ASTRONOMY 220C

ADVANCED STAGES OF STELLAR EVOLUTION AND NUCLEOSYNTHESIS

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Author	Title	Approximate price	Comments
O. R. Pols	*Stellar Structure and Evolution*	Free (pdf available)	Good overall introduction to stellar evolution at the upper division undergraduate level.
Collins	*Stellar Structure and Evolution*	Free (on line)	Web based graduate level text
Kippenhahn and Weigert	*Stellar Structure and Evolution*	$88.67 (hardcover)	Great “introductory” textbook (more for 220A though)
Branch and Wheeler	*Supernova Explosions*	$109.81	Excellent recent (2017) book on just the supernova par of the course. Expensive. Can be “rented” for half price.
Don Clayton	*Principles of Stellar Evolution and Nucleosynthesis*	$<40.85 (paperback)	A classic. Great on nuclear physics and basic stellar physics. Good on the s-process, but quite dated otherwise. Used $10.00. Buy it.

<u>**Lecture 1**</u>

Overview

Time Scales, Temperature-density Scalings, Critical Masses

I. Preliminaries

Stars are gravitationally confined thermonuclear reactors.

The life of any star is a continual struggle between the force of gravity, seeking to reduce the star to a point, and pressure, which holds it up. A balance is maintained.

So long as they remain non-degenerate and have not encountered the any instabilities, overheating leads to expansion and cooling. Cooling, on the other hand, leads to contraction and heating. Hence stars are stable. The Virial Theorem works.

But, since ideal gas pressure depends on temperature, stars must remain hot. By being hot, they are compelled to radiate. In order to replenish the energy lost to radiation, stars must either contract or obtain energy from nuclear reactions. Since nuclear reactions change their composition, stars must evolve.

The Virial Theorem implies that if a star (with constant density in this example) is neither too degenerate nor too relativistic (radiation or pair dominated)

$$\frac{1}{2}\frac{3GM^2}{5R} \sim 2MN_A kT \quad \text{(for an ideal ionized hydrogen, constant T)}$$

$$T \sim \frac{3GM}{20 N_A k R}$$

$$R \sim \left(\frac{3M}{4\pi\rho}\right)^{1/3} \qquad \text{for constant density}$$

So
$$\mathrm{T} \sim \left(\frac{3}{20}\right)\left(\frac{4\pi}{3}\right)^{1/3} \frac{GM^{2/3}}{N_A k} \rho^{1/3} = 0.093 \frac{GM^{2/3}}{N_A k} \rho^{1/3}$$

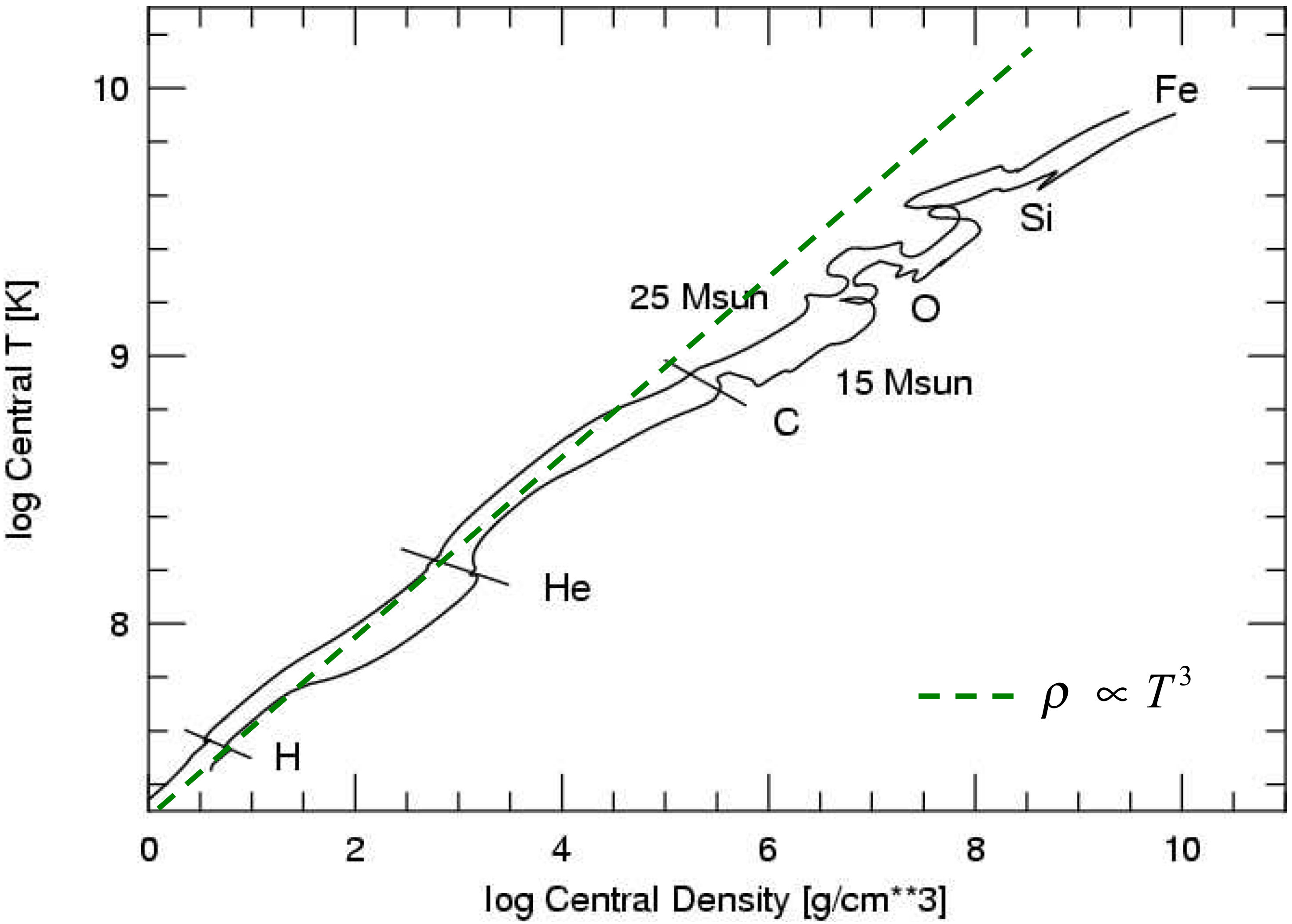
This is about 5 MK for the average temperature of the sun, while the central temperature is about 2 to 3 times greater.

If the central temperature has the same sort of scaling as the average,

$$\mathrm{T}_c \sim \frac{GM^{2/3}}{N_A k} \rho_c^{1/3}$$

That is as stars of ideal gas contract, they get hotter and since a given fuel (H, He, C etc) burns at about the same temperature, more massive stars will burn their fuels at lower density, i.e., higher entropy. $\rho \propto M^{-2} T^3$

Central Conditions



at death the iron cores of massive stars are somewhat degenerate

That is, as a star of given mass evolves, its central temperature rises roughly as the cube root of its central density

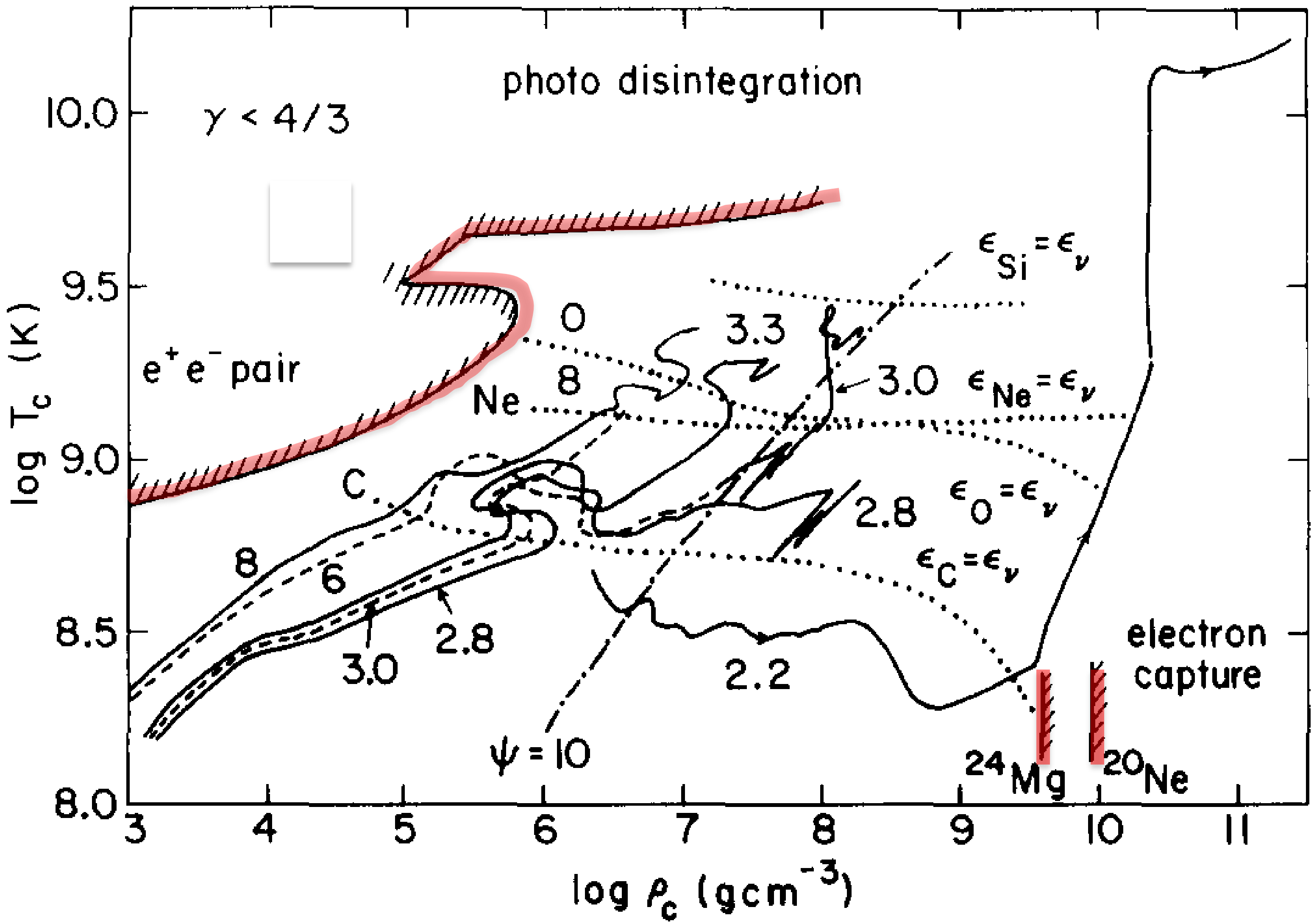
Burning Processes

(e.g., 20 solar masses)

