Lecture 12

Advanced Stages of Stellar Evolution – II
Silicon Burning and NSE

Silicon Burning

Silicon burning proceeds in a way different from any nuclear process discussed so far. It is analogous, in ways, to neon burning in that it proceeds by photodisintegration and rearrangement*, but it involves many more nuclei and is quite complex.

The reaction $^{28}Si + {}^{28}Si \rightarrow ({}^{56}Ni)^*$ does not occur owing to the large Coulomb inhibition. Rather a portion of the silicon (and sulfur, argon, etc.) “melt” by photodisintegration reactions into a sea of neutrons, protons, and alpha-particles. These lighter constituents add onto the remaining silicon and heavier elements, gradually increasing their mean atomic weight until species in the iron group are most abundant.

Carbon burning	Heavy ion fusion
Neon Burning	Photodisintegration rearrangement
Oxygen burning	Heavy ion fusion
Silicon burning	Photodisintegration rearrangement

*Basically the temperature threshold for removing an alpha from ^{24}Mg is reached before that of $^{28}Si + {}^{28}Si$

Initial Composition

The initial composition is mostly Si and S, but which isotopes of Si and S dominate depends upon whether one is discussing the inner core or less dense locations farther out in the star. It is quite different for silicon core burning in a presupernova star and the explosive variety of silicon burning we shall discuss later.

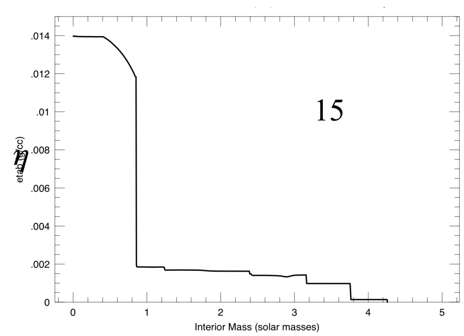
In the *center of the star*, one typically has, after oxygen burning, and a phase of electron capture that goes on between oxygen depletion and silicon ignition:

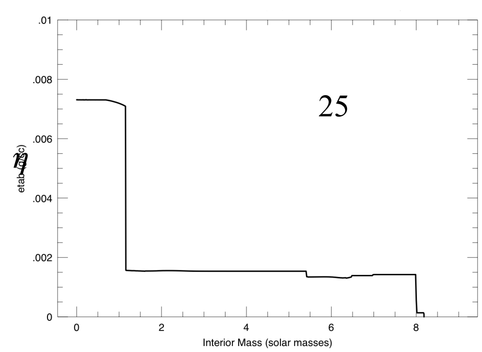
^{30}Si, ^{34}S, ^{38}Ar and a lot of other less abundant nuclei. High η

Farther outside of the core where silicon might burn explosively, one has species more characteristically with Z = N

^{28}Si, ^{32}S, ^{36}Ar, ^{40}Ca, etc.

Historically, Si burning was discussed for a ^{28}Si rich composition. Low η





Neutron excess after oxygen core depletion in 15 and 25 solar mass stars.

The inner core is becoming increasingly “neutronized”, especially for the lower mass stars. The process accelerates during silicon burning

The nucleosynthesis of the inner core would be very strange were it to be ejected (it is not). η ~ 0.002 – 0.004 is good. 0.01 is not

Quasi-equilibrium

This term is used to describe a situation where groups of adjacent isotopes, but not all isotopes globally, have come into equilibrium with respect to the exchange of n, p, α, and γ.

It began in neon burning with $^{20}\mathrm{Ne} + \gamma \rightleftharpoons {}^{16}\mathrm{O} + \alpha$ and continues to characterize an increasing number of nuclei during oxygen burning. In silicon burning, it becomes the rule rather than the exception.

A typical "quasiequilibrium cluster" might include the equilibrated reactions :

$$^{28}\mathrm{Si} \underset{n}{\rightleftarrows} {}^{29}\mathrm{Si} \underset{n}{\rightleftarrows} {}^{30}\mathrm{Si} \underset{p}{\rightleftarrows} {}^{31}\mathrm{P} \underset{p}{\rightleftarrows} {}^{32}\mathrm{S} \underset{\alpha}{\rightleftarrows} {}^{28}\mathrm{Si}$$

By which one means $\mathrm{Y}(^{28}\mathrm{Si})\,\mathrm{Y_n}\,\rho\,\lambda_{n\gamma}(^{28}\mathrm{Si}) \approx \mathrm{Y}(^{29}\mathrm{Si})\,\lambda_{\gamma n}(^{29}\mathrm{Si})$

$$\mathrm{Y}(^{30}\mathrm{Si})\mathrm{Y_n}\rho\,\lambda_{n\gamma}(^{29}\mathrm{Si}) \approx \mathrm{Y}(^{30}\mathrm{Si})\,\lambda_{\gamma n}(^{30}\mathrm{Si})$$

etc.

