Lecture 6

p+p, Helium Burning and Energy Generation

Proton-proton reaction:

$$p(p,e^{+}\nu_e)^2\mathrm{H} \quad (+0.42\,\mathrm{MeV})$$

This cross section is far too small (~10^{-47} cm^2 at 1 MeV) to measure in the laboratory, but it does have a nearly constant, calculable S-factor.

The theory is straightforward, but complex (e.g., Clayton 366 - 368) because it includes a strong interaction and weak interaction happening in rapid succession .

Two stages:

- Temporarily form diproton (initial wave function is same as for proton scattering). Initial diproton must have J = 0 because can't have protons in identical states.

- Diproton experiences a weak interaction (with a spin flip) to make deuteron.

We shall be terse in our classroom discussion of this reaction, chiefly because it involves a lot of concepts we have not discussed so far (weak decays, axial/vector currents, etc), but also because it is unimportant in massive stars. Read Adelberger, RMP, pages 1272 – 1275 for background. This is given at the class website.

$$S(0)=6\pi^2 m_p c\alpha \ln 2\frac{\Lambda^2}{\gamma^3}\left(\frac{G_A}{G_V}\right)^2 \frac{f_{pp}^R}{(ft)_{0^+\to 0^+}}(1+\delta)^2$$

where α is the fine structure constant, m_p is the mass of the proton, c is the speed of light, G_V and G_A are the Fermi and axial vector weak-coupling constants, $\gamma = (2\mu E_D) =$ 0.23161 fm^{-1} is the deuteron binding wave number, μ is the proton-neutron reduced mass and E_D is the deuteron binding energy, $(\hbar=1)$, f_{pp}^R is the phase space factor, $(ft)_{0^+\to 0^+}$ is the (ft) value for the superallowed $0^+ \to 0^+$ transitions, Λ is proportional to the overlap of the *pp* and deuteron wave functions, and δ is a small correction to the nuclear force for the exchange of heavier mesons.

Λ^2 is given by the overlap integral between the initial *pp* wave function and the final state deuteron wave function. The wave functions are determined by integrating Schroedinger's equation for the two nucleon system with an assumed nuclear potential. The potential for the *pp* wave function must fit the data on proton-proton scattering. Five different potentials* were explored by Kamionkowski and Bahcall (1994) and give results consistent with the quoted error bar. The deuteron wave function must be consistent with the deuteron binding energy and other experimental constraints. Seven different possibilities were explored. The overall error is in Λ^2 is about 0.2%.

(ft) and G_A/G_V are determined by measurements of weak decay in a variety of nuclei and especially the lifetime of the free neutron. The standard value for the latter is 887 +- 2.0 seconds. The weak decay here is of the Gamow-Teller type ($\Delta J = 0,1$), not Fermi ($\Delta J = 0$). GT is mediated by the axial current (A). Fermi is mediated by the vector current (V).

The other factors are either accurately measurable (deuteron BE), straightforward to calculate (f_{pp}), or complicated and not very important (δ).

*square well, Gaussian, exponential Yukawa, and repuslive core

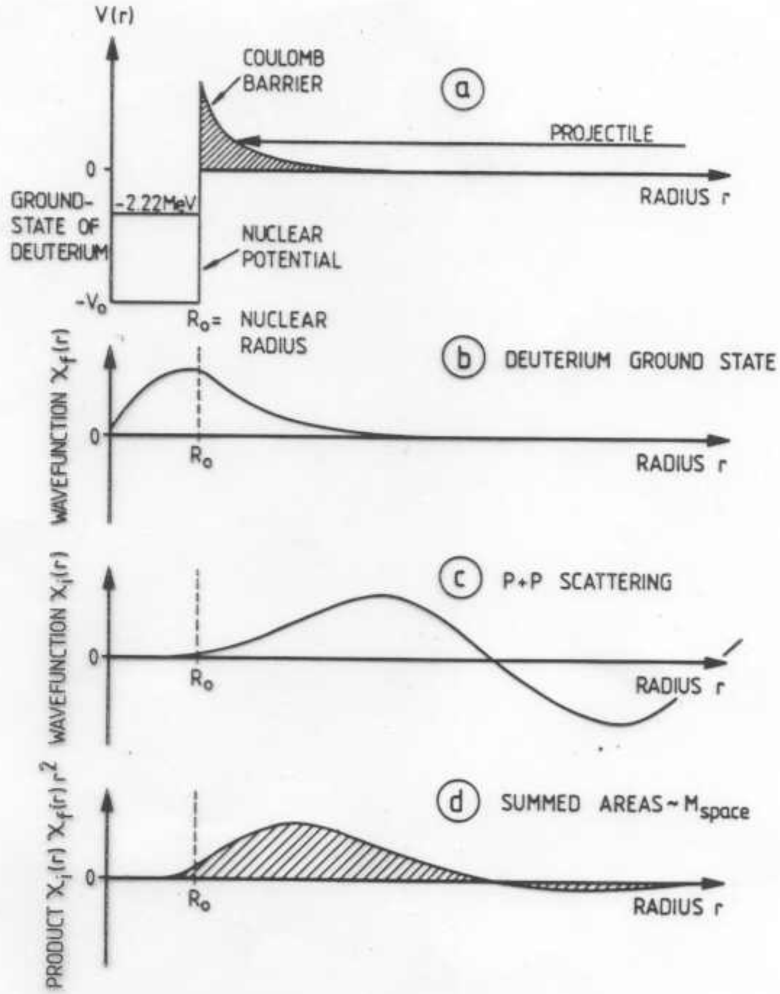


FIGURE 6.4. Shown schematically are a few ingredients used in the numerical evaluation of the space matrix element M_{space} for the $p + p \to d + e^+ + \nu$ reaction. The potential is shown in (*a*), where, for a given nuclear radius R_0, the observed binding energy of the deuterium determines the potential depth V_0. The deuterium radial wave function $\chi_f(r)$ is determined by the potential $V(r)$. Because of the loosely bound ground state, $\chi_f(r)$ extends far outside R_0 with appreciable amplitudes (*b*). The initial wave function $\chi_i(r)$ is obtained from $p + p$ elastic scattering data, which gives (*c*) a small amplitude for $r \leq R_0$ and has the usual oscillating pattern of a plane wave for $r \gg R_0$. The radial integrand in M_{space} (*d*) then has its major contributions in regions far outside R_0 (*hatched areas*).

The overlap is insensitive to the form of nuclear potential assumed inside a few fm and is determined by the tail of the potential at the nuclear surface.

This is highly constrained by proton scattering experiments.

History:

Bethe and Critchfield (1938)
Salpeter (1952)

Solar fusion cross sections

Eric G. Adelberger *et al.*, Institute for Nuclear Theory
Workshop on Solar Fusion Reactions

Reprint No. 590 from
Reviews of Modern Physics
Volume 70, No. 4, October 1998

Published by The American Physical Society
through the American Institute of Physics

Putting in best values (1998)

$$S(0)=4.00\times10^{-25}\ \text{MeV barns}\left(\frac{(ft)_{0^+\to0^+}}{3073\ \text{sec}}\right)^{-1}\left(\frac{\Lambda^2}{6.92}\right)$$