Fuel	Main Product	Secondary Products	Temp (10^9 K)	Time (yr)
H	He	^{14}N	0.02	10^7
He	C,O	^{18}O,^{22}Ne s- process	0.2	10^6
C	Ne, Mg	Na	0.8	10^3
Ne	O, Mg	Al, P	1.5	3
O	Si, S	Cl, Ar K, Ca	2.0	0.8
Si	Fe	Ti, V, Cr Mn, Co, Ni	3.5	1 week

More generally for helium cores of constant mass, 2.2, 2.8, 3.0, 3.3, 6 and 8 M_O (Nomoto and Hashimoto 1988)

$M_{ZAMS} \approx 10 - 25\, M_\odot$



It turns out that M_{He} =35 M_O will just brush the e+e- pair instability

These instabilities (cross hatched lines) have dramatic consequences for the star:

- Pair instability can lead to pulsations (pulsational pair-instability supernovae), explosion (pair-instability supernovae), or collapse to a black hole

- Electron capture can rob the core of pressure support and cause collapse to a neutron star – resulting in a supernova)

- Photodisintegration can also cause collapse to a neutron star or black hole and make a supernova

There are critical masses for all these occurrences

Supernovae are powered by one of two sources:

strong force

- Thermonuclear - white dwarf explosions and pair instability

gravity the weakest force

- Gravitational collapse – *aka* "core collapse" - a fraction of the binding energy of the neutron star or black hole transported by neutrinos or rotation and magnetic fields

What kind of supernova you see depends on the properties of the star (and its surroundings) in which these instabilities operate

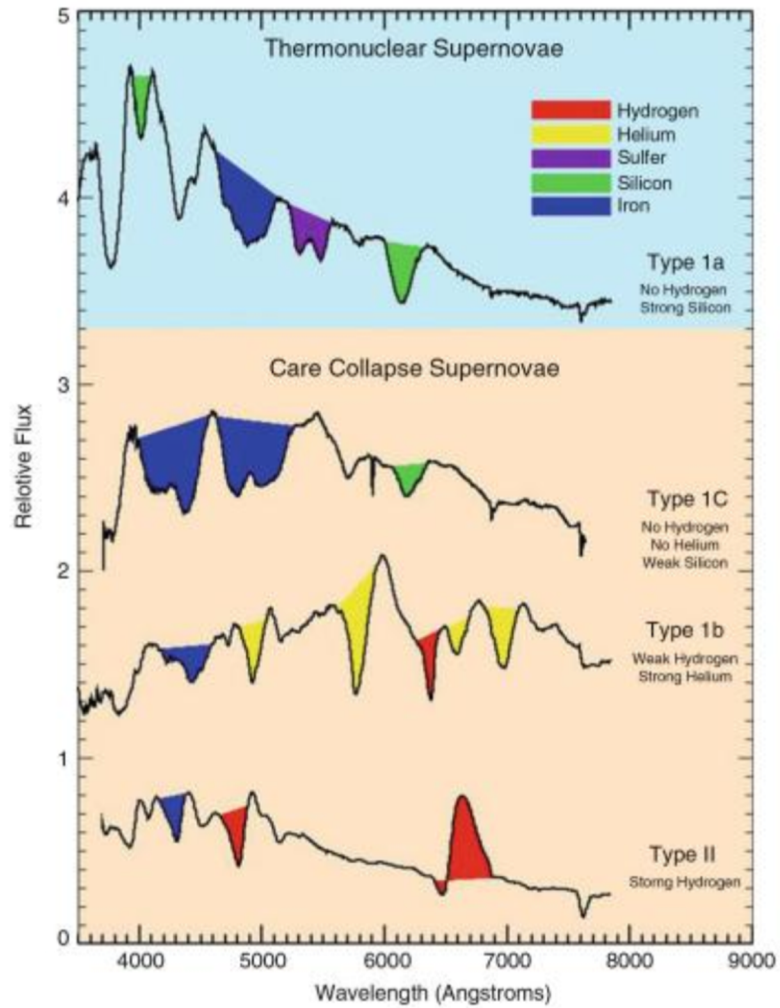
- White dwarf – explosion shatters the star, but by the time the debris expand enough to let the light out the initial explosion energy has been degraded to essentially nothing. Entirely a radioactive display

- Giant star – enough energy is retained (1%) that when the supernova expands and releases it (100 AU), the supernova stays bright for months

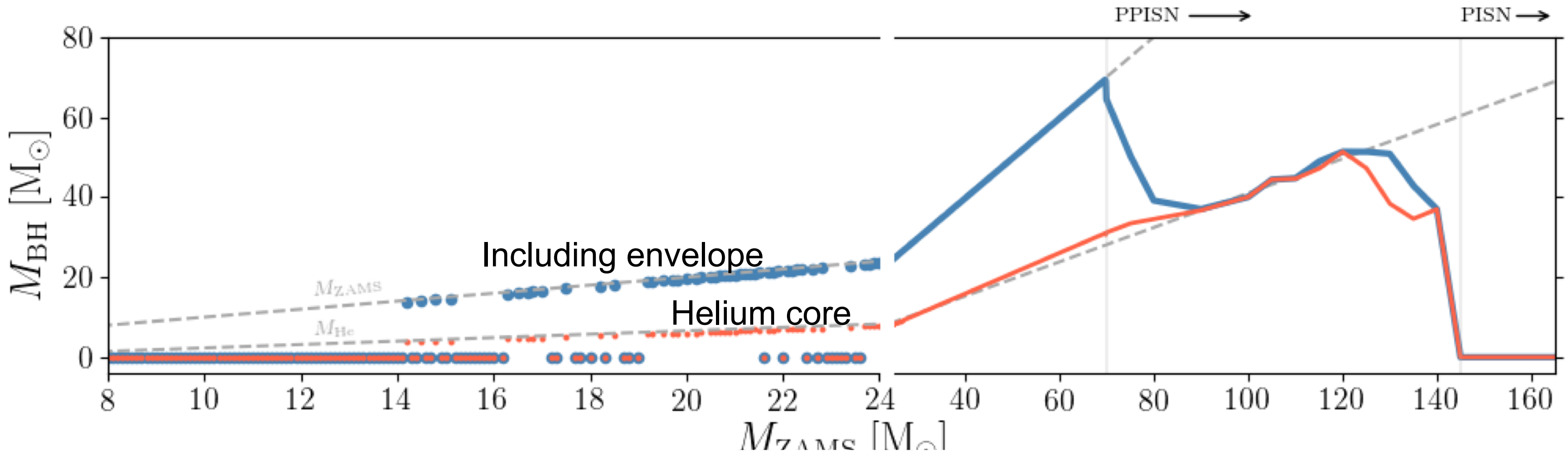
- Wolf-Rayet star – like a white dwarf, the display is chiefly radioactive with perhaps some early activity from the explosion, but the explosion mechanism is collapse.

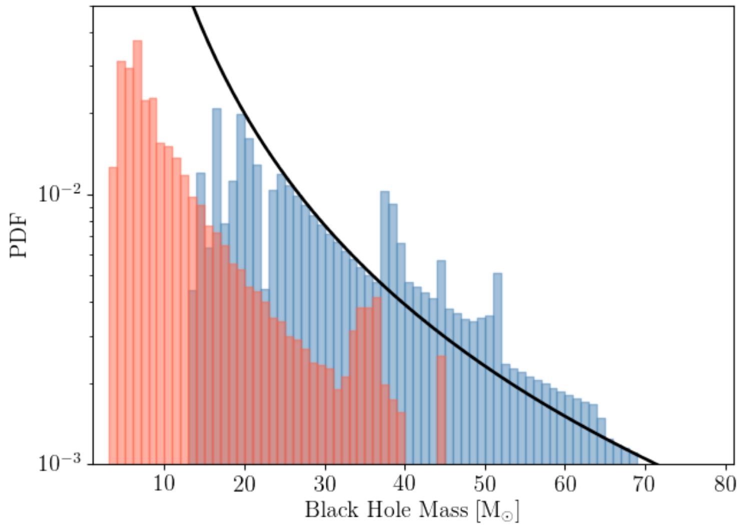
- Magnetar, circumstellar interaction, and pair instability for special cases

Similarly the light curve and spectrum depend on the properties of the star that blew up, especially whether it had a hydrogen envelope or not. Obviously white dwarfs and Wolf-Rayet stars will not make Type II supernovae.



Pair instability SNe and white dwarf explosions leave behind nothing. The rest leave an interesting distribution of compact remnants