Late during oxygen burning, many isolated clusters grow and merge until, at silicon ignition, there exist only two large QE groups

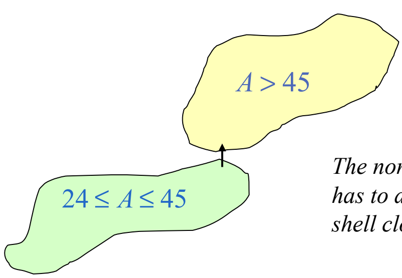
$24 \le A \le 45$ $\quad\quad$ $46 \le A \le 60$ (at least)

Reactions below ^{24}Mg, e.g., ^{20}Ne(α,γ)^{24}Mg and ^{12}C(α,γ)^{16}O are, in general, *not* in equilibrium with their inverses at oxygen depletion (exception, ^{16}O(α,γ)^{20}Ne which has been in equilibrium since neon burning).

Within the groups heavier than A = 24, except at the boundaries of the clusters, the abundance of any species is related to that of another by successive application of the Saha equation.

$$e.g., \quad \frac{Y(^{40}Ca)}{Y(^{28}Si)} = \left(\frac{Y(^{32}S)}{Y(^{28}Si)}\right)\left(\frac{Y(^{36}Ar)}{Y(^{32}S)}\right)\left(\frac{Y(^{40}Ca)}{Y(^{36}Ar)}\right)$$
$$= \left(\frac{\rho Y_\alpha \lambda_{\alpha\gamma}(^{28}Si)}{\lambda_{\gamma\alpha}(^{32}S)}\right)\left(\frac{\rho Y_\alpha \lambda_{\alpha\gamma}(^{32}S)}{\lambda_{\gamma\alpha}(^{36}Ar)}\right)\left(\frac{\rho Y_\alpha \lambda_{\alpha\gamma}(^{36}Ar)}{\lambda_{\gamma\alpha}(^{40}Ca)}\right)$$
$$= f(T, Q_{\alpha\gamma})\ \rho^3\, Y_\alpha^3 \quad\quad etc.$$

The situation at the end of oxygen burning is that there are two large QE groups coupled by non-equilibrated links near A = 45.



The non-equilibrated link has to do with the double shell closure at Z = N = 20

Early during silicon burning these two groups merge and the only remaining non-equilibrated reactions are for A < 24 (Mg).

Within that one group,(A > 23), which contains ^{28}Si and the vast majority of the mass, one can evaluate any abundance relative to e.g., ^{28}Si

$$\boxed{Y(^A Z) = C(^A Z, \rho, T_9)\, Y(^{28}Si)\, Y_\alpha^{\delta_\alpha} Y_p^{\delta_p} Y_n^{\delta_n}}$$

where δ_α = largest integer $\le \frac{Z-14}{2}$

Need 6 parameters: Y_α, Y_p, Y_n and $Y(^{28}Si)$ plus T and ρ, but …

$$\delta_n = N - 14 - 2\delta_\alpha$$
$$\delta_p = Z - 14 - 2\delta_\alpha$$

$e.g., {}^{35}Cl$ 17 protons; 20 neutrons

$$\delta_\alpha = 1 \quad \delta_p = 1 \quad \delta_n = 2$$

^{40}K 19 protons 21 neutrons

$$\delta_\alpha = 2 \quad \delta_p = 1 \quad \delta_n = 3$$

$$C(^AZ) = (\rho N_A)^{\delta_\alpha+\delta_p+\delta_n} C'(^AZ)$$

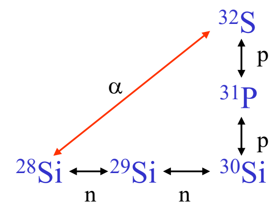
$$C'(^AZ) = \left(5.942\times10^{33}\, T_9^{3/2}\right)^{-(\delta_\alpha+\delta_p+\delta_n)} \frac{G(^AZ)}{G(^{28}Si)}\, 2^{-(\delta_p+\delta_n)} \left(\frac{A}{28}\right)^{3/2} \left(\frac{1}{4}\right)^{3\delta_\alpha/2} \exp(Q/kT)$$

where

$$Q = BE(^AZ) - BE(^{28}Si) - \delta_\alpha BE(\alpha)$$

i.e., the energy required to dissociate the nucleus ^AZ into ^{28}Si and δ_α alpha particles. The binding energy of a neutron or proton is zero.

Moreover there exist loops like:



For constant ρ and T

$$Y(^{32}S) = const \bullet Y(^{30}Si) Y_p^2$$
$$Y(^{30}Si) = const' \bullet Y(^{28}Si) Y_n^2$$
$$Y(^{32}S) = const'' \bullet Y(^{28}Si) Y_\alpha$$
$$Y_\alpha \propto \left(\frac{Y(^{32}S)}{Y(^{28}Si)}\right) \propto \left(\frac{Y(^{32}S)}{Y(^{30}Si)}\right)\left(\frac{Y(^{30}Si)}{Y(^{28}Si)}\right) \propto Y_p^2 Y_n^2$$

$$\boxed{Y_\alpha = C_\alpha(T_9,\rho)\, Y_n^2 Y_p^2}$$

$$C_\alpha = \frac{1}{2}\,(5.94\times10^{33})^{-3}\, \rho^3 T_9^{9/2} \exp(328.36/T_9)$$

where 328.36 = BE(α)/kT

This reduces the number of independent variables to 5, but wait ...