$3.78\pm0.15\times10^{-25}$ in Bahcall (1968)

$$\times\left(\frac{G_A/G_V}{1.2654}\right)^2\left(\frac{f_{pp}^R}{0.144}\right)\left(\frac{1+\delta}{1.01}\right)^2$$

theoretical

$$S(0)=4.00\times10^{-25}\ (1\pm0.007^{+0.020}_{-0.011})\ \text{MeV barns}$$

statistical

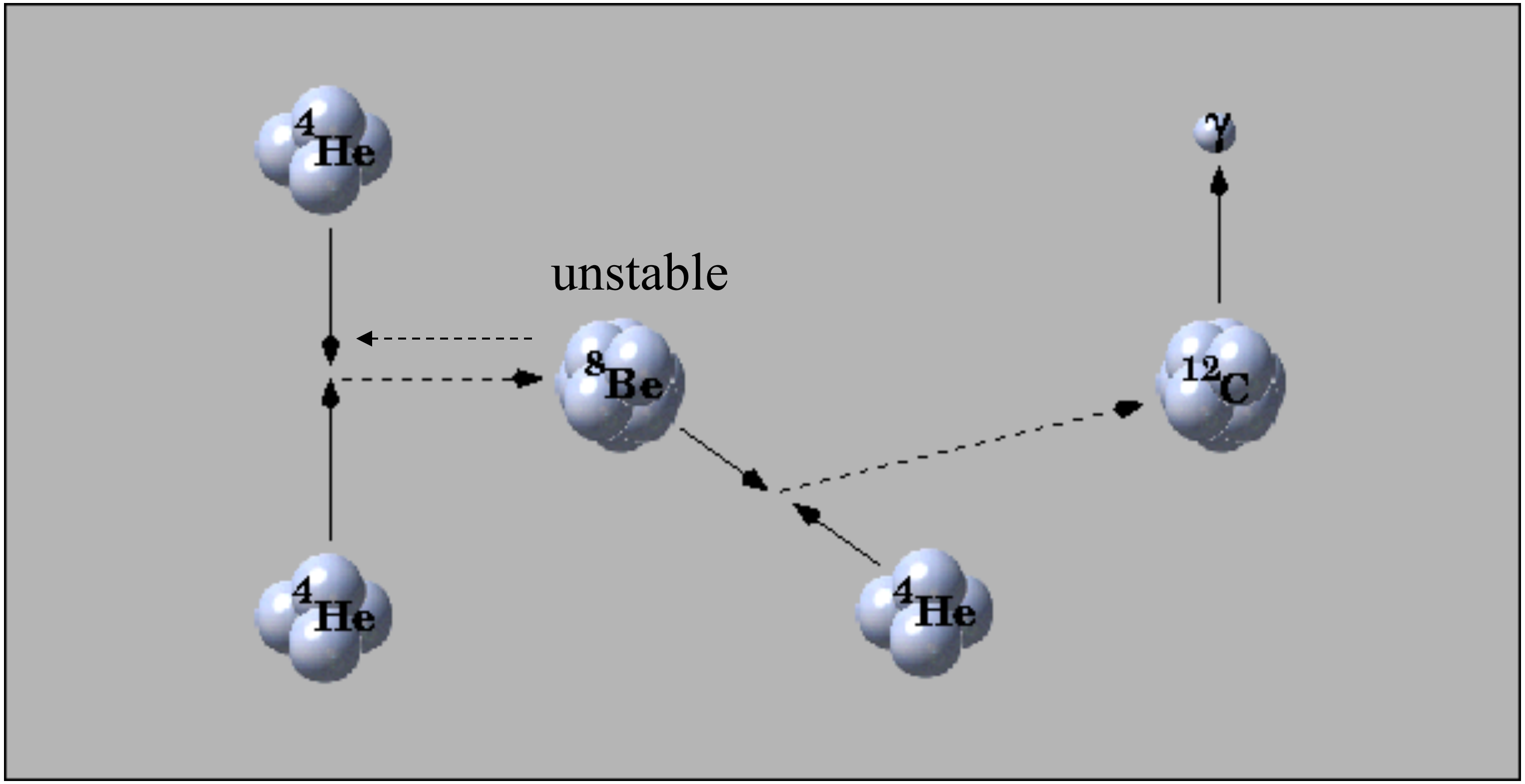
Adelberger et al. RMP, (1998)

TABLE I. Best-estimate low-energy nuclear reaction cross-section factors and their estimated 1σ errors.

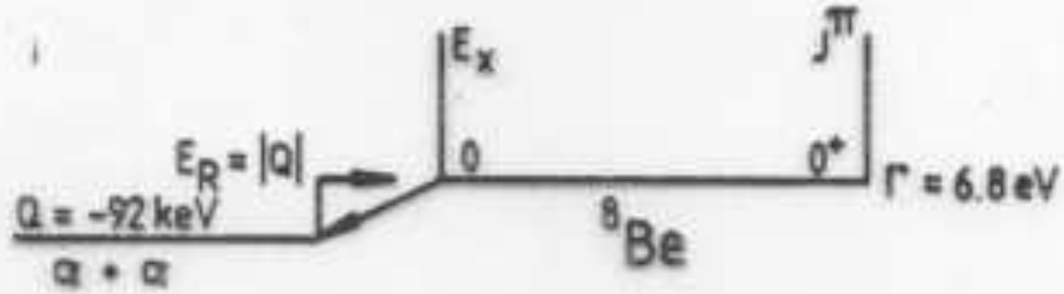
Reaction	$S(0)$ (keV b)	$S'(0)$ (b)
$^{1}H(p,e^{+}\nu_e)^{2}H$	$4.00(1\pm 0.007^{+0.020}_{-0.011})\times 10^{-22}$	4.48×10^{-24}
$^{1}H(pe^{-},\nu_e)^{2}H$	Eq. (19)	
$^{3}He(^{3}He,2p)^{4}He$	$(5.4\pm 0.4)^{a}\times 10^{-3}$	
$^{3}He(\alpha,\gamma)^{7}Be$	0.53 ± 0.05	-3.0×10^{-4}
$^{3}He(p,e^{+}\nu_e)^{4}He$	2.3×10^{-20}	
$^{7}Be(e^{-},\nu_e)^{7}Li$	Eq. (26)	
$^{7}Be(p,\gamma)^{8}B$	$0.019^{+0.004}_{-0.002}$	See Sec. VIII.A
$^{14}N(p,\gamma)^{15}O$	$3.5^{+0.4}_{-1.6}$	See Sec. IX.A.5

Helium Burning

Helium burning is a two-stage nuclear process in which two alpha-particles temporarily form the ground state of unstable $^{8}Be^{*}$. Occasionally the $^{8}Be^{*}$ captures a third alpha-particle before it flies apart. No weak interactions are involved.