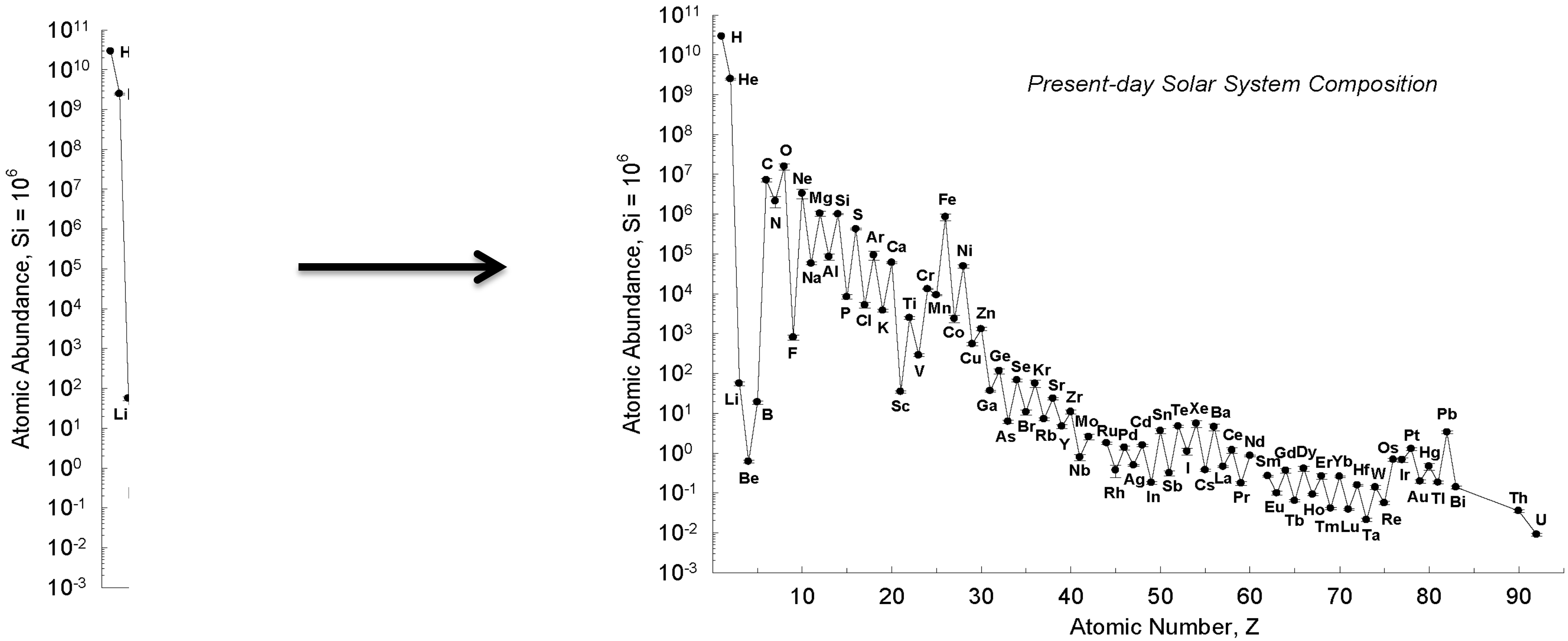


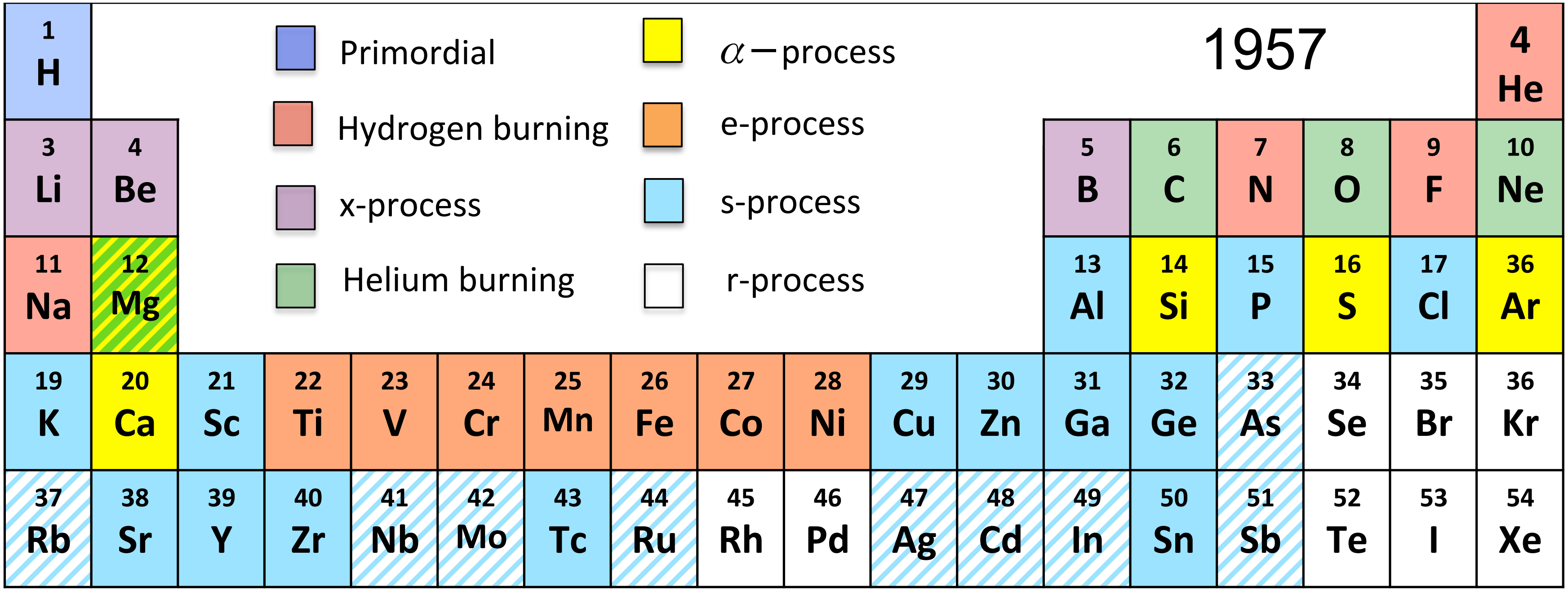


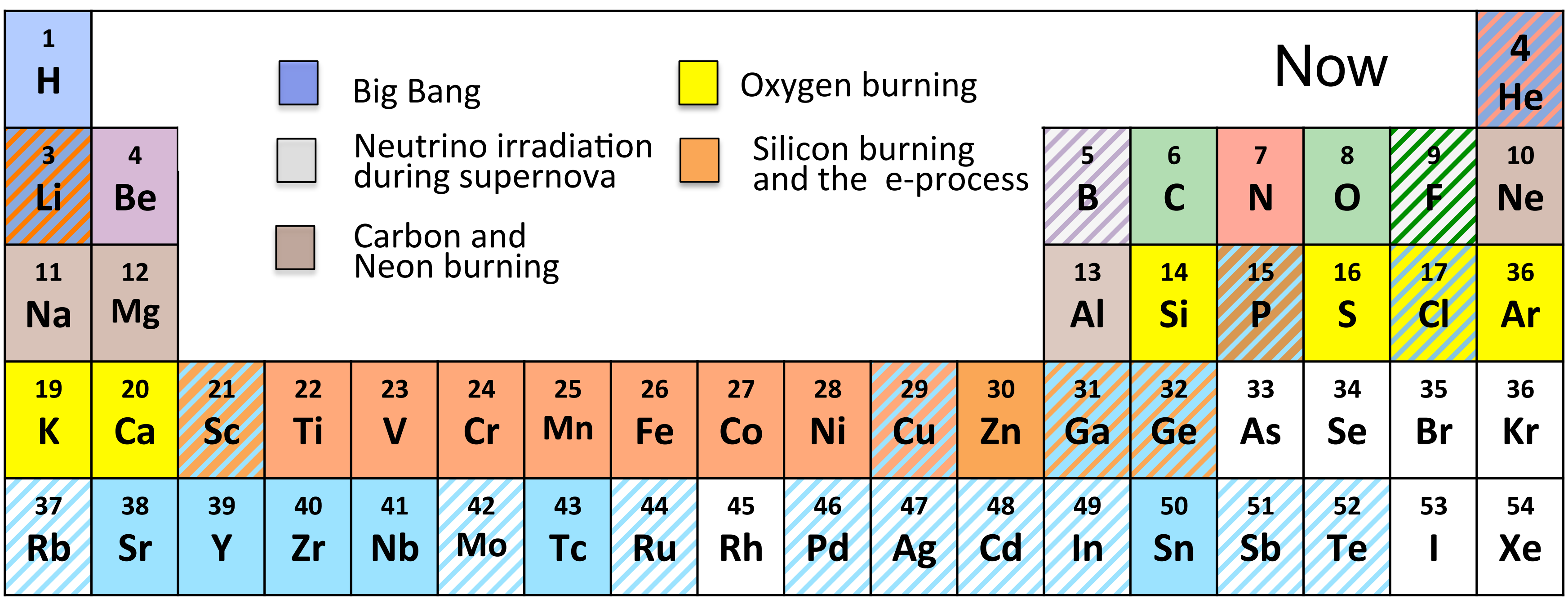
Sukhbold and Woosley (2018)

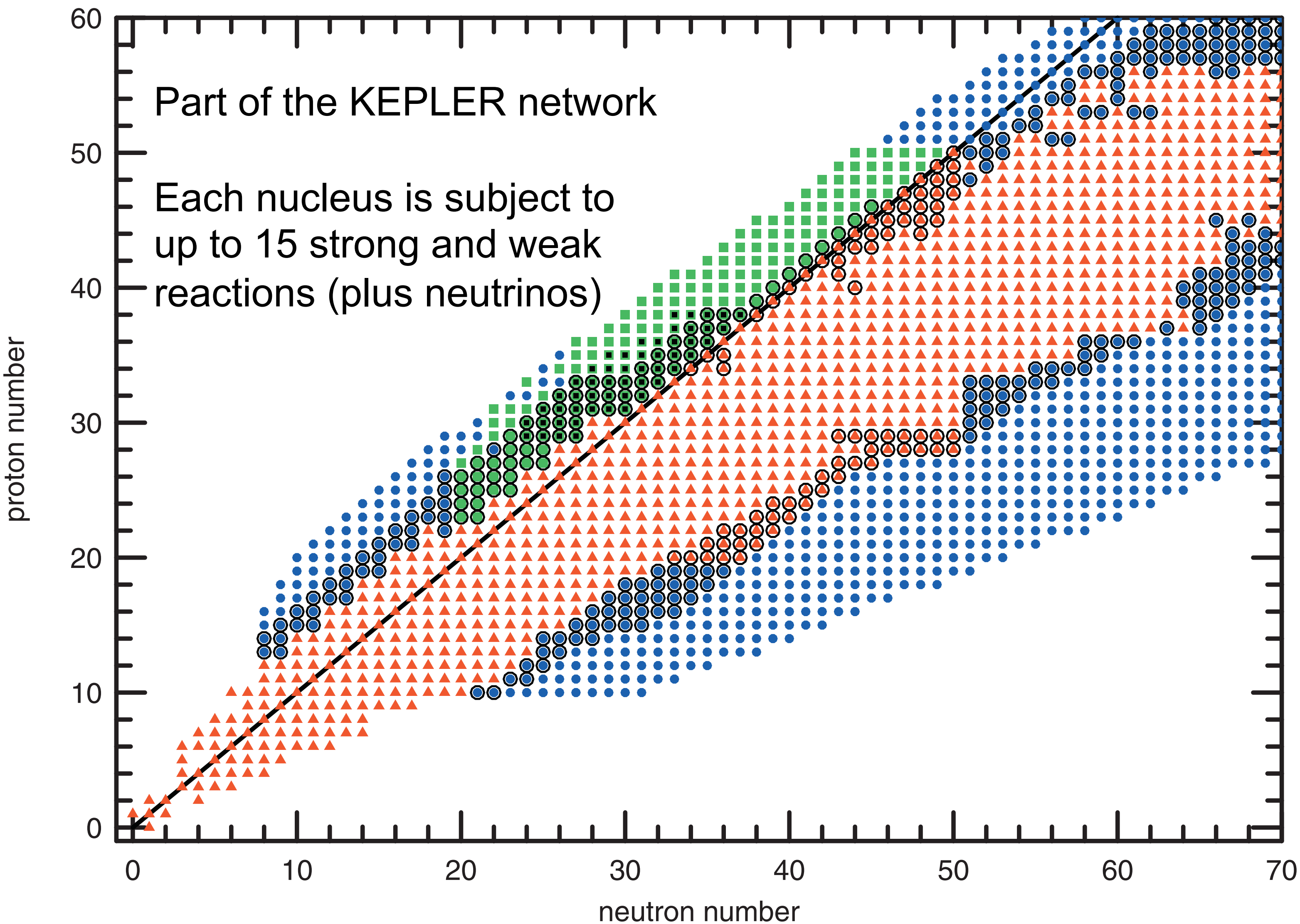
Stars inherently turn light elements into heavier ones

Massive stars and the supernovae (and the neutron stars that they make) are responsible for the synthesis of most of the elements heavier than helium. (Some are made by lighter stars and one or two by cosmic ray spallation)