The large QE cluster that includes nuclei from A = 24 through at least A = 60 contains most of the matter (^{20}Ne, ^{16}O, ^{12}C, and α are all small), so we have the additional two constraints

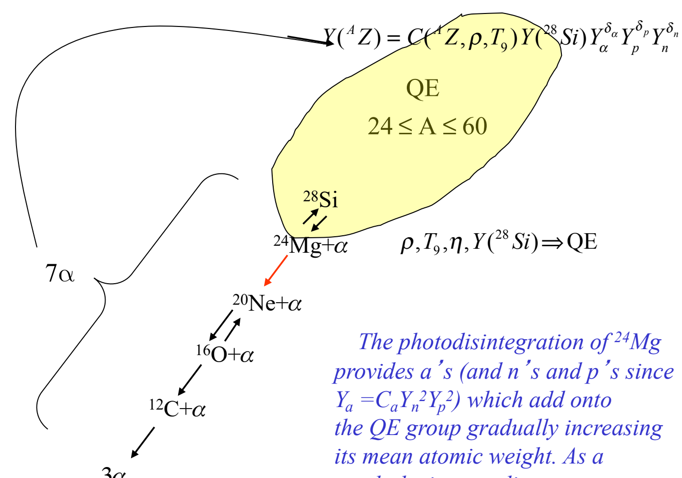
$$\sum_{60\geq A\geq 24} A_i Y_i \approx 1 \qquad \text{mass conservation}$$

$$\sum_{60\geq A\geq 24} (N_i - Z_i)\, Y_i \approx \eta \qquad \text{charge conservation}$$

The first equation can be used to eliminate one more unknown, say Y_p, and the second can be used to replace Y_n with an easier to use variable, η. *Thus 4 variables now specify the abundances of all nuclei heavier than magnesium* . These are

$$\boxed{\rho,\ T_9,\ \eta,\ \text{and } Y(^{28}Si)}$$

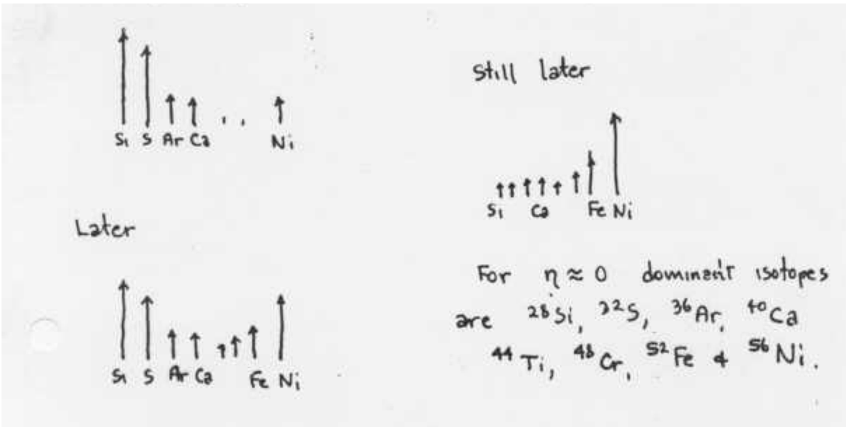
For low η, the cluster evolves at a rate given by ^{24}Mg(γ,α)^{20}Ne



The photodisintegration of ^{24}Mg provides a's (and n's and p's since $Y_\alpha = C_\alpha Y_n^2 Y_p^2$) which add onto the QE group gradually increasing its mean atomic weight. As a result the intermediate mass group, Si-Ca gradually "melts" into the iron group.

Nature of the burning:

Lighter species melt away while the iron group grows



Energetics:

Suppose ^{28}Si burns to ^{56}Ni. To rough approximation

$2(^{28}\mathrm{Si}) \rightarrow\ ^{56}\mathrm{Ni}$ (n.b. not fusion of 2 silicons)

$$q_{nuc} = 9.65 \times 10^{17}\,[1/2\,(483.982 - 236.536)/28]$$
$$= 1.9 \times 10^{17}\ \mathrm{erg\ g^{-1}} \quad \text{(not much)}$$

But this assumes ^{28}Si burns to ^{56}Ni (small η approximation)

This is misleading because, except explosively (later), silicon burning does not produce ^{56}Ni. There has been a lot of electron capture during oxygen burning and more happens in silicon burning. The silicon that burns is not ^{28}Si, but more typically ^{30}Si.

E.g., Si ignition in a 15 $M_\odot$ star	$\eta_c \approx 0.07$	$Y_e \approx 0.46$
Si depletion	$\eta_c \approx 0.13$	$Y_e \approx 0.44$

Under these conditions silicon burning produces ^{54}Fe, ^{56}Fe, ^{58}Fe and other neutron rich species in the iron group.