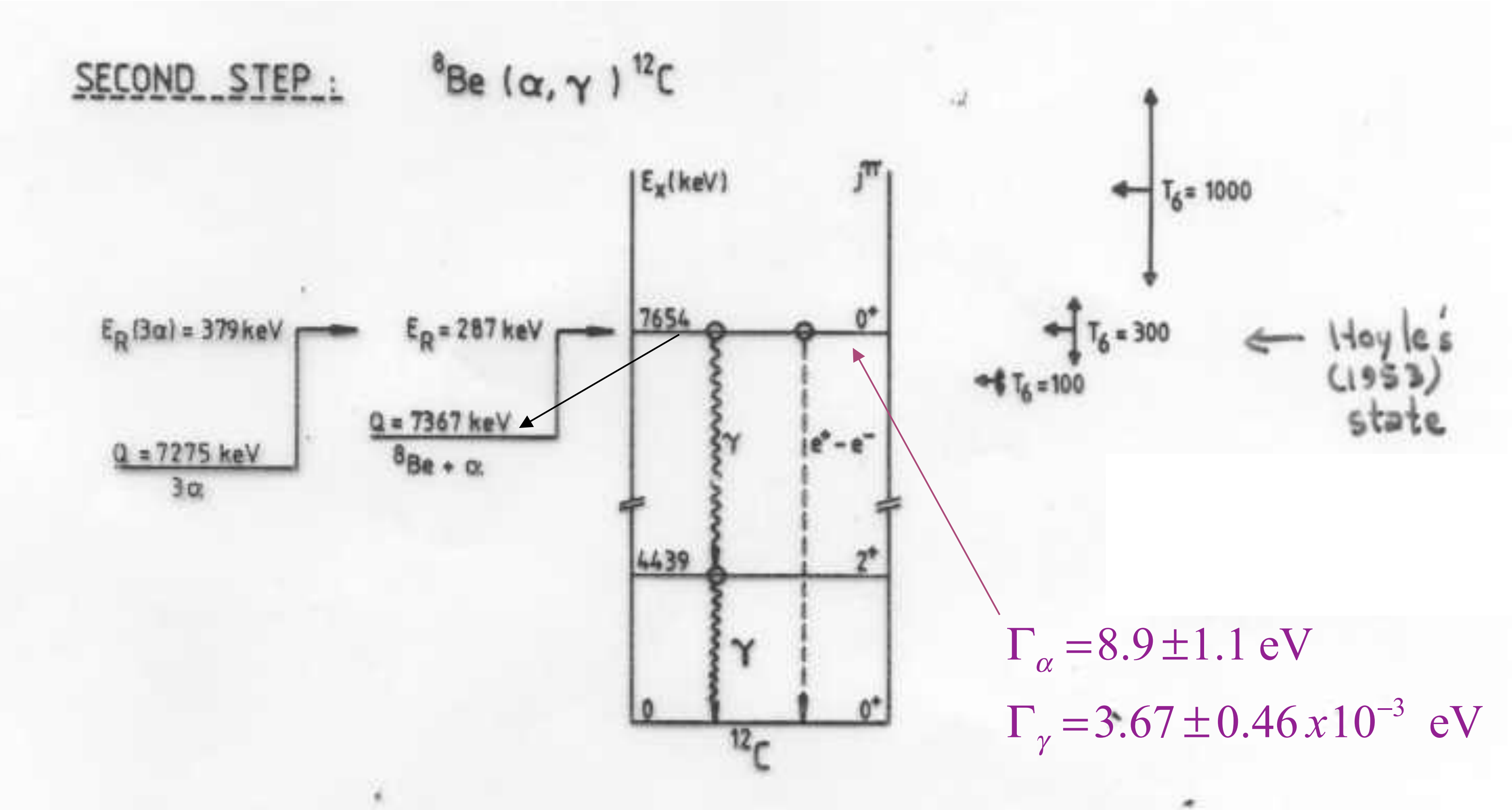


FIRST STEP: $\alpha + \alpha \rightleftharpoons {}^{8}Be$



The ground state of $^{8}Be^{*}$ is unbound by 92 keV to α-decay. It has a width $\Gamma_\alpha = 6.8$ eV and a lifetime of

$$\tau = \frac{\hbar}{\Gamma} = \frac{6.58 x 10^{-22}\ \text{MeV s}}{6.8 x 10^{-6}\ \text{MeV}} = 9.7 x 10^{-17}\ \text{sec}$$



$$\Gamma_{\alpha} = 8.9 \pm 1.1 \text{ eV}$$

$$\Gamma_{\gamma} = 3.67 \pm 0.46\, x 10^{-3} \text{ eV}$$

The 7.654 MeV excited state of ^{12}C plays a critical role in the 3α reaction. Its α-width is much greater than its photon width, so it predominantly decays back to $^{8}Be^{*}$, setting up an equilibrium abundance of $^{12}C^{*}$. Γ_{γ} is augmented by a small contribution from pair production.

Recall the Saha equation: (e.g., Clayton p 29). For example, for ionized and neutral hydrogen:

$$\frac{n(\mathrm{H\ II})\ n_e}{n(\mathrm{H\ I})} = \frac{G(\mathrm{H\ II})\ \ g_e}{G(\mathrm{H\ I})}\left[\frac{\left(2\pi m_e kT\right)^{3/2}}{h^3}\right]\exp(-\chi_r / kT)$$

The same thermodynamic arguments (equilibrium, chemical potential, etc.) also give a *nuclear* Saha equation. In particular, the equilibrium concentration of an unbound transitory $^8\mathrm{Be}^*$ nucleus is given by

$$\frac{n_\alpha^2}{n(^8\mathrm{Be})} = \left(\underbrace{\frac{G_\alpha^2}{G(^8\mathrm{Be}^*)}}_{1}\right)\left(\underbrace{\frac{\left(2\pi \hat{A} kT\right)^{3/2}}{N_A^{3/2}\ h^3}}_{5.94\times 10^{33}\ \hat{A}^{3/2}\ T_9^{3/2}}\right)\exp\left(\frac{-Q_{\alpha\gamma}(^4\mathrm{He})}{kT}\right)$$

$$Q_{\alpha\gamma}(^4\mathrm{He}) = BE(^8\mathrm{Be}^*)\text{-}2\,BE(\alpha)=56.4995\text{-}2(28.2957)$$

$$= -0.0919\ \mathrm{MeV}$$

$$Q_{\alpha\gamma}(^4\mathrm{He})/kT = -0.0919\ \times 11.6045/T_9\ = -1.066/T_9$$

$$n({}^8\mathrm{Be}^*) = \left(5.94\times10^{33}\, 2^{3/2}\, T_9^{3/2}\right)^{-1} n_\alpha^2 \exp(-1.066 / T_9)$$

$$n({}^8\mathrm{Be}^*) = n_\alpha^2\, T_9^{-3/2}\, (5.95\times10^{-35})\exp(-1.066/T_9)\ \mathrm{cm}^{-3}$$

$$\hat{A} = \frac{4\times4}{4+4} = 2$$

or, since $n \equiv \rho N_A Y$ and $Y = \dfrac{X}{A}$

$$X({}^8\mathrm{Be}^*) = 1.79\times10^{-11}\, \frac{\rho X_\alpha^2}{T_9^{3/2}}\, e^{-1.066/T_9}$$

For example, at 2×10^8 K, $\rho = 10^3\ \mathrm{g\,cm^{-3}}, X_\alpha = 1$

$$X({}^8\mathrm{Be}^*) \approx 10^{-9}$$

This works because the dominant decay mode of ${}^8\mathrm{Be}^*$ is to the same products from which it is assembled, i.e.,

$$\alpha + \alpha \rightleftarrows {}^8\mathrm{Be}^*$$

The time scale for establishing this equilibrium is very short.

Now consider the excited state of ^{12}C at 7.6542 MeV. It also has as its dominant width, $\Gamma_\alpha >> \Gamma_\gamma$. That is $^8\mathrm{Be}^* + \alpha \rightleftarrows {}^{12}\mathrm{C}^*$ where we have denoted the excited state as $^{12}\mathrm{C}^*$.

$$\frac{n(^8Be^*)n_\alpha}{n(^{12}C^*)} = 5.94\times10^{33}\mathrm{T}_9^{3/2}\left(\frac{4\bullet8}{4+8}\right)^{3/2} \mathrm{e}^{-\mathrm{Q}_{\alpha\gamma}(^8Be^*)/\mathrm{kT}}$$

$$\mathrm{n}(^{12}\mathrm{C}^*) = (5.94\times10^{33})^{-1}\,\mathrm{T}_9^{-3/2}\left(\frac{12}{32}\right)^{3/2} n(^8Be^*)n_\alpha \exp\left(-0.287/kT\right)$$

$$\mathrm{Q}_{\alpha\gamma}(^8Be^*) = BE(^{12}C) - BE(^8Be^*) - BE(\alpha) - 7.6542$$