1957
Primordial
Hydrogen burning
x-process
Helium burning
α − process
e-process
s-process
r-process
1 H
4 He
3 Li
4 Be
5 B
6 C
7 N
8 O
9 F
10 Ne
11 Na
12 Mg
13 Al
14 Si
15 P
16 S
17 Cl
36 Ar
19 K
20 Ca
21 Sc
22 Ti
23 V
24 Cr
25 Mn
26 Fe
27 Co
28 Ni
29 Cu
30 Zn
31 Ga
32 Ge
33 As
34 Se
35 Br
36 Kr
37 Rb
38 Sr
39 Y
40 Zr
41 Nb
42 Mo
43 Tc
44 Ru
45 Rh
46 Pd
47 Ag
48 Cd
49 In
50 Sn
51 Sb
52 Te
53 I
54 Xe


Now
Big Bang
Neutrino irradiation during supernova
Carbon and Neon burning
Oxygen burning
Silicon burning and the e-process
1 H
4 He
3 Li
4 Be
5 B
6 C
7 N
8 O
9 F
10 Ne
11 Na
12 Mg
13 Al
14 Si
15 P
16 S
17 Cl
36 Ar
19 K
20 Ca
21 Sc
22 Ti
23 V
24 Cr
25 Mn
26 Fe
27 Co
28 Ni
29 Cu
30 Zn
31 Ga
32 Ge
33 As
34 Se
35 Br
36 Kr
37 Rb
38 Sr
39 Y
40 Zr
41 Nb
42 Mo
43 Tc
44 Ru
45 Rh
46 Pd
47 Ag
48 Cd
49 In
50 Sn
51 Sb
52 Te
53 I
54 Xe



During the evolution of a massive star its constituent nuclei are subject to a vast array of nuclear reactions involving, protons, neutrons, alpha-particles, photons, electrons, positrons, and neutrinos. The KEPLER nuclear data deck for stellar nucleosynthesis studies Includes 5442 nuclei and 105,000 reactions (plus their inverses). Most (non-r-process) studies use about 1/3 of this.

STELLAR PHYSICS

The evolution of stars, supernovae, and explosive transients is governed by gravity, thermodynamics, and hydrodynamics as embodied in two equations (Landau and Lifshitz 1959) – plus many subsidiary conditions – opacity, mixing, transport rotation, etc

Force or momentum

$$\frac{dv}{dt} = -4\pi r^2 \frac{\partial P}{\partial m} - \frac{Gm}{r^2} + \frac{4\pi}{r}\frac{\partial Q}{\partial m},$$

Euler equation

energy

$$\frac{d\epsilon}{dt} = -4\pi P \frac{\partial}{\partial m}(vr^2) + 4\pi Q \frac{\partial}{\partial m}\left(\frac{v}{r}\right) - \frac{\partial L}{\partial m} + \dot{S};$$

where Q $\equiv \frac{4}{3}\eta_v r^4 \frac{\partial(v/r)}{\partial r}$ and η_v is the dynamic viscosity and the rest of the terms have their usual meaning. ε is the internal energy per gram and $\frac{\partial L}{\partial m}$ is the flux of energy due to convection or diffusion. Rotation and magnetic fields are neglected but could be included.

You may know better the equations of stellar structure (Q =0) in Lagrangian coordinates

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho}$$ continuity

$dm = 4\pi r^2 \rho \, dr \; - 4\pi r^2 \rho v \, dt$

$$\frac{\partial L}{\partial m} = S_{nuc} - \frac{\partial \varepsilon}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}$$

$$\Rightarrow \frac{\partial \rho}{\partial t} = -r^{-2}\frac{\partial\left(\rho r^2 v\right)}{\partial r}$$

see A.Weiss notes

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial v}{\partial t}$$

$$\frac{\partial T}{\partial m} = \frac{GmT}{4\pi r^4 P}\nabla$$

$$\nabla = \nabla_{rad} = \frac{3\bar{\kappa} L(m) P}{16\pi c G m a T^4}$$ or

$$\nabla = \nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_S$$

$\nabla_r < \nabla_{ad}$ is stable

Or neglecting time dependent terms and expressing in radial (Eulerian) coordinates

$$\frac{dm}{dr} = 4\pi r^2 \rho$$ mass conservation

$$\frac{dP}{dr} = -\frac{Gm(r)\rho}{r^2}$$ hydrostatic equilibrium

$$\frac{dL}{dr} = 4\pi r^2 \rho S_{nuc}$$ energy generation

$$\frac{dT}{dr} = \frac{3\bar{\kappa}\rho}{16\pi ac T^3}\frac{L(r)}{r^2} = \frac{1}{4\pi r^2 K_{cond}} L(r)$$ diffusion

In addition there is an equation that describes the mixing of composition

$$\frac{\partial X_i}{\partial t} = \frac{\partial}{\partial m}\left[\left(4\pi r^2\rho\right)^2 D\frac{\partial X_i}{\partial r}\right]$$

Ions are mixed by convection (time-dependent) and by other processes – rotation, semiconvection, and convective overshoot – but not by diffusion.

The transport physics enters in to the calculation of D. See Podsiadlowski notes.

TIME SCALES

These equations have associated with them four time scales. The shortest is the time required to approach and maintain hydrostatic equilibrium. Stars not in a state of dynamical implosion or explosion maintain a balance between pressure and gravity on a few sound crossing times. The sound crossing time is typically comparable to the free fall time scale.

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

$$P_{cent} \sim \frac{GM\rho}{2R}$$

and

$$c_s = \gamma\left(\frac{P}{\rho}\right)^{1/2} \sim \left(\frac{GM}{R}\right)^{1/2}$$

$$v_{esc} \sim \left(\frac{2GM}{R}\right)^{1/2}$$

So the escape speed and the sound speed in the deep interior are comparable. A shock wave can thus lead to mass ejection.

The free fall time scale is ~R/v_{esc} so

$$\tau_{esc} \sim \frac{R}{v_{esc}} = \left(\frac{R^3}{2GM}\right)^{1/2} \qquad \frac{M}{R^3} \sim \frac{4\pi\rho}{3}$$

$$\sim \left(\frac{3}{8\pi G\rho}\right)^{1/2} = \frac{R}{\dot{R}}$$

The number out front depends upon how the time scale is evaluated. The e-folding time for the density is 1/3 of this

$$\tau_{HD} \sim \left(\frac{1}{24\pi G\rho}\right)^{1/2} \sim \frac{446}{\sqrt{\rho}}\ \text{sec}$$

$$3\frac{\dot{R}}{R} = \frac{\dot{\rho}}{\rho} = \frac{1}{\tau_{HD}}$$

which is often used to describe explosions as well as collapse.

The second relevant adjustment time, moving up in scale, is the thermal time scale given by radiative diffusion, appropriately modified for convection. This is the time it takes for a star to come into and maintain thermal steady state, e.g., for the energy generated in the interior to balance that emitted in the form of radiation at the surface. (not all stars are in thermal steady state)

$$\tau_{therm} = \frac{R^2}{D_{therm}}$$

where D is the diffusion coefficient. In the simplest case, D is characteristically 1/3 times a length scale (e.g. the mean free path) times a characteristic speed (e.g., the speed of light).

If most of the energy does not reside in radiation this may be multiplied by a dimensionless correction factor.

The thermal diffusion coefficient (also called the thermal diffusivity) is defined as

$$D_T = \left(\frac{conductivity}{heat\ capacity}\right) = \frac{K}{C_P \rho}$$

where K appears in Fourier's equation

heat flow = $-K\ \nabla T$

For radiative diffusion the "conductivity", K, is given by

$$K = \frac{4acT^3}{3\kappa\rho} \qquad \text{(see Clayton 3-12)}$$

where κ is the opacity ($\mathrm{cm^2\ g^{-1}}$), thus

$$D_T = \left(\frac{4a\,cT^3}{3\kappa C_P \rho^2}\right)$$

where C_P is the heat capacity ($\mathrm{erg\ g^{-1}\ K^{-1}}$)

D_T here thus has units $\mathrm{cm^2\ s^{-1}}$

Note that

$$D \approx \left(\frac{c}{\kappa\rho}\right)\left(\frac{aT^4}{\rho C_P T}\right) = \left(\frac{c}{\kappa\rho}\right)\left(\frac{radiation\ energy\ content}{total\ heat\ content}\right)$$

If radiation energy density is a substantial fraction of the internal energy (not true in the general case), $D \sim c/\kappa\rho$ with κ the opacity, and taking advantage of the fact that in massive stars electron scattering dominates so that $\kappa = 0.2$ to $0.4\ cm^2\ g^{-1}$, the thermal time scale then scales like

$$t_{rad} \approx \frac{R^2\kappa\rho}{c} \sim \left(\frac{0.2R^2}{c}\right)\left(\frac{3M}{4\pi R^3}\right) \propto \frac{M}{R}$$

however, massive stars have convective cores so the thermal time is generally governed by the diffusion time in their outer layers. Since the dimensions are still (several) solar radii while the densities are less and the opacity about the same, the radiative time scales are somewhat less than the sun (1.7×10^5 yr; Mitalas and Sills, ApJ, 401, 759 (1992)).