Suppose $\left(\dfrac{56}{30}\right)\ ^{30}\mathrm{Si} \rightarrow\ ^{56}\mathrm{Fe}$ $\qquad$ $i.e., q_{nuc} = 9.65\times10^{17} \sum \dfrac{X_i}{A_i} BE(A_i)$

$$q_{nuc} = 9.65 \times 10^{17}\left[\,(492.26)/56 - (255.62)/30\,\right]$$
$$= \boxed{2.6 \times 10^{17}\ \mathrm{erg\ g^{-1}}}$$ which is closer to correct for Si core burning than 1.9×10^{17} erg g^{-1}

An approximation to energy generation is derived in Appendix 2. See also next page

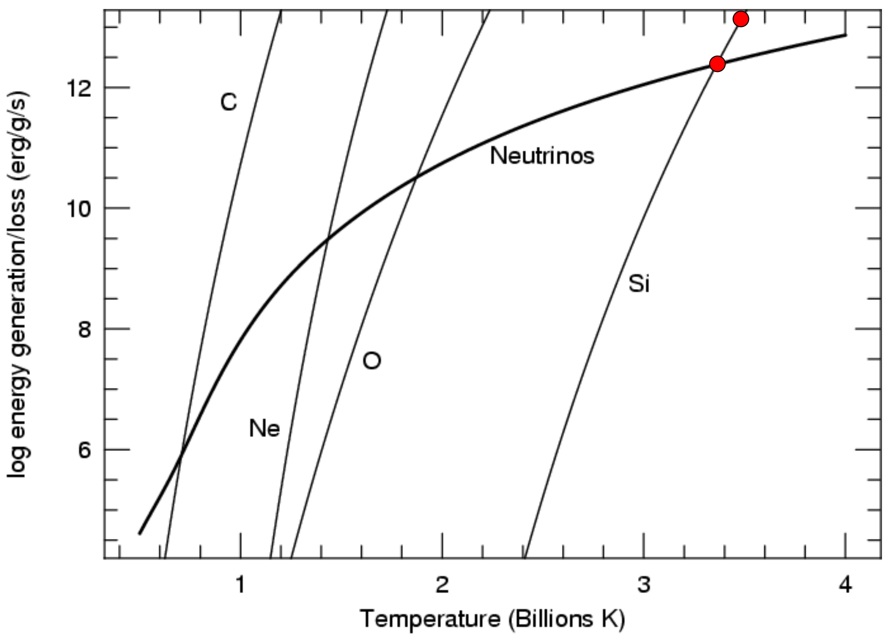
It depends on T^{47}

This approximation implies that Si buring will achieve balanced power at 3.5 billion degrees with a generation rate approximately 10^{13} erg g^{-1} s^{-1}

The approximate lifetime is thus

$$\frac{q\ \Delta X(^{28}Si)}{\varepsilon_{nuc}} \sim \frac{2.6\times10^{17}\ (1)}{10^{13}} \sim 7 \text{ hours}$$

Shell burning and convection can lengthen this to days to weeks



Nucleosynthesis

Basically, silicon burning in the star's core turns the products of oxygen burning (Si, S, Ar, Ca, etc.) into the most tightly bound nuclei (in the iron group) for a given neutron excess, η.

The silicon-burning nucleosynthesis that is ejected by a supernova is produced explosively, and has a different composition dominated by ^{56}Ni.

The products of silicon-core and shell burning in the core are both so neutron-rich (η so large) that they need to be left behind in a neutron star or black hole. However, even in that case, the composition and its evolution is critical to setting the stage for core collapse and the supernova explosion that follows.

Silicon burning nucleosynthesis

Following Si-burning at the middle of a 25 solar mass star:

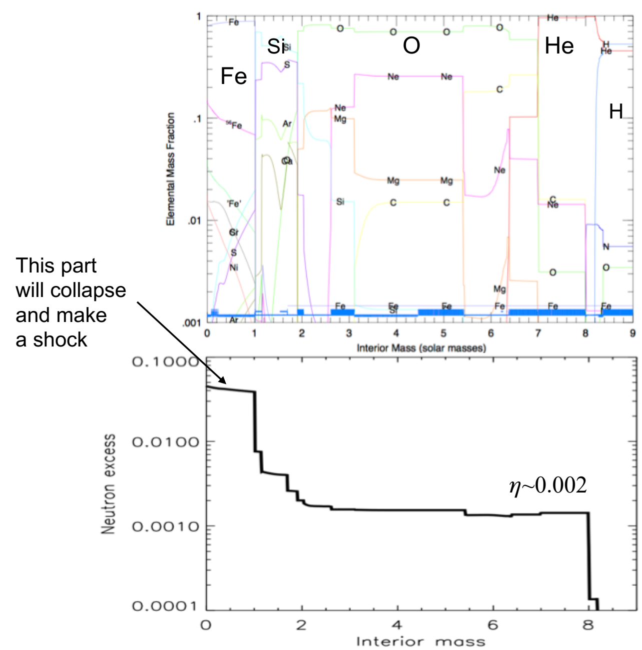
^{54}Fe	0.487
^{58}Ni	0.147
^{56}Fe	0.141
^{55}Fe	0.071
^{57}Co	0.044