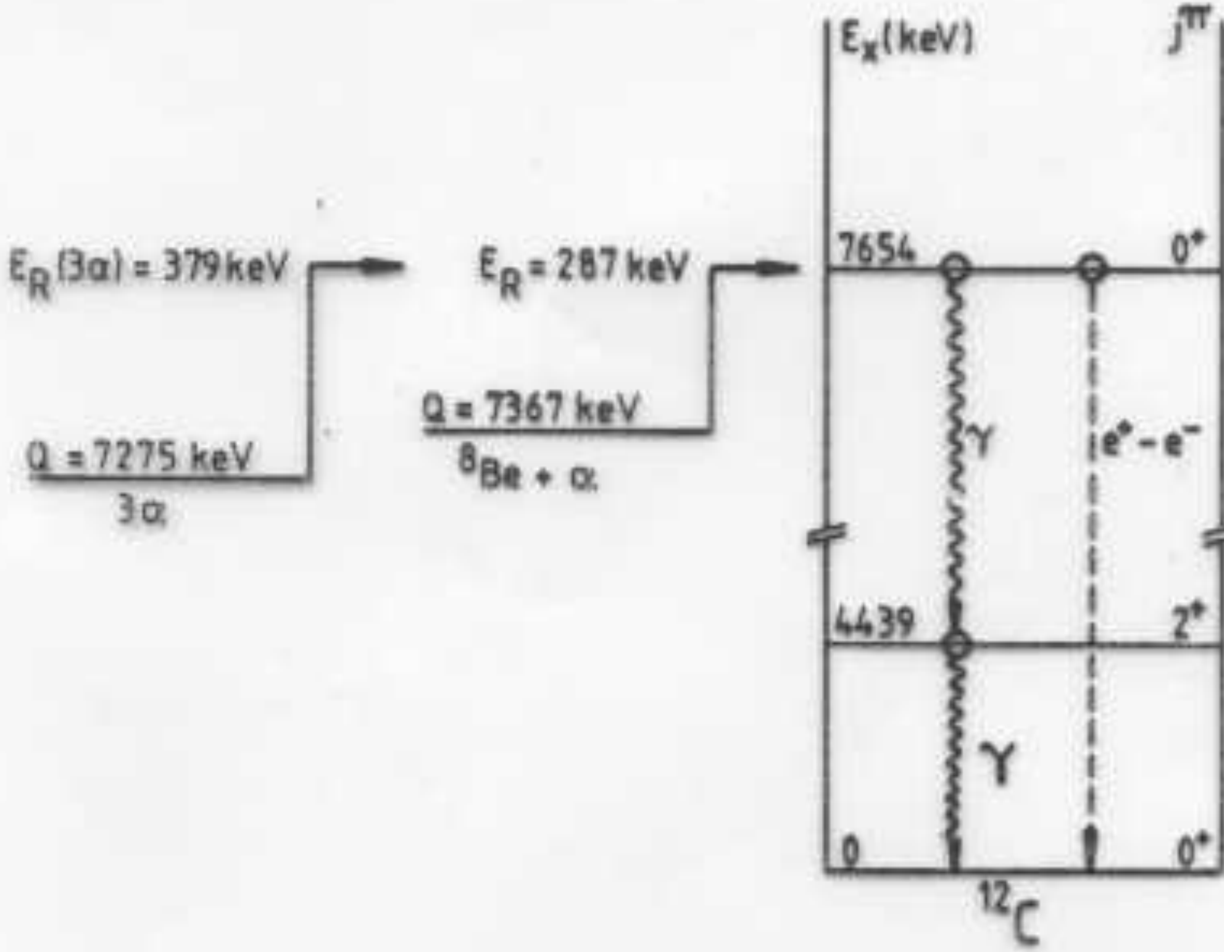
$$= 92.1617 - 56.4995 \text{ -}28.2957 \text{ - } 7.6542 \text{ MeV}$$

$$= \text{-}0.2870 \text{ MeV} \quad (^*1/\mathrm{k} = 11.6045 \Rightarrow \text{-}3.330)$$

$$\mathrm{n}(^{12}\mathrm{C}^*) = 3.87\times10^{-35}\,\mathrm{T}_9^{-3/2}\, n(^8Be^*)n_\alpha \exp\left(-3.330/T_9\right)$$

$$= 3.87\times10^{-35}\,(5.95\times10^{-35})\mathrm{T}_9^{-3}\mathrm{n}_\alpha^3 \exp(-3.330/T_9 \text{-}1.066/\mathrm{T}_9)$$

$$= 2.303\times10^{-69}\,\mathrm{T}_9^{-3}\mathrm{n}_\alpha^3 \exp(-4.396/T_9)$$



$E_R(3\alpha) = 379\,keV$
$Q = 7275\,keV$
3α
$E_R = 287\,keV$
$Q = 7367\,keV$
$^8Be + \alpha$
$E_x(keV)$
J^π
7654
0^+
γ
$e^+ - e^-$
4439
2^+
γ
0
0^+
^{12}C

The number of ^{12}C formed permanently per second is

$$\mathrm{R}_{3\alpha} = n(^{12}C^{*})\,\frac{\Gamma_{rad}}{\hbar}$$

Γ_{rad} is the one thing besides binding energies and excited state energy that has to be measured

$$\Gamma_{\mathrm{rad}} = 3.41 \pm 1.12 \times 10^{-3}\ \mathrm{eV}\ \ (1976)$$

$$= 3.67 \pm 0.46 \times 10^{-3}\ \mathrm{eV}\ (1988)$$

$$= 3.64 \pm 0.5\ \ \mathrm{meV}\ (1990)$$

$$\Gamma_{\mathrm{e}^{\pm}} = 60.5 \pm 3.9\ \ \mu\mathrm{eV}$$

see article by Hale (1997). Current error about 10% (Sam Austin 2013)

This gives:

$$R_{3\alpha} = 1.28\times10^{-56}\,T_9^{-3}\,n_\alpha^3 \exp(-4.396/T_9)\ \mathrm{cm}^{-3}\,\mathrm{sec}^{-1}$$

$$\frac{dn_{12}}{dt} = R_{3\alpha} \qquad \frac{dn_\alpha}{dt} = -3R_{3\alpha}$$

converting to our standard, Y_i notation

$$n_\alpha = \rho N_A Y_\alpha \qquad Y_\alpha = \frac{X(^4\mathrm{He})}{4}$$

$$n_{12} = \rho N_A Y_{12} \qquad Y_{12} = \frac{X(^{12}\mathrm{C})}{12}$$

$$\frac{dY_{12}}{dt} = \rho^2 Y_\alpha^3(\lambda_{3\alpha}/3!) \qquad \frac{dY_\alpha}{dt} = -3\rho^2 Y_\alpha^3(\lambda_{3\alpha}/3!)$$

where

$$\lambda_{3\alpha} = 3!\times N_A^2\,R_{3\alpha} = 2.79\times10^{-8}\,T_9^{-3}\exp(-4.396/T_9)\ \mathrm{cm}^6\,\mathrm{gm}^{-2}\ \mathrm{Mole}^{-2}\,\mathrm{sec}^{-1}$$

(the units are such that $\rho^2\,Y_\alpha^3\,\lambda_{3\alpha}$ has units of Mole/s)

The current value is due to Caughlan and Fowler (1988) using mesurements from Sam Austin