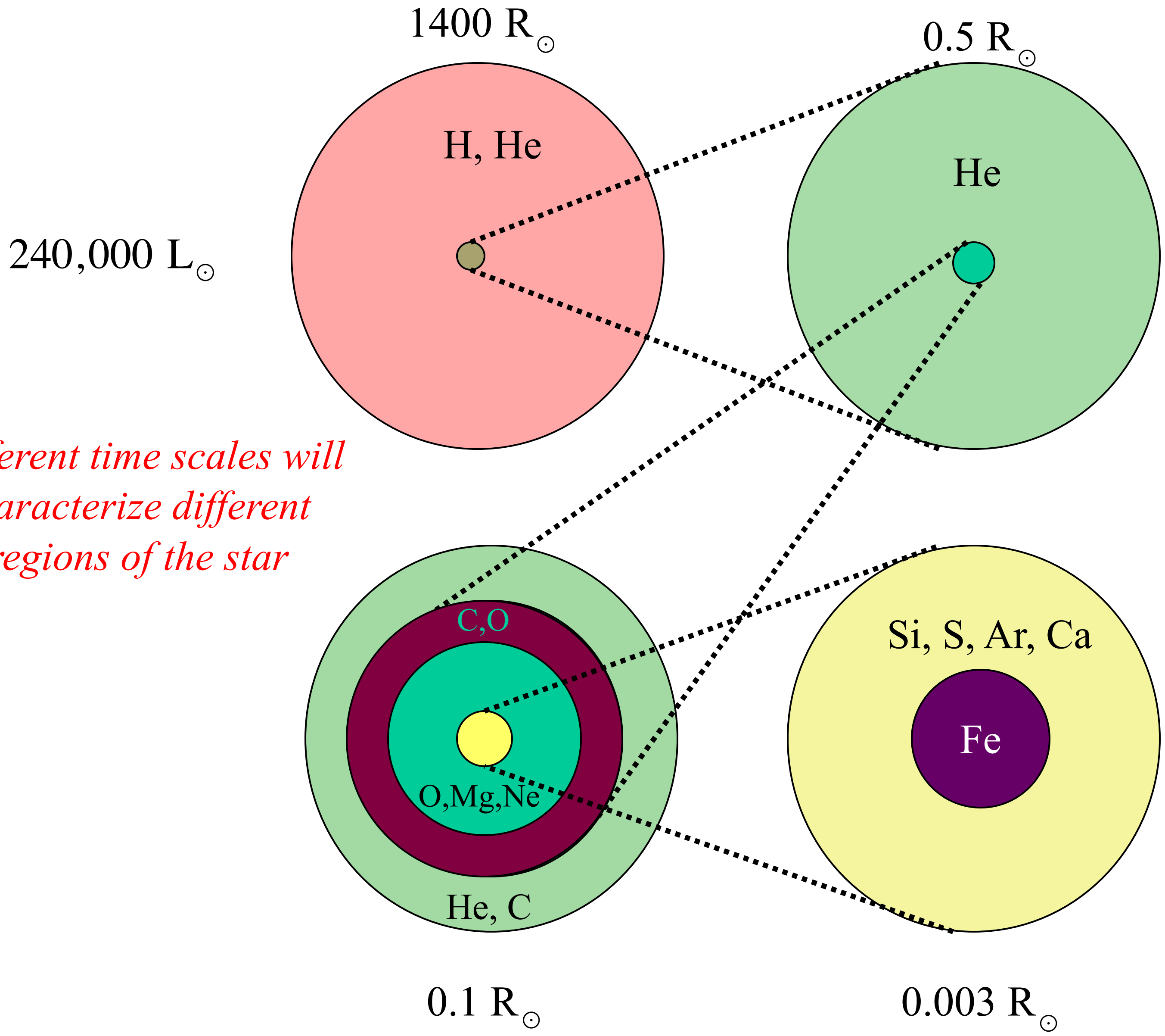
A closely related time scale is the Kelvin Helmholtz time scale

$$\tau_{KH} \approx \frac{GM^2}{2RL} \propto \frac{M^{5/3}\rho^{1/3}}{L}$$

$i.e.\ R \sim (M/\rho)^{1/3}$

Except for very massive stars, L on the main sequance is proportional to M to roughly the power 2 to 4, and ρ decreases with M so the Kelvin Helmholtz time scale is shorter for more massive stars. Note that there are numerous Kelvin Helmholtz time scales for massive stars since they typically go through six stages of nuclear burning. During the stages after helium burning, L in the heavy element core is given by pair neutrino emission and the Kelvin Helmhotz time scale becomes quite short - e.g. a protoneutron star evolves in a few seconds.

$25\ M_\odot$ Presupernova Star (typical for $9 - 130\ M_\odot$)



Different time scales will characterize different regions of the star

Finally, there is the nuclear time,

$$\tau_{\rm nuc} = \left(\frac{1}{X}\frac{dX}{dt}\right)^{-1},$$

where X is the mass fraction of the chief combustible fuel.

Usually, $\tau_{\rm HD} < \tau_{\rm thermal} < \tau_{\rm KH} < \tau_{\rm nuc}$. During the late stages of massive stellar evolution, however, the inequality $\tau_{\rm thermal} < \tau_{\rm nuc}$ actually begins to break down. During "explosive nucleosynthesis" in a supernova, there is near equality between $\tau_{\rm nuc}$ and τ_{HD}.

The life of a (non-degenerate) star is then typically a series of nuclear burning stages separated by periods of Kelvin-Helmholtz contraction. Hydrostatic equilibrium is maintained throughout the interior and thermal steady state is maintained if $\tau_{\rm therm}$ is short enough.

e.g., the sun is in thermal steady state. A presupernova star is not.

Examples of Time Scales

- In stellar explosions, the relevant time scale is the hydrodynamic one.

- Explosive nucleosynthesis happens when $\tau_{\mathrm{HD}} \sim \tau_{\mathrm{nuc}}$

- In between stages of nuclear burning, $\tau_{\mathrm{KH}} < \tau_{\mathrm{nuc}}$. The evolution occurs on a Kelvin Helmholtz time

- In a massive presupernova star the nuclear time scale in the inner core is less than the thermal time scale in the envelope. $\tau_{\mathrm{nuc}} < \tau_{\mathrm{thermal}}$. Thus the outer layers are not in thermal equilibrium with the interior. The core evolves like a separate star.

- In a supernova of Type I maximum light occurs when the age is equal to the diffusion time

Examples of Time Scales

- In the pulsational pair instability supernova the time between pulses is the Kelvin Helmholtz time.

- In Type Ia supernovae, the runaway occurs when the convection (i.e., thermal) time scale equals the nuclear time scale.

- Rotation and accretion can add additional time scales. E.g. Eddington Sweet vs nuclear. Accretion vs nuclear. Convection also has its own time scale.

Summary

Time scale	Value in sun	Scaling
τ – HD	30 min	$(R^3/GM)^{1/2}$
τ – diffusion	2 x 10^5 y	κ M/(Rc)
τ – KH	3 x 10^7 y	$GM^2/(RL)$
τ – nuc	6 x 10^9 y	qM/L

Relevant Stellar Physics

- **Equation of state** – $P(\rho,T,X_i)$, $\varepsilon(\rho,T,X_i)$

Perhaps the best understood part except at super nuclear density. Can be complicated when the electrons are semidegenerate and semirelativistic and in the presence of partial ionization but modern computers handle it easily.

- **Opacity** – $\kappa(\rho,T,X_i)$

Easy for electron scattering – though again problematic for partial ionization. Complex otherwise but there are tables. Uncertain at low temperature where dust and molecules form. Uncertain for heavy element hydrogen-free compositions

Stellar Physics

- **Mass Loss** – $\dot{M}(L, R, X_i)$, *binary mass exchange*

Perhaps the greatest uncertainty affecting modern studies of stellar evolution. Moderately well understood for line driven mass loss in hot stars. Very poorly understood for giant stars and for mass exchanging binaries. Very uncertain for massive stars (e.g. Eta Carina). Scaling with metallicity uncertain.

- **Convection**

Moderately well understood in slowly evolving stars far from convective boundaries. Poorly understood at convective boundaries (overshoot, undershoot), in semiconvective regions, in rotating stars, and in stars where the convective time scale is close to the evolutionary time scale

Stellar Physics

- **Nuclear Physics** – $S_{nuc}(T,\rho,X_i)$

Becoming well understood. Still a few critical reaction rates poorly determined.

- **Rotation and Magnetic Fields**

Rotation affects compositional mixing which affects the overall evolution. Processes understood qualitatively but not very quantitative. Angular momentum transport by magnetic torques during the evolution is a source of great uncertainty which affects our understanding of the explosion process and compact remnant properties.

Stellar Physics

- **Explosion Physics**

An area of great activity and uncertainty for decades.
A variety of mechanisms operate involving thermonuclear instability, neutrino transport, rotation, and magnetic fields. There are basic physics problems afflicting each of them - except for pair-instability supernovae.
E.g., flame physics and detonation, 3D neutrino transport, and magnetic instabilities.

Stellar Physics

- **Abundances**

The initial star has a composition given by all the activity that went on before it was born. A common assumption is that modern stars are born with a composition like the sun. Other "metallicities" require some assumptions about how different elemental ratios scale.

- **Chaos**

A recent realization is that even for well defined physics and initial conditions the final outcome of the evolution of a given massive star may be at some level indeterminate.

Some Critical Masses

Central Conditions for Polytropes

So long as a star is in hydrostatic equilibrium, it satisfies

$$\frac{dP}{dr} = -\frac{GM(r)\rho(r)}{r^2}.$$

If the density is assumed to be constant,

$$\rho = \text{const} = \frac{3M}{4\pi R^3},$$

direct integration implies

$$P_c = \frac{1}{2}\frac{GM\rho_c}{R} \propto \frac{M^2}{R^4}$$

where here $\rho_c = \bar{\rho} = \rho$.

It follows that

$$\frac{P_c}{\rho_c} = \frac{1}{2}GM\left(\frac{4\pi\rho_c}{3M}\right)^{1/3}$$

since

$$\frac{1}{R} = \left(\frac{4\pi\rho_c}{3M}\right)^{1/3}$$

and

$$\frac{P_c^3}{\rho_c^4} = \frac{4\pi}{24}G^3M^2$$

for constant density

More generally, for a polytrope of index n,
$P \propto \rho^\gamma$; $\gamma = (n+1)/n$, see e.g., Clayton, Eq. 2-313

$$P_c = \frac{4\pi R^2 G}{(n+1)\zeta_1^2}\rho_c^2$$

$$\bar{\rho} = -\frac{3}{\zeta_1}\left(\frac{df}{d\zeta}\right)_{\zeta_1} \rho_c$$

$$= \frac{3M}{4\pi R^3}$$

where ζ_1 is the Emden constant given, e.g., in Table 2.5 of Clayton.

From this it follows that

$$\frac{P_c^3}{\rho_c^4} = 4\pi G^3 \left(\frac{M}{\phi}\right)^2$$

where ϕ is a constant given by solution of the polytropic equation for index n,

$$\phi = (n+1)^{3/2}\zeta_1^2\left(\frac{df}{d\zeta}\right)_{\zeta_1}$$

$$n = 0 \qquad \phi = 4.8988 \;=\sqrt{24}$$
$$n = 3 \qquad \phi = 16.145$$
$$n = 3/2 \qquad \phi = 10.73$$

most stars have
$1.5 < n < 3$

Now, if P is P_{ideal} (NR, ND, ionized),

$$P_{\text{ideal}} = \frac{N_A k}{\mu} \rho T$$

where μ is the mean molecular weight

Aside: Abundance nomenclature

In general the mass fraction of a species "i" is X_i. The number density of i is then

$$n_i = \rho N_A \frac{X_i}{A_i}$$

with A_i the atomic mass number (integer) of isotope i and N_A, Avogadro's number, 6.02205×10^{23} particles/mole, or approximately the reciprocal mass of the nucleon in grams.