Neutron-rich nuclei in the iron peak.
$Y_e = 0.4775$

Following *explosive* Si-burning in a 25 solar mass supernova, interesting species produced at Y_e = 0.498 to 0.499.

product	parent
^{44}Ca	^{44}Ti
^{47,48,49}Ti	^{48,49}Cr
^{51}V	^{51}Cr
^{55}Mn	^{55}Co
^{50,52,53}Cr	^{52,53}Fe
^{54,56,57}Fe	^{56,57}Ni
^{59}Co	^{59}Cu
^{58,60,61,62}Ni	^{60,61,62}Zn

^{44}Ti and $^{56.57}$Ni are important targets of γ-ray astronomy



25 Solar Mass
Pop I
at Si-depletion
(Woosley and Heger 2007)

Nuclear Statistical Equilibrium

As the silicon abundance tends towards zero (though it never becomes microscopically small), the unequilibrated reactions below A = 24 finally come into equilibrium

$$^{24}Ne(\alpha,\gamma)^{24}Mg \quad \leftrightarrow \quad ^{24}Mg(\gamma,\alpha)^{20}Ne$$

$$^{16}O(\alpha,\gamma)^{20}Ne \quad \leftrightarrow \quad ^{20}Ne(\gamma,\alpha)^{16}O \quad \text{(for a long time already)}$$

$$^{12}C(\alpha,\gamma)^{16}O \quad \leftrightarrow \quad ^{16}O(\gamma,\alpha)^{12}C$$

$$3\alpha \rightarrow ^{12}C \quad \leftrightarrow \quad ^{12}C(\gamma,\alpha)2\alpha$$

The 3α reaction is the last to equilibrate. Once this occurs, ***every*** isotope is in equilibrium with every other isotope by strong and electromagnetic reactions (but not by weak interactions)

Until the temperature becomes very high ($T \geq 10^{10}\,K$)
The most abundant nuclei are those with large binding energy per nucleon and "natural" values of η. For example,

$\eta = 0$ $\quad ^{56}$Ni
0.037 $\quad ^{54}$Fe
0.071 $\quad ^{56}$Fe
etc.

In general, the abundance of an isotope peaks at its natural value for η. E. g.,

$$\eta(^{54}\text{Fe}) = \frac{N-Z}{A} = \frac{28-26}{54} = 0.0370$$

$$\eta(^{56}\text{Fe}) = \frac{N-Z}{A} = \frac{30-26}{56} = 0.0714$$

In particular, $Y(^{28}Si) = f(T,\rho)\, Y_\alpha^7$ with the result that now only $\boxed{\text{3 variables, } \rho,\ T_9, \text{and } \eta}$ specify the abundances of everything

$$Y(^AZ) = C(^AZ,\rho,T_9)\; Y_n^N Y_p^Z$$

$$\boxed{C(^AZ,\rho,T_9) = \left(\rho N_A\right)^{A-1} C'(^AZ,T_9)}$$

$$C'(^AZ,T_9) = \frac{G(^AZ,T_9)}{2^A}\,\theta^{1-A} \exp\left[BE(^AZ)/kT\right]$$

$$\theta = 5.943 \times 10^{33}\, T_9^{3/2}$$

$G(^AZ,T_9)$ is the temperature-dependent partition function.

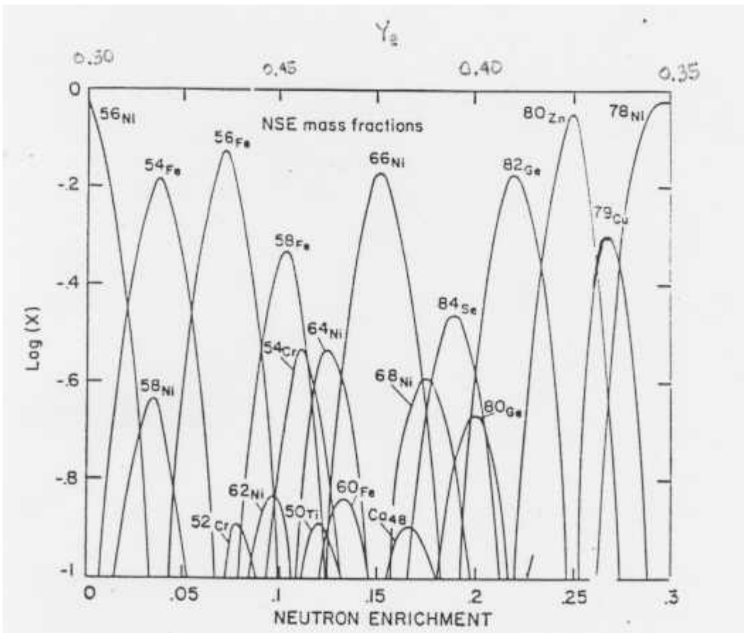
At low T

$$G(^AZ,T_9) = \sum \left(2J_i+1\right) e^{-E_i/kT}$$

At high T, though (see earlier discussion of nuclear level density)

$$\boxed{G(^AZ,T_9) \approx \frac{\pi}{6akT}\quad e^{a(kT)}}$$

$$a \approx \frac{A}{9}\ \text{MeV}^{-1}$$



The resultant nucleosynthesis is most sensitive to η.