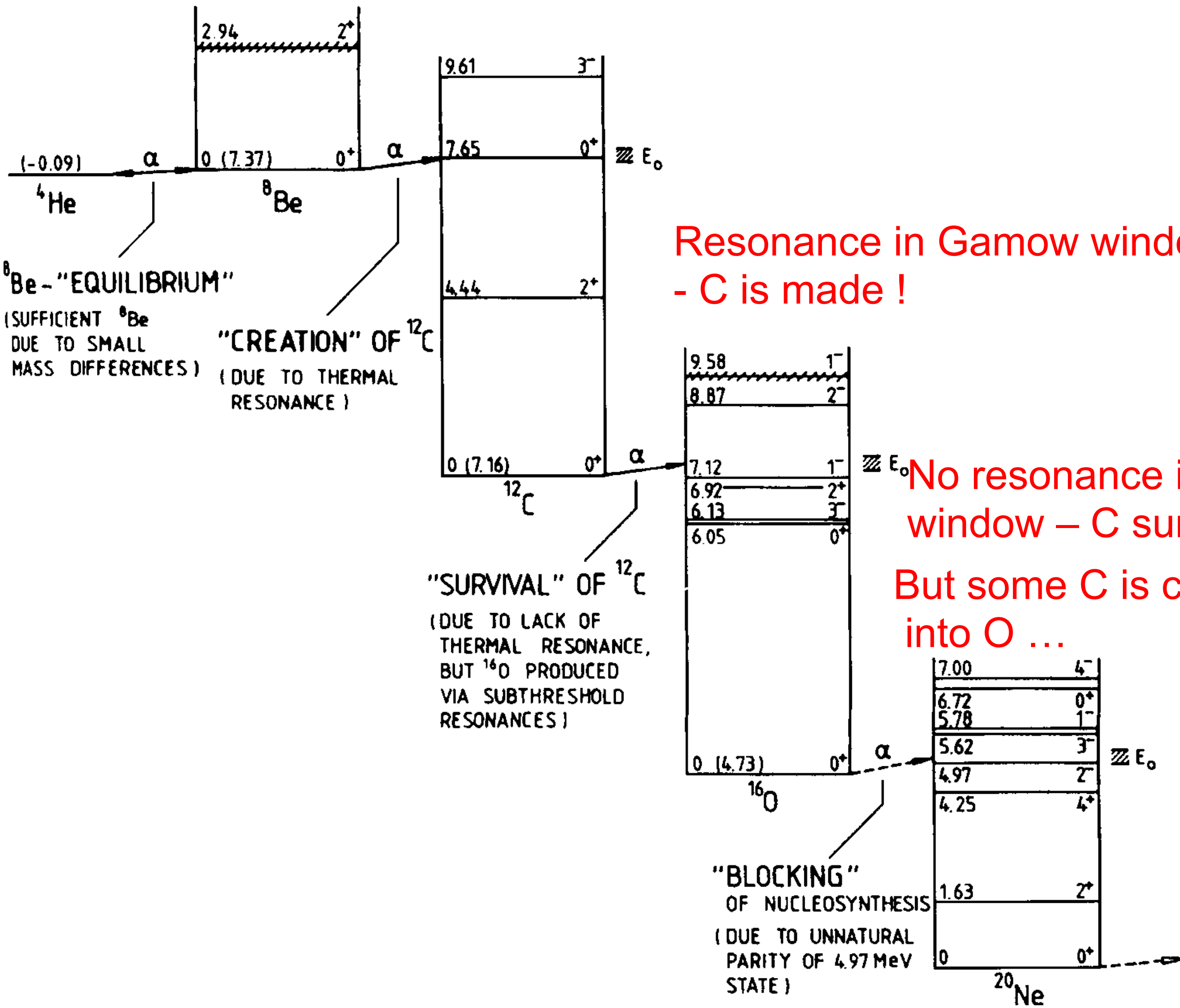
$$\lambda_{3\alpha} = 2.79 \times 10^{-8}\ T_9^{-3} \exp(-4.396 / T_9)$$

Slight revisions to Γ_γ here

$$\left.\begin{array}{cc} T_9 & \dfrac{d \ln \lambda}{d \ln T} \\ 0.1 & 41 \\ 0.2 & 19 \\ 0.3 & 12 \end{array}\right\} = \frac{4.396}{T_9} - 3$$

Unlike most reactions in astrophysics, the temperature dependence here is not determined by barrier penetration but by the Saha equation. In fact, at high temperature ($T_9 > 1.5$) the rate saturates and actually begins to decline slowly as the resonance slips out of the Gamow window.

Helium burning 2 – the $^{12}C(\alpha,\gamma)$ rate

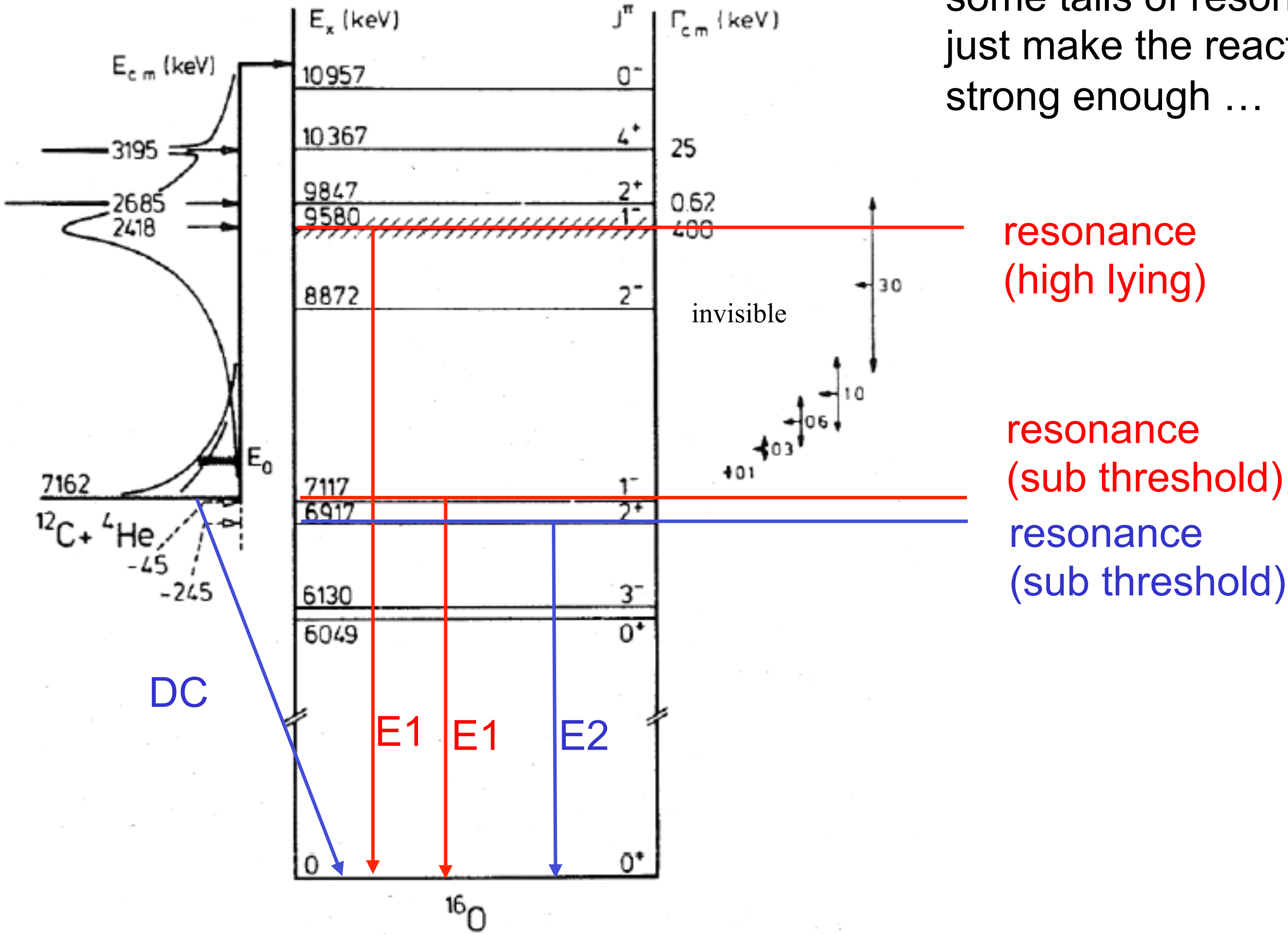


Resonance in Gamow window - C is made !

No resonance in Gamow window – C survives !