In this class we will extensively use the notation

$$Y_i = \frac{X_i}{A_i}$$

where Y_i is like a dimensionless number density

$$Y_i = \frac{n_i}{\rho N_A}$$

Similarly we can define an electron abundance variable

$$Y_e = \frac{n_e}{\rho N_A}$$

The total gas pressure for an ideal, non-relativistic, non-degenerate ionized gas is then

$$P_{\text{ideal}} = \rho N_A kT \left[\Sigma Y_i + Y_e\right]$$

which implies

$$\mu = \left[\Sigma Y_i + Y_e\right]^{-1}$$

Also the mean atomic weight, $\bar{A}$, is given

Actually the dimensions of Y are Mole/gm and N_A has dimensions particles per Mole.

$$P = \Sigma\, n_i\, kT \;=\; \frac{N_A k}{\mu} \rho T$$

by

$$\bar{A} = \frac{\Sigma\, n_i A_i}{\Sigma\, n_i} = \frac{\rho N_A \Sigma\, Y_i A_i}{\rho N_A \Sigma\, Y_i} = \frac{\Sigma\, X_i}{\Sigma\, Y_i}$$

$$= (\Sigma\, Y_i)^{-1}$$

X_i= mass fraction of species "i"

Similarly

$$Y_e = \Sigma\, Z_i Y_i$$

$$\rho N_A Y_e = n_e = \sum Z_i n_i$$

$$= \rho N_A \sum Z_i Y_i$$

and

$$\mu = (\Sigma\,(1 + Z_i) Y_i)^{-1}$$

$$0.5 < \mu < 2$$

Some examples:

a) Pure hydrogen:

$$Y_H = 1 \qquad \bar{A} = 1 \qquad Y_e = 1$$

$$\mu = (1+1)^{-1} = \frac{1}{2}$$

$$P_{\text{ideal}} = 2\rho N_A kT$$

(The limit μ=2 is achieved as A goes to infinity and Z = A/2, i.e. electrons dominate)

b) 75% H, 25% He:

$$\bar{A} = \left(0.75 + \frac{0.25}{4}\right)^{-1} \qquad = \left(\sum Y_i\right)^{-1}$$
$$= 1.23$$

$$Y_e = 0.75 + (2)(\frac{0.25}{4}) \qquad = \sum Z_i Y_i$$
$$= 0.875$$

$$\mu = \left[(1+1)(0.75) \; + \; (1+2)(\frac{0.25}{4})\right]^{-1} \qquad = \left((1+Z_i)Y_i\right)^{-1}$$
$$= 0.5926$$

$$P_{\mathrm{ideal}} = 1.69\,\rho N_A kT$$

As an exercise to the reader, for pure helium, $\bar{A} = 4$, $Y_e = 0.50$, $\mu = 4/3$, and $P_{\mathrm{ideal}} = 0.75\rho N_A kT$. For a mixture of 50% ^{12}C and 50% ^{16}O, $\bar{A} = 13.71$, $Y_e = 0.50$ (as it always does for a gas of isotopes having neutron number = proton number), μ

For advanced stages of evolution where $\bar{A} > 1$, most of the pressure is due to the electrons (and radiation)

$= 1.745$, and $P_{\rm ideal} = 0.573 \rho N_A k T$, (0.50 from e$^-$; 0.073 for ions).

Back to the main discussion:

$$\frac{P_c^3}{\rho_c^4} \propto M^2$$

thus implies for an ideal gas equation of state

$$P_c \propto \rho_c \frac{T_c}{\mu}$$

$$\boxed{\frac{T_c^3}{\rho_c} \propto M^2 \mu^3}$$

This would suggest that the ratio would *increase* as the star evolved and μ became greater.

Therefore, for a given temperature, as might be necessary to burn a given fuel, for example, the central density will be lower for a star of higher mass. And, in fact, for a given constant mass and composition, so long as the star closely resembles a single polytrope, and the pressure remains ideal, the central density will scale as

$$\boxed{\rho_c \propto \left(\frac{T_c}{\mu}\right)^3}$$

$$T_c \propto \mu \rho_c^{1/3}$$

Since μ increases as the fuel burns to heavier ashes, the relation $\rho \propto T^3$ works pretty well at the stars center but tends to be an over-estimate. The onset of degeneracy or near relativistic motion of the electrons at high temperature can also cause deviations.

Now, especially for massive stars, the radiation pressure will not be negligible. One traditionally defines a quantity

$$\beta = \frac{P_{\rm ideal}}{P_{\rm ideal} + P_{\rm rad}} = \left(\frac{P_{ideal}}{P_{total}}\right)$$

$$P_{\rm tot} = P_{\rm ideal}/\beta$$

$$P_{tot} \propto \frac{P_{ideal}}{\beta} \propto \rho_c \frac{T_c}{\mu\beta}$$

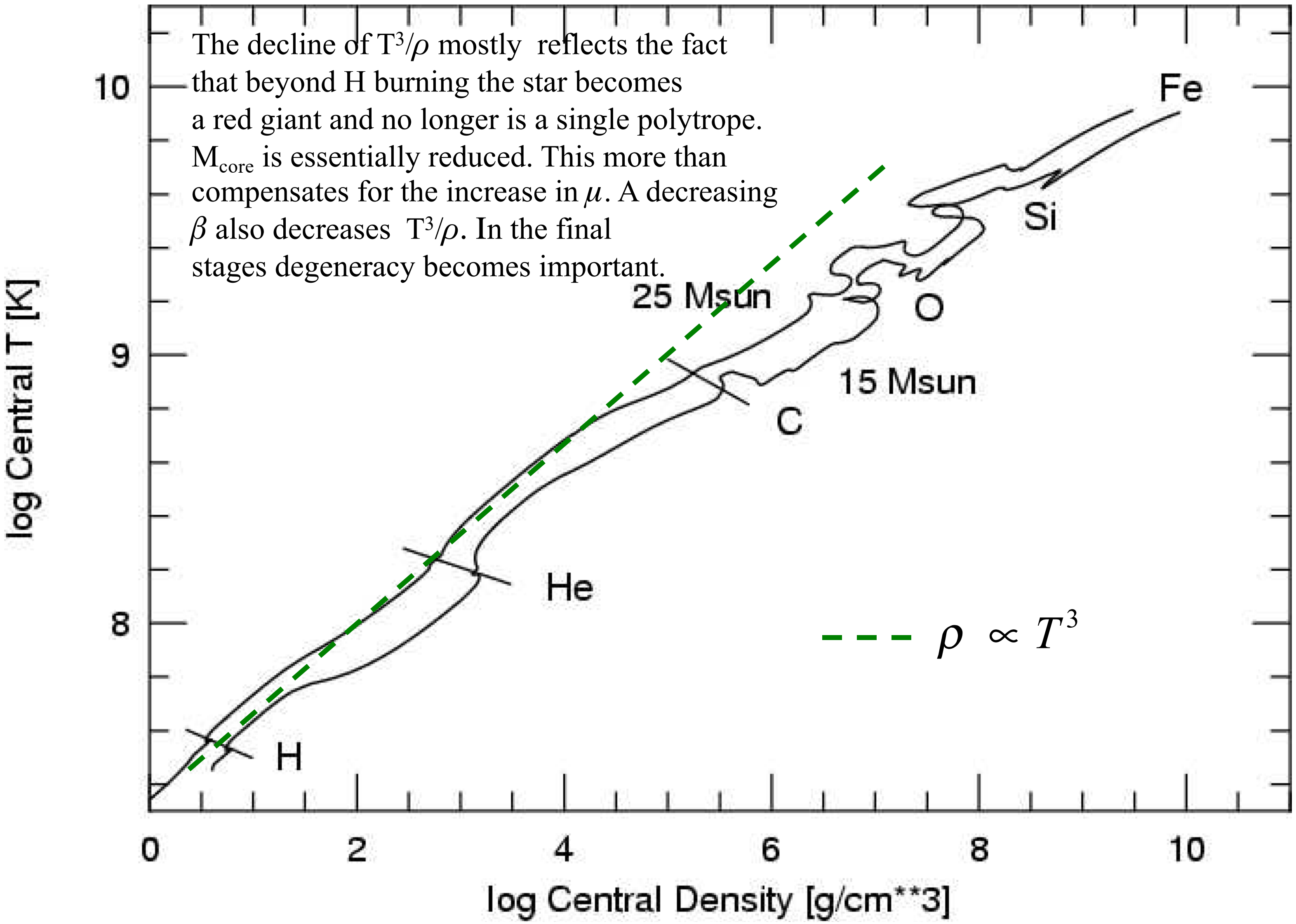
Then $P_c^3/\rho_c^4 \propto M^2$, implies

$$\frac{T_c^3}{\rho_c} \propto M^2 \beta^3 \mu^3,$$

that is, so long as beta doesn't change much, one gets the same relation as before.

Decrease in β as star evolves acts to lower T^3/ρ.

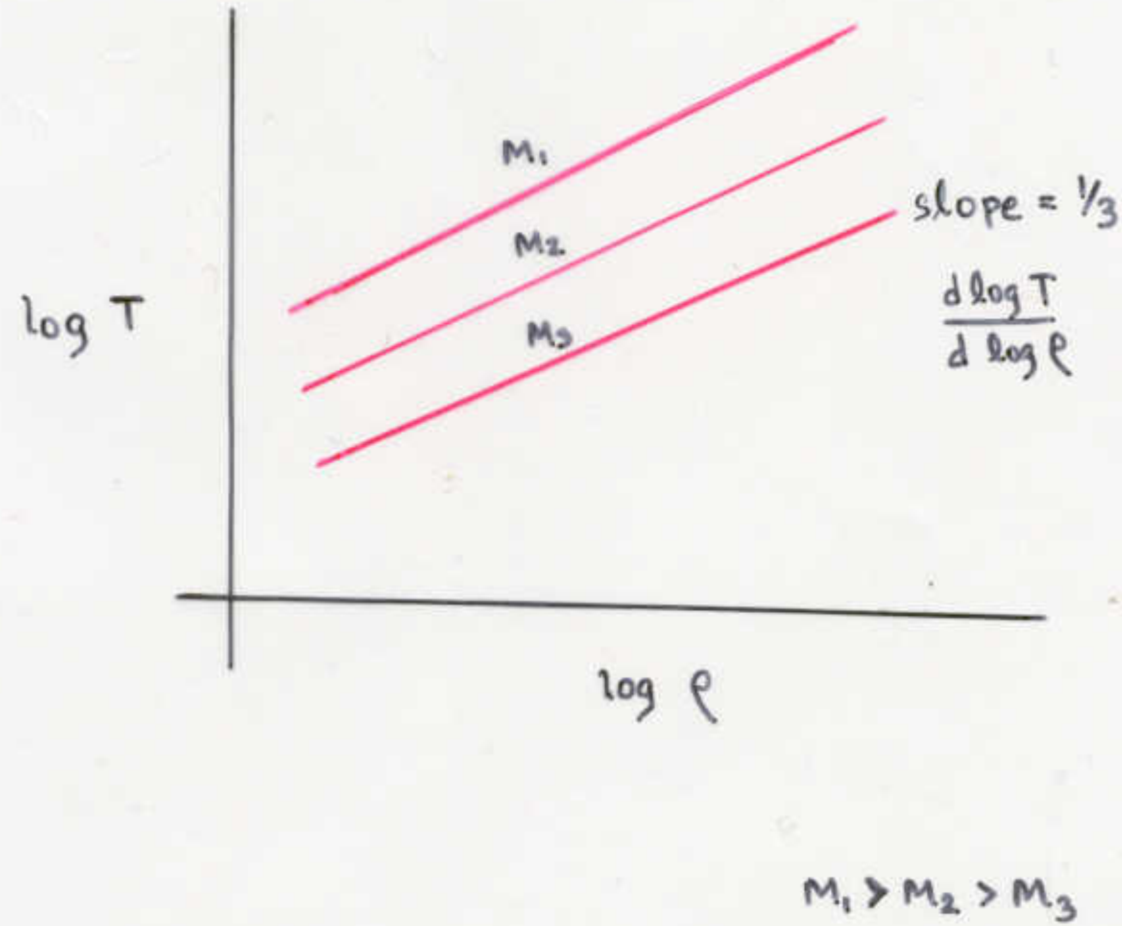
As before but now the relation for T_c is more correctly derived.