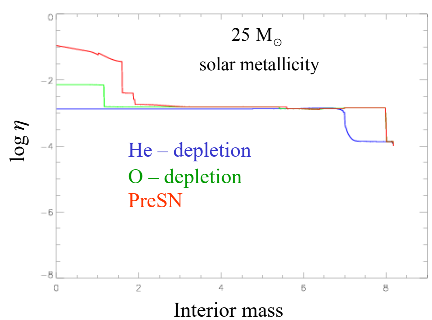
True Equilibrium

If the weak interactions were also to be balanced, (e.g., neutrino capture occurring as frequently on the daughter nucleus as electron capture on the parent), one would have a state of true equilibrium. Only two parameters, ρ and T, would specify the abundances of everything. The first time this occurred in the universe was for temperatures above 10 billion K in the Big Bang.

However, one can also have a *dynamic weak equilibrium* where neutrino emission balances anti-neutrino emission, i.e., when

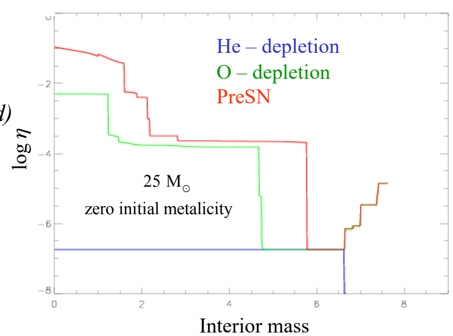
$$\frac{dY_e}{dt} = 0$$

This could occur, and for some stars does, when electron-capture balances beta-decay globally, but not on individual nuclei. The abundances would be set by ρ and T, but would also depend on the weak interaction rate set employed.



The distribution of neutron excess, η, within two stars of 25 solar masses (8 solar mass helium cores) is remarkably different. In the Pop I star, η is approximately 1.5 x 10^{-3} everywhere except in the inner core (destined to become a collapsed remnant)

In the Pop III (Z = 0) star the neutron excess is essentially zero at the end of helium burning (some primordial nitrogen was created) Outside of the core η is a few x 10^{-4}, chiefly from weak interactions during carbon burning. Note some primary nitrogen production at the outer edge where convection has mixed ^{12}C and protons.



Weak Interactions

Electron capture, and at late times beta-decay, occur for a variety of isotopes whose identity depends on the star, the weak reaction rates employed, and the stage of evolution examined. During the late stages it is most sensitive to η, the neutron excess.

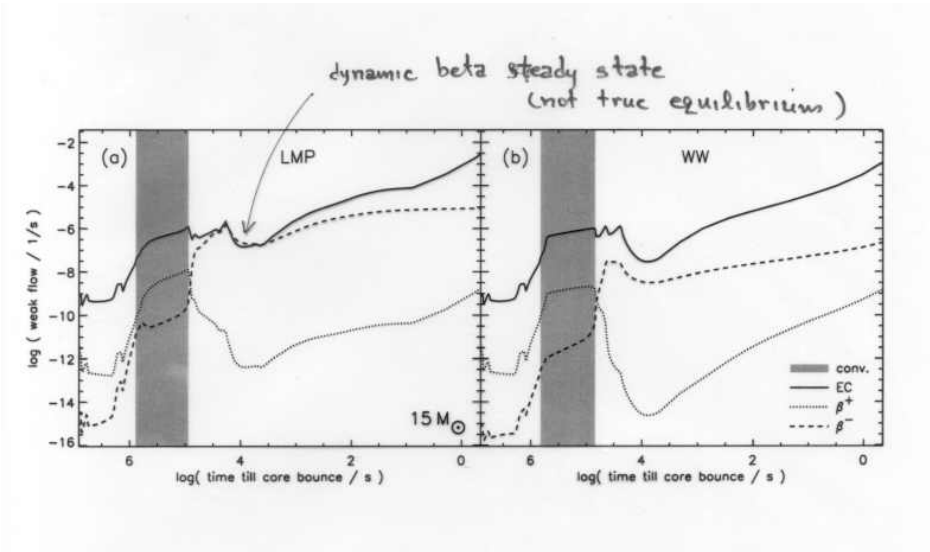
Aside from their nucleosynthetic implications, the weak interactions determine Y_e, which in turn affects the structure of the star. The most important isotopes changing Y_e are not generally the most abundant, but those that have some combination of significant abundance and favorable nuclear structure (especially Q-value) for weak decay.

From silicon burning onwards these weak decays provide neutrino emission that competes with and ultimately dominates that from thermal processes (i.e., pair annihilation).