But some C is converted into O …

some tails of resonances just make the reaction strong enough …



complications:

- very low cross section makes direct measurement impossible
- subthreshold resonances cannot be measured at resonance energy
- Interference between the E1 and the E2 components

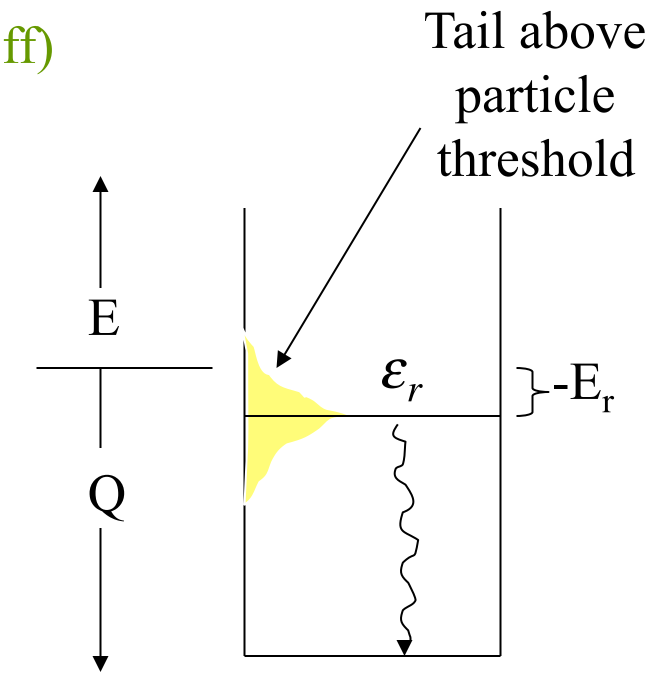
Sub-threshold resonances

(See Rolfs and Rodney, *Cauldrons in the Cosmos*, p. 185ff)

$$\sigma(E) = \pi \lambda\!\!\!^{-2} \omega \frac{\Gamma_1(E)\,\Gamma_2(E+Q)}{(E-E_r)^2 + [\Gamma(E)/2]^2}$$

E.g, 1 is an α-particle and 2 is a photon. Γ_1 is the probability that the α penetrates to the nuclear surface. Γ_2 is the photon width evaluated at E + Q. e.g., for dipole radiation

$$\Gamma_2 = \left(\frac{E+Q}{\varepsilon_r}\right)^3 \Gamma_\gamma(\varepsilon_r)$$



An excited state of a compound nucleus lies E_r below the threshold of the reaction, Q. The excited state is known to decay by γ emission and is characterized by a width Γ_γ. Because of this width the state extends energetically to both sides of E_r on a rapidly decreasing scale.

Uncertainty in the $^{12}C(\alpha,\gamma)$ rate is the single most important nuclear physics uncertainty in astrophysics

Affects:
- C/O ratio → further stellar evolution (C-burning or O-burning ?)
- iron (and other) core sizes (outcome of SN explosion)

More than 30 experiments in past 30 years …

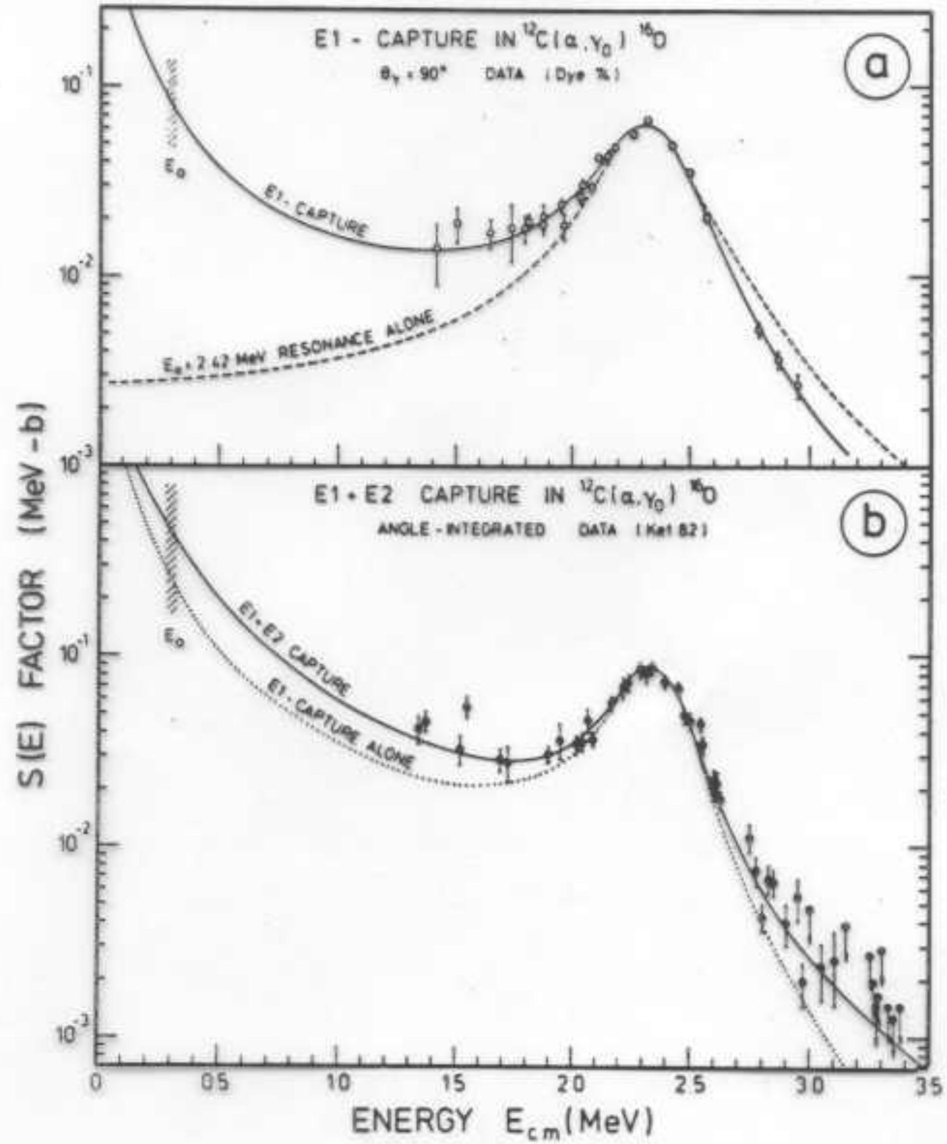
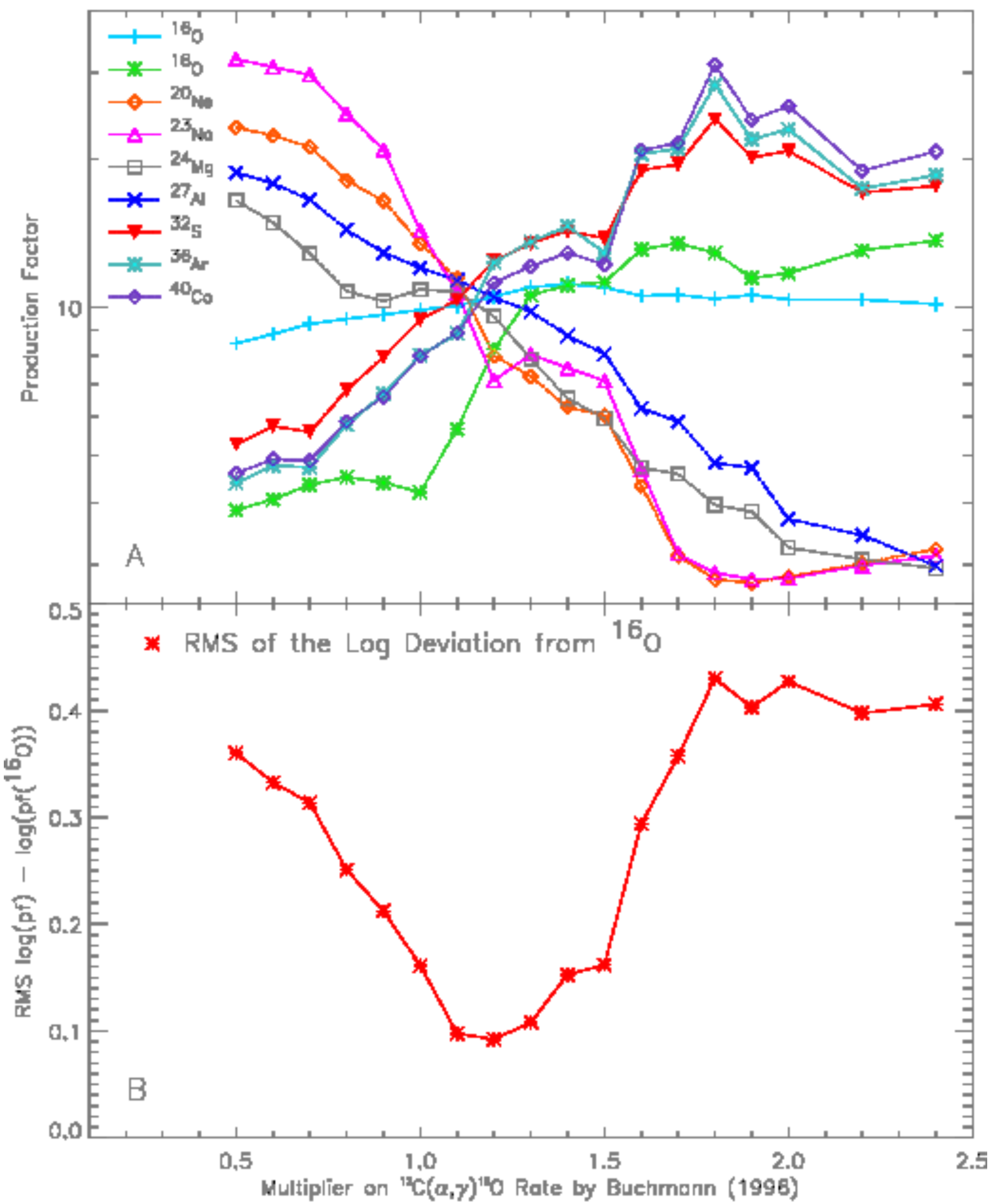