CRITICAL MASSES

Dropping, for now, the explicit dependence on μ and β, a contracting protostar of constant mass, or the contracting core of a massive star in between burning stages, so long as that core has an approximately constant polytropic index, will obey $T_c \propto \rho_c^{1/3} M^{2/3}$. Contraction leads to heating. The greater weight of the more compact configuration requires more pressure to hold it up and the pressure rises by increasing both T and ρ. A plot of log T_c *vs.* log ρ_c gives a straight line with an upward slope of 1/3. Lines for larger mass will lie above those for lower mass. As the density grows ever higher, three possibilities emerge: a) collapse to a black hole; b) a dynamical event of some sort (e.g., neutron star formation) or c) the onset of degeneracy. For now, we are most interested in c).

Ideal gas plus radiation



log T
log ρ
M1
M2
M3
slope = 1/3
d log T / d log ρ
M1 > M2 > M3

A completely degenerate gas can be characterized by an equation of state of the form $P_{\rm deg} = K_\gamma(\rho Y_e)^\gamma$ with γ between 4/3 and 5/3.

The case $\gamma = 4/3$ has a well known singularity. For an n = 3 polytrope, which is appropriate here,

$$\frac{P_c^3}{\rho_c^4} = 4\pi G^3 \left(\frac{M}{16.14}\right)^2 = \frac{K_{4/3}^3 \rho_c^4 Y_e^4}{\rho_c^4}$$

note cancellation of ρ_c

$$M = \left(\frac{20.745 K_{4/3}^3}{G^3}\right)^{1/2} Y_e^2$$

$$K_{4/3} = 1.244 \times 10^{15} \text{ dyne cm}^{-2}$$

$$\frac{M}{M_\odot} = 5.80\ Y_e^2$$

The Chandrasekhar Mass

$= 1.45\, M_\odot$ if $Y_e = 0.50$ neglecting Coulomb corrections and relativistic corrections. 1.39 $M_\odot$ if they are included.

For lower densities and hence degenerate cores significantly less than the Chandrasekhar mass non-relativistic degeneracy pressure gives another solution (γ = 5/3)

$$\frac{P_c^3}{\rho_c^4} = 4\pi G^3 \left(\frac{M}{10.73}\right)^2$$

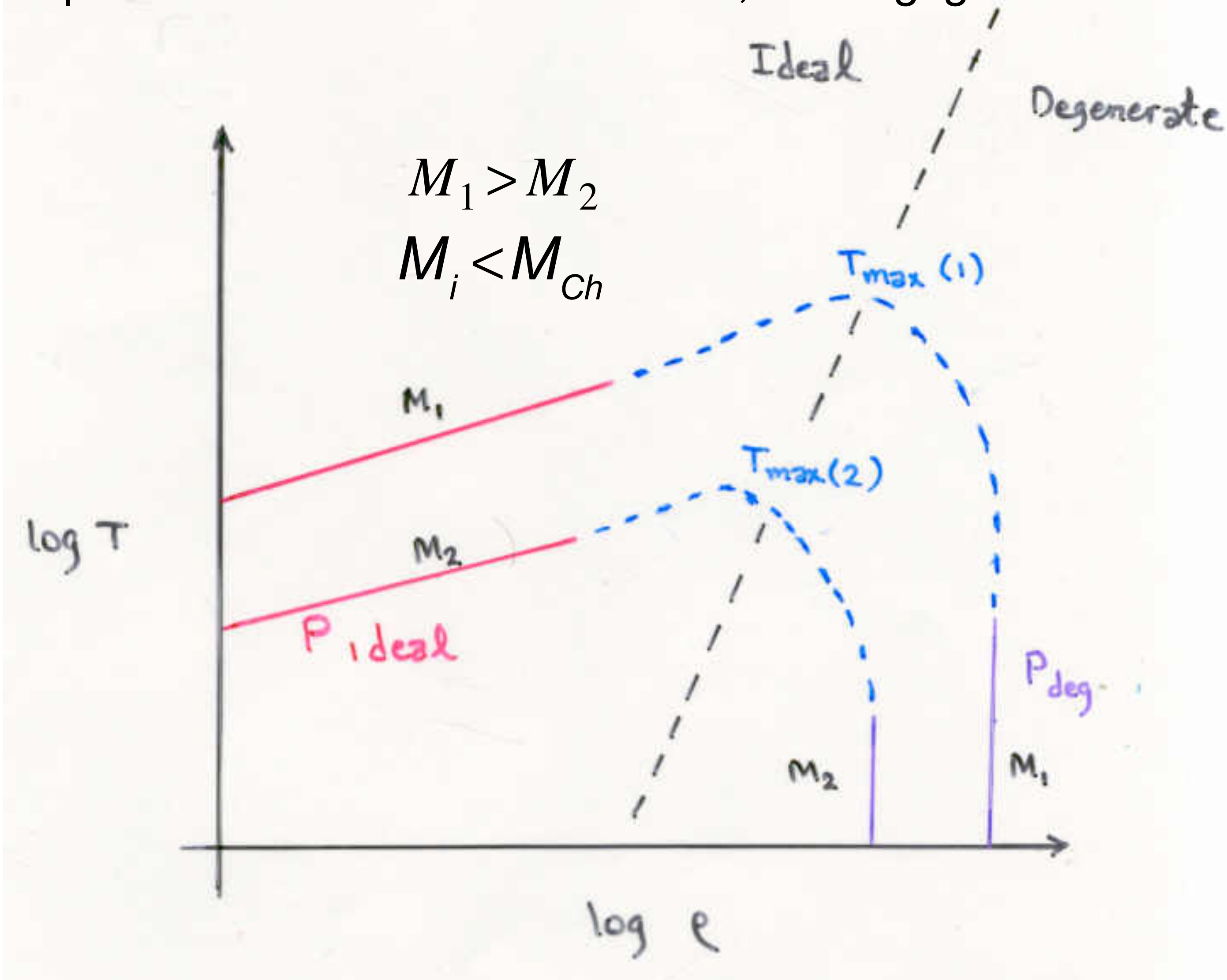
$$\rho_c = \frac{4\pi G^3 M^2}{K_{5/3}^3 Y_e^5 (10.73)^2}$$

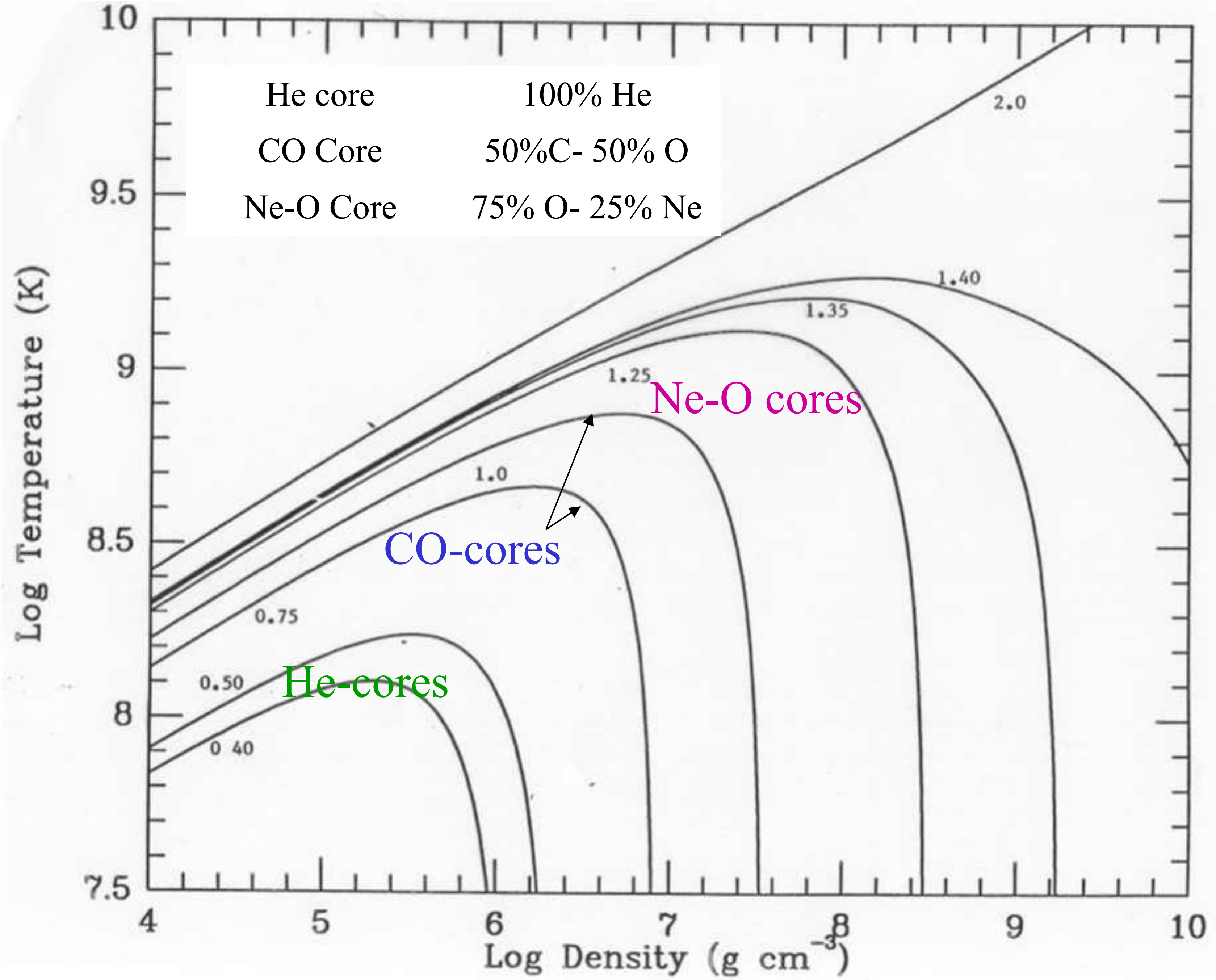
$$= 4.05 \times 10^6 \left(\frac{0.5}{Y_e}\right)^5 \left(\frac{M}{M_\odot}\right)^2 \text{ g cm}^{-3}$$

This implies, for each mass, a stable permanent configuration of fixed ρ_c independent of T..