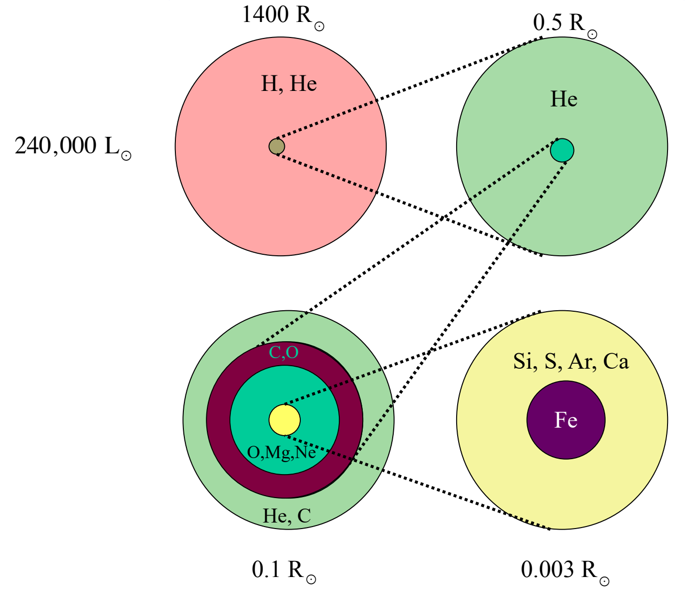
	O-depletion	O-shell	Si-ignition	Si-shell
T(10^9K)	2.26	1.90	2.86	3.39
ρ(g cm^{-3})	1.2 x 10^7	2.8 x 10^7	1.1 x 10^8	4.5 x 10^7
Y_e	0.498	0.495	0.489	0.480
e-capture	^{35}Cl, ^{37}Ar	^{35}Cl, ^{33}S	^{33}S,^{35}Cl	^{54,55}Fe
β-decay	^{32}P, ^{36}Cl	^{32}P,^{36}Cl	^{32}P, ^{28}Al	^{54,55}Mn

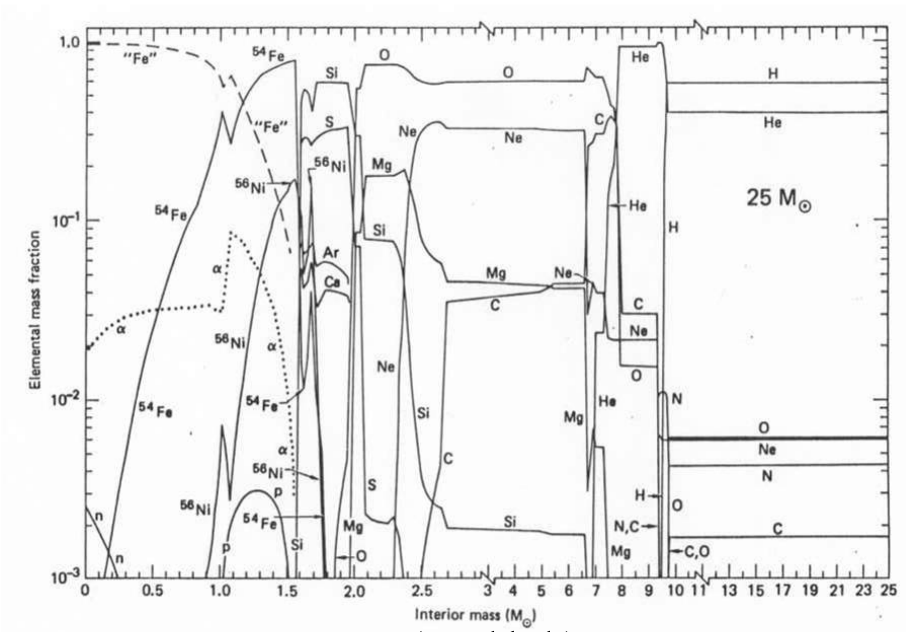
	Si-depletion	Si-shell burn	Core contraction	PreSN
T(10^9 K)	3.78	4.13	3.55	7.16
ρ (g cm^{-3})	5.9 x 10^7	3.2 x 10^8	5.4 x 10^8	9.1 x 10^9
Y_e	0.467	0.449	0.445	0.432
e-capture	^{54,55}Fe	^{57}Fe, ^{61}Ni	^{57}Fe, ^{55}Mn	^{65}Ni, ^{59}Fe
β-decay	^{54}Mn, ^{53}Cr	^{56}Mn,^{52}V	^{62}Co,^{58}Mn	^{64}Co, ^{58}Mn

15 solar mass star (Heger et al 2001)



25 $M_\odot$ Presupernova Star (typical for 9 - 130 $M_\odot$)