FIGURE 7.10. (*a*) The E1 capture yield in $^{12}C(\alpha, \gamma_0)^{16}O$ is shown in $S(E)$ factor form toge.her with a theoretical analysis (Koo74). The data cannot be explained by the $E_R = 2.42$ MeV resonance alone. They require an additional contribu.ion from the $E_R = -45$ keV subthreshold resonance,



Woosley and Weaver, Physics Reports (2007)

Prediction:

S(300 keV) = 170 keV-barns

See also Woosley & Weaver, *Phys. Reports*, **227**, 65, (1993)

Buchmann, L. 1996, ApJ, 468, L127 gives fits good at both low and hi T

Kunz et al., ApJ, 567, 643, (2002)

$S_{E1}(300\ \text{keV}) = 76 +/- 20\ \text{keV b}$

$S_{E2}(300\ \text{keV}) = 85 +/- 30\ \text{kev b}$

$S_{casc}(300\ \text{keV}) = 4 +/- 4\ \text{keV b}$

$S_{tot}(300\ \text{keV}) = 165 +/- 50\ \text{keV b}$

This corresponds to about 1.2*Buchmann and is what we are using this year.

Helium Burning

$$\frac{dY_\alpha}{dt} = -3\rho^2 Y_\alpha^3 \lambda_{3\alpha}/6 - Y_\alpha Y(^{12}C)\rho\lambda_{\alpha\gamma}(^{12}C)$$

$$\frac{dY(^{12}C)}{dt} = \rho^2 Y_\alpha^3 \lambda_{3\alpha}/6 - Y_\alpha Y(^{12}C)\rho\lambda_{\alpha\gamma}(^{12}C)$$