This is the well known central density mass relation for (non-relativistic) white dwarfs

Specifying a core mass gives a maximum temperature achieved before degeneracy supports the star. If that maximum temperature exceeds a critical value, burning ignites





He core
100% He
CO Core
50%C- 50% O
Ne-O Core
75% O- 25% Ne
2.0
1.40
1.35
1.25
Ne-O cores
1.0
CO-cores
0.75
0.50
He-cores
0 40
Log Temperature (K)
Log Density (g cm^{-3})
10
9.5
9
8.5
8
7.5
4
5
6
7
8
9
10

Critical Temperatures

Given by "balanced power" for post-helium burning phases and stellar models or polytropes for H and He.

Fuel	Main Product	Secondary Products	Temp (10^9 K)	Time (yr)
H	He	^{14}N	0.02	10^7
He	C,O	^{18}O, ^{22}Ne s- process	0.2	10^6
C	Ne, Mg	Na	0.8	10^3
Ne	O, Mg	Al, P	1.5	3
O	Si, S	Cl, Ar K, Ca	2.0	0.8
Si	Fe	Ti, V, Cr Mn, Co, Ni	3.5	1 week

The calculations shown give critical masses:

C	~1.0
Ne	~1.25
O	1.39
Si	1.39

More detailed and physical calculations exist in the literature, see especially Nomoto and Hashimoto (1986). The following should be regarded as standard

Fuel	Min. Mass
He	0.25
C	1.06
Ne	1.37
O	1.39
Si	1.39

These are core masses. The corresponding main sequence masses are larger.

All stars with main sequence mass above the Chandrasekhar mass could in principle go on to burn Si. In fact, that never happens below ~8 solar masses. Stars develop a red giant structure with a low density surrounding a compact core. The convective envelope "dredges up" helium core material and causes it to shrink. Only for stars above about 8 or 9 solar masses does the He core stay greater than the Chandrasekhar mass after helium burning.

Main Sequence Critical Masses

0.08	$M_\odot$	Lower limit for hydrogen ignition
0.45	$M_\odot$	helium ignition
7.25	$M_\odot$	carbon ignition
9.00	$M_\odot$	neon, oxygen, silicon ignition (off center)
10.5	$M_\odot$	ignite all stages at the stellar center
~ 70	$M_\odot$	First encounter the pair instability (neglecting mass loss)
~35	$M_\odot$	Lose envelope if solar metallicity star

These are for single stars calculated with the KEPLER code including semiconvection and convective overshoot mixing but ignoring rotation. With rotation the numbers may be shifted to slightly lower values. Low metallicity may raise the numbers slightly since less initial He means a smaller helium core. Other codes give different results typically to within 1 solar mass. Results for binaries will differ.

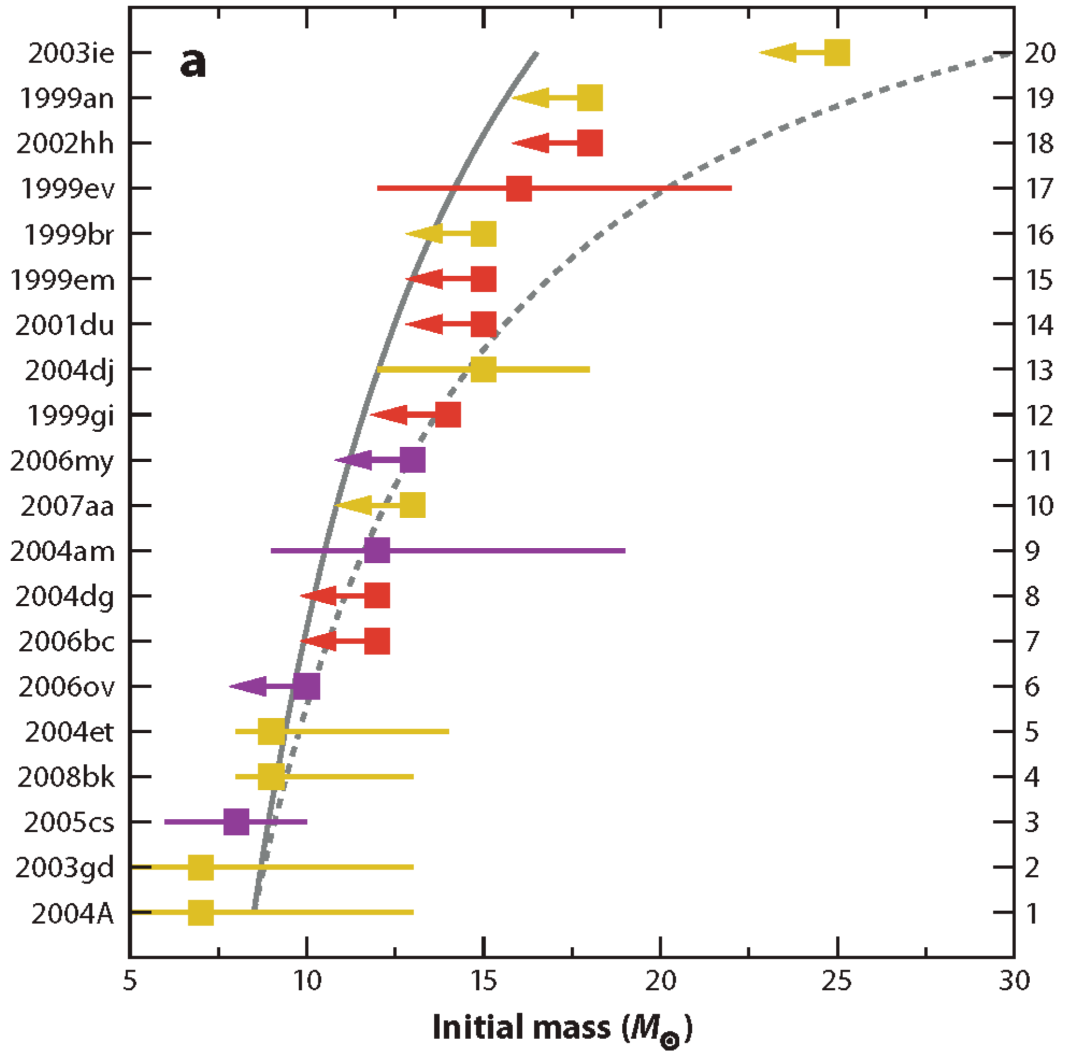
Between 8 and 10.5 solar masses the evolution can be quite complicated and code dependent owing to the combined effects of degeneracy and neutrino losses. Off-center ignition is the norm for the post-carbon burning stages.

Mass loss introduces additional uncertainty, especially with regard to final outcome. Does a 8.5 solar mass main sequence star produce a NeO white dwarf or an electron-capture superno.

Above 9 solar masses an iron core eventually forms – on up to the pair instability limit.

Mass (solar masses)	End point	Remnant
< 7 to ~8	planetary nebula	CO white dwarf
~8 to ~11	degenerate core neutrino-powreed low energy SN	Ne-O WD below 9? neutron star above 9
~11 - ~20	neutrino-powered normal supernova; SN Ibc in binary. Islands of explosion at higher mass	neutron stars and black holes
20 - 70	without mass loss probably no SN (unless rotationally powered); with mass loss SN Ibc	black hole if enough mass loss neutron star
70 – 150	pulsational pair SN if low mass loss	black hole
150 - 260	pair instability SN if low mass loss	none
> 260	pair induced collapse if low mass loss	black hole

Presupernova stars – Type IIp and II-L



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The solid line is for a Salpeter IMF with a maximum mass of 16.5 solar masses. The dashed line is a Salpeter IMF with a maximum of 35 solar masses

Minimum mass supernova

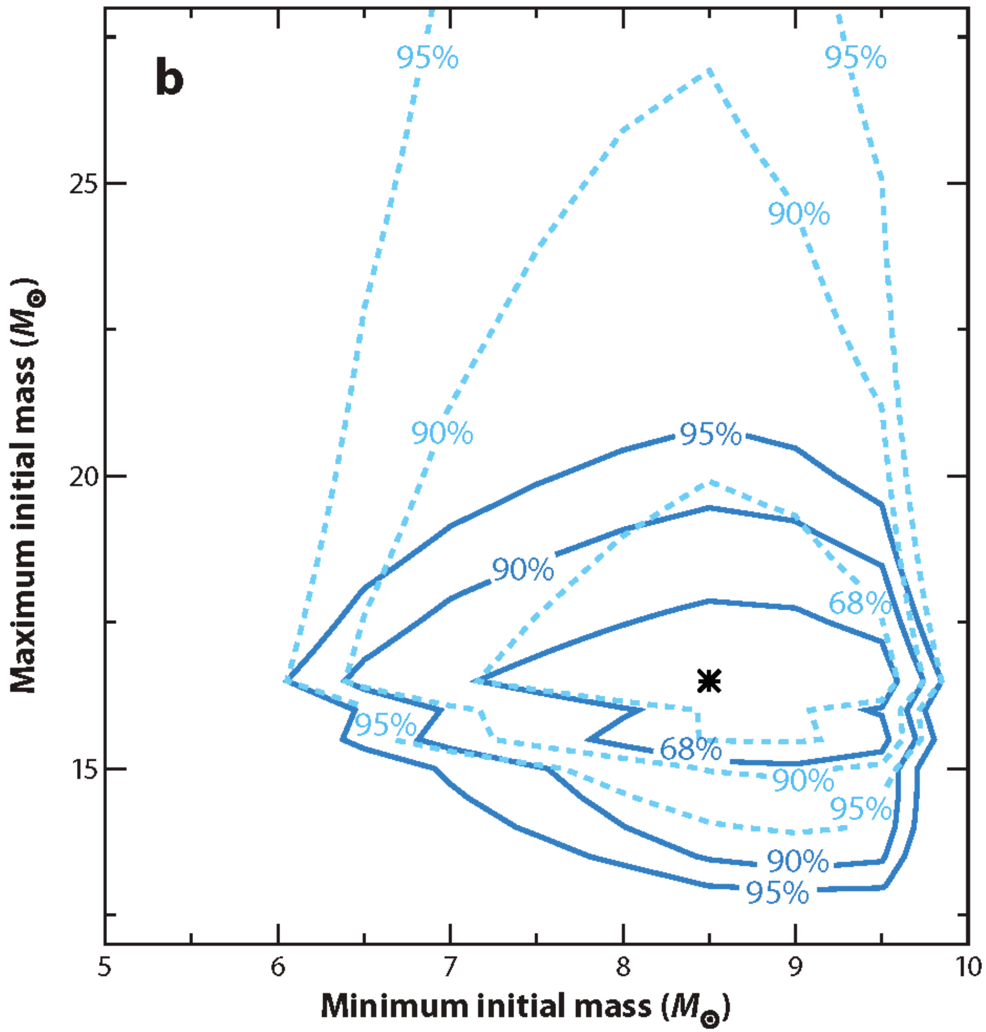

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Fig. 6b

Based on the previous figure. Solid lines use observed preSN only. Dashed lines include upper limits.