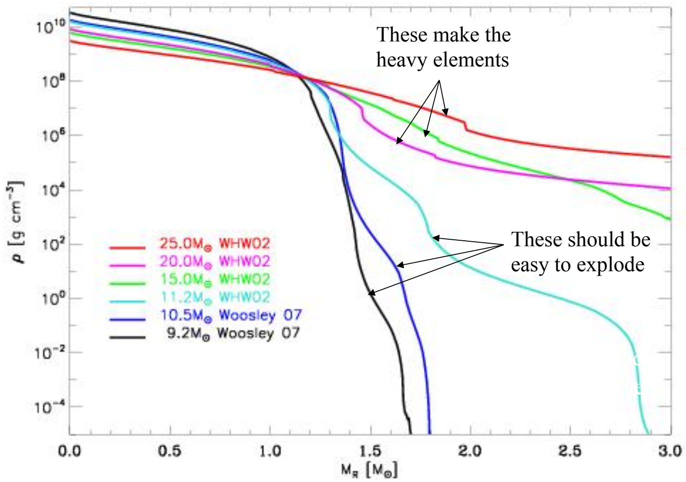
- *The density structure surrounding the iron core*
- *The rotation rate of the core and that material*

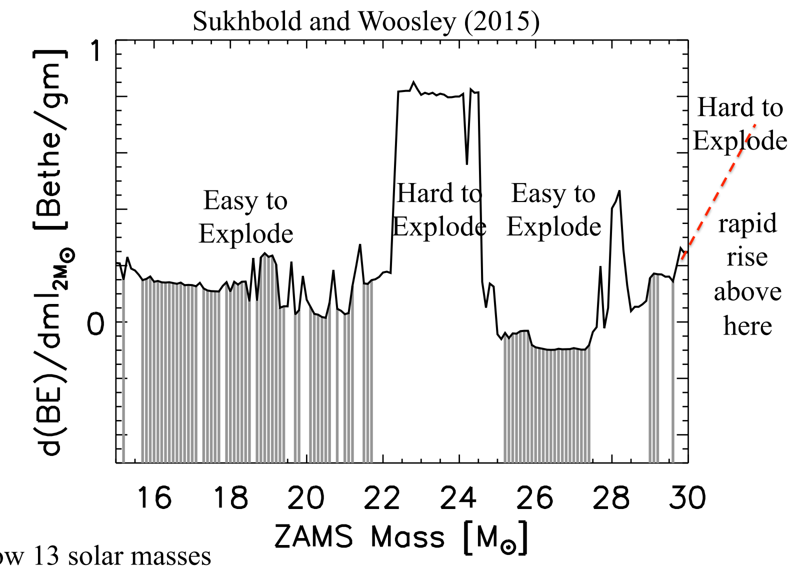
The density structure depends on the entropy of presupernova cores (TBD). Higher entropy cores occur for higher masses and are less degenerate and less centrally condensed.

Density Profiles of Supernova Progenitor Cores



Density Profiles of Supernova Progenitor Cores





Below 13 solar masses
very easy to explode

Mass (solar masses)	End point	Remnant
< 7 to ~8	planetary nebula	CO white dwarf
~8 to ~11	degenerate core low energy SN	Ne-O WD below 9? neutron star above 9
~11 - ~20	neutrino powered supernova; SN Ibc in binary. Islands of explosion at higher mass	neutron stars and black holes
20 - 85	without mass loss probably no SN (unless rotationally powered); with mass loss SN Ibc	black hole if enough mass loss neutron star
85 – 150	pulsational pair SN if low mass loss	black hole
150 - 260	pair instability SN if low mass loss	none
> 260	pair induced collapse if low mass loss	black hole