$$\frac{dY(^{16}O)}{dt} = Y_\alpha Y(^{12}C)\rho\lambda_{\alpha\gamma}(^{12}C)$$

For binary reactions, $\lambda \equiv N_A \langle \sigma v \rangle$

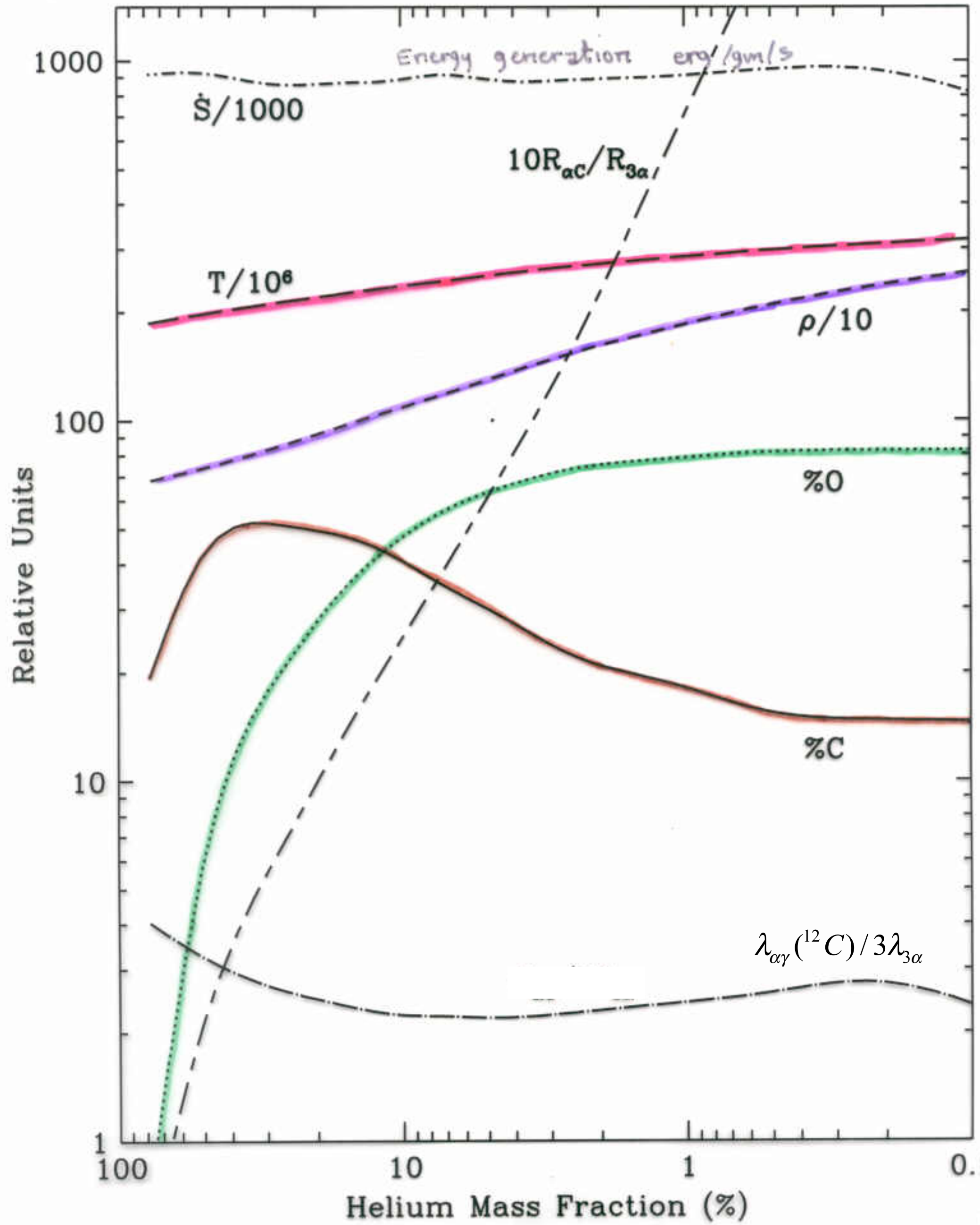
For Y_{12} small or ρ large

$$\alpha \rightarrow {}^{12}C$$

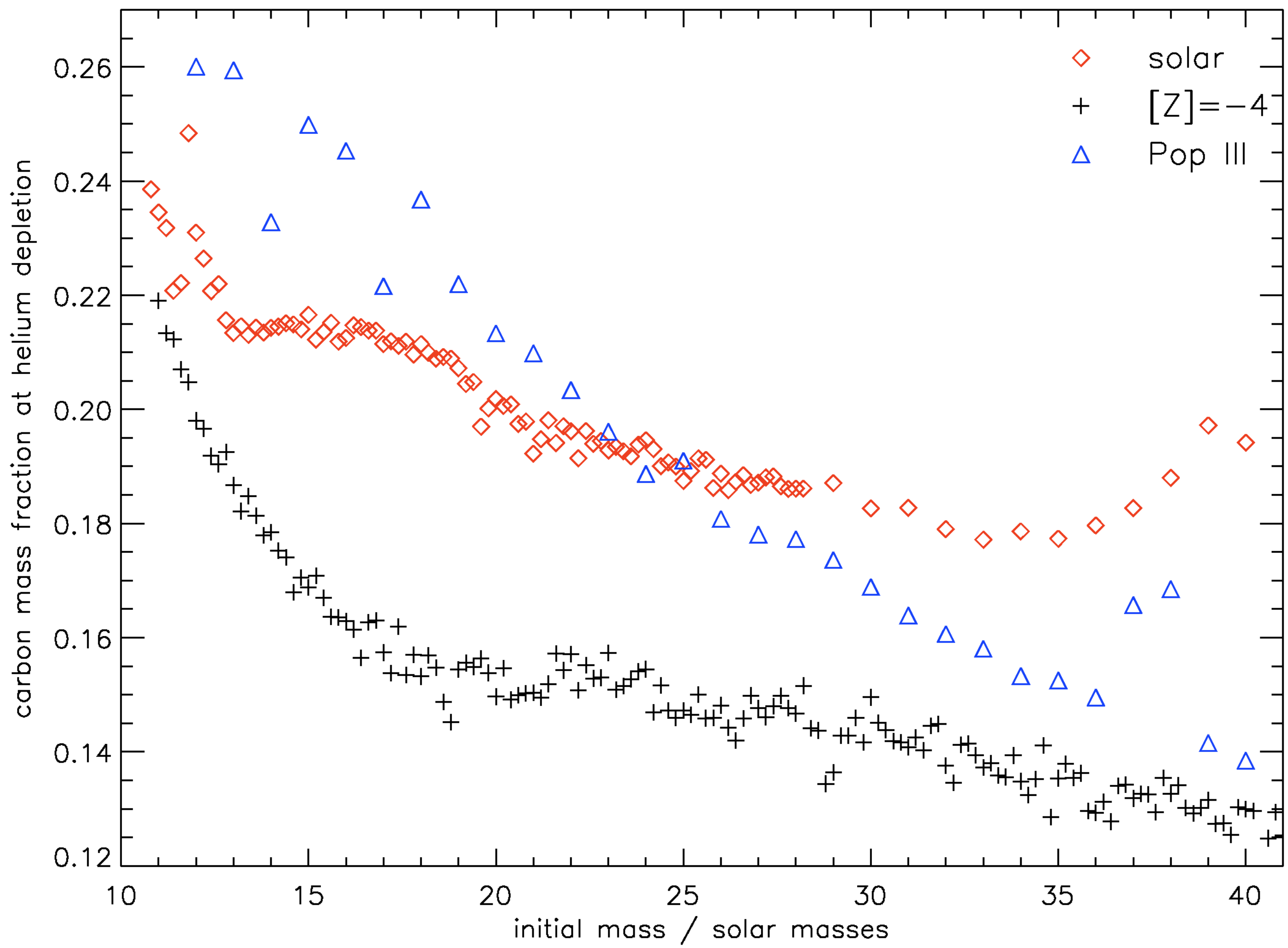
For Y_{12} large or ρ small

$$\alpha \rightarrow {}^{16}O$$

In a 15 solar mass star:



Because of the tendency of $\frac{\rho}{T^3}$ to decrease with increasing mass and the near constancy of helium burning temperatures, massive stars make a decreasing ratio of carbon to oxygen as M increases. Variation with Z reflects the different extent of convection during He burning resulting from e.g., mass loss in solar Z stars, red vs. blue supergiant



Nuclear Energy Yield

When an arbitrary composition, $\{Y_i\}$, rearranges by nuclear reactions to a new composition, $\{Y_i'\}$, where $Y_i' = Y_i + \delta Y_i$, there is a change in internal energy that can be positive or negative

$$q_{nuc} = 1.602\times10^{-6}\, N_A \sum(\delta Y_i)(BE_i) \;\; - q_\nu \quad \text{erg/gm}$$

Here 1.602×10^{-6} is the conversion factor from MeV (which are the units of BE) to erg and the q_ν corrects for any neutrinos that might be emitted by weak interactions or thermal processes (like pair annihilation). If there are no weak interactions and thermal neutrino losses are negligible, e.g., in helium burning, $q_\nu = 0$.

Example: Hydogen burning

a) 100% $^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He}$ $\quad \delta \mathrm{Y}(^{1}\mathrm{H}) = -1 \quad \mathrm{BE}(^{1}\mathrm{H}) = 0$

$$\delta \mathrm{Y}(^{4}\mathrm{He}) = \frac{1}{4} \qquad \mathrm{BE}(^{4}\mathrm{He}) = 28.296\,\mathrm{MeV}$$

$$\mathrm{q} = 9.65\times10^{17}\left(\frac{28.296}{4}\right) = 6.83\times10^{18}\ \mathrm{erg\ g^{-1}}$$

b) 70% $^{1}\mathrm{H}$; 30% $^{4}\mathrm{He} \rightarrow {}^{4}\mathrm{He}$ $\quad \delta \mathrm{Y}(^{1}\mathrm{H}) = -0.7$

$$\delta Y(^{4}\mathrm{He}) = \frac{1}{4} - \frac{0.3}{4}$$

$$\mathrm{q} = 9.65\times10^{17}\left(\frac{1}{4} - \frac{0.3}{4}\right)28.296 = 4.78\times10^{18}\ \mathrm{erg\ g^{-1}}$$

$$\left.\begin{array}{l}\mathrm{BE}(^{12}\mathrm{C}) = 92.162\ \mathrm{MeV} \\ \mathrm{BE}(^{16}\mathrm{O}) = 127.619 \\ \mathrm{BE}(\alpha) = 28.296\ \mathrm{MeV}\end{array}\right\}\ \text{values for helium burning}$$

A related quantity, the energy generation rate is given by

$$\boxed{\dot{\varepsilon}_{nuc} = 9.65\times10^{17}\sum\frac{dY_i}{dt}(BE_i) - q_{\nu,weak} - q_{\nu,thermal}\ \ \mathrm{erg\ g^{-1}\ sec^{-1}}}$$

Both these expressions are only good for strong interactions. In a weak interaction one has to worry about n and p mass differences, electron masses created and destroyed, as well as the mean neutrino energy loss.

A correct expression uses the atomic mass excesses. To within a constant

$$\dot{\varepsilon}_{nuc} = -\sum_i \frac{dY_i}{dt} M_i(^A Z)\ [-\text{neutrino losses}] \qquad \text{where}$$

$$M_i = \mathrm{A}(931.49) + \Delta \quad \text{MeV} \qquad \text{and}$$

$$BE = Z\Delta_H + N\Delta_n - \Delta(^A Z) \quad \text{and} \quad \sum_i \frac{dY_i}{dt} A_i = 0 \quad \text{so that}$$

$$\dot{\varepsilon}_{nuc} = \sum_i \frac{dY_i}{dt} BE_i - \sum_i \frac{dY_i}{dt}\left(Z_i\Delta_H + N_i\Delta_n\right) \ [-\text{neutrino losses}]$$

In the absence of weak interactions the second term may be dropped. (this includes the energy that the positrons deposit when they annihilate in positron emission).