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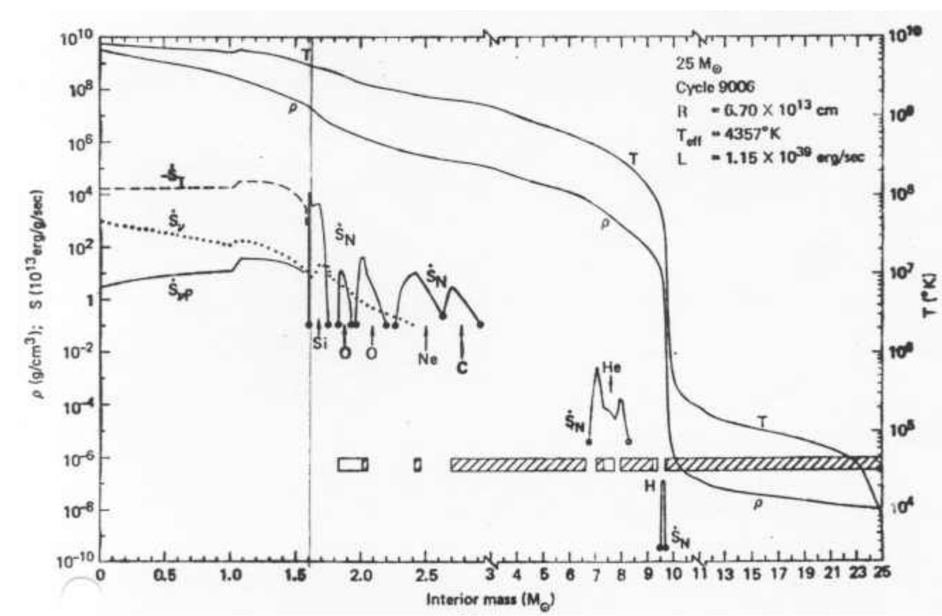

As silicon shells, typically one or at most two, burn out, the iron core grows in discontinuous spurts and approaches instability.

Pressure is predominantly due to relativistic electrons. As they become increasingly relativistic, the structural adiabatic index of the iron core hovers precariously near 4/3. The presence of non-degenerate ions has a stabilizing influence, but the core is rapidly losing entropy to neutrinos and becoming degenerate making the concept of a Chandrasekhar Mass relevant.

In addition to neutrino losses there are also two other important instabilities:

- *Photodisintegration* – which takes energy that might have provided pressure and uses it instead to pay a debt of negative nuclear energy generation.
- *Electron capture* – since pressure is dominantly from electrons, removing them reduces the pressure.

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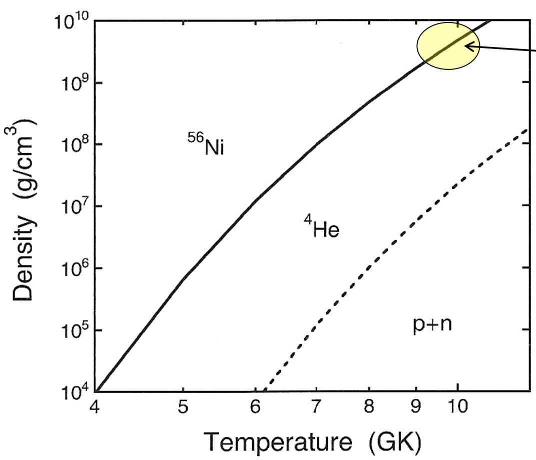
Dominant constituent in NSE (η approximately 0)

510 | *5 Nuclear Burning Stages and Processes*



Typical presupernova central conditions

The transition from the iron group to He and especially from He to 2n + 2p absorbs an enormous amount of energy.

Figure needs correcting for high temperature nuclear partition functions.

See also Clayton Fig 7-9 and discussion

Photodisintegration:

from earlier in this lecture

$$Y_\alpha = C_\alpha(T_9,\rho)\, Y_n^2 Y_p^2$$

$$C_\alpha = \frac{1}{2}(5.94\times10^{33})^{-3}\,\rho^3 T_9^{9/2}\exp(328.36/T_9)$$

where 328.36 = BE(α)/kT $\Rightarrow C_\alpha$ increases rapidly as T $\downarrow$

Setting Y_α = 1/8 (i.e., X = ½ divided by 4) and Y_p=Y_n=1/4 gives the line for the helium-nucleon transition on the previous page.

The Ni-α transition is given by Clayton problem 7-11 and eqn 7-22. It comes from solving the Saha equation for NSE (see previous discussion these notes) for the case

$$X(^{56}Ni) = X_\alpha = 0.5 \Rightarrow Y(^{56}Ni) = 1/112;\ Y_\alpha = 1/8$$
$$Y(^{56}Ni) = C_{56}(T_9)\,\rho^{12} Y_\alpha^{13}$$

The photodisintegration of one gram of ^{56}Ni to one gram of α-particles absorbs:

$$\begin{aligned} q_{nuc} &= 1.602\times10^{-6}\, N_A \sum(\delta Y_i)(BE_i)\ - q_\nu \ \text{ erg/gm} \\ &= 9.64\times10^{17}\left[\left(-\frac{1}{56}\right)(483.993)+\left(\frac{1}{4}\right)(28.296)\right] \\ &= -1.51\times10^{18}\ \text{erg g}^{-1} \end{aligned}$$

Similarly, the photodisintegration of one gram of α's to one gram of nucleons absorbs:

$$\begin{aligned} q_{nuc} &= 9.64\times10^{17}\left[-\left(\frac{1}{4}\right)(28.296)+0\right] \\ &= -6.82\times10^{18}\ \text{erg g}^{-1} \end{aligned}$$

essentially undoing all the energy released by nuclear reactions since the zero age main sequence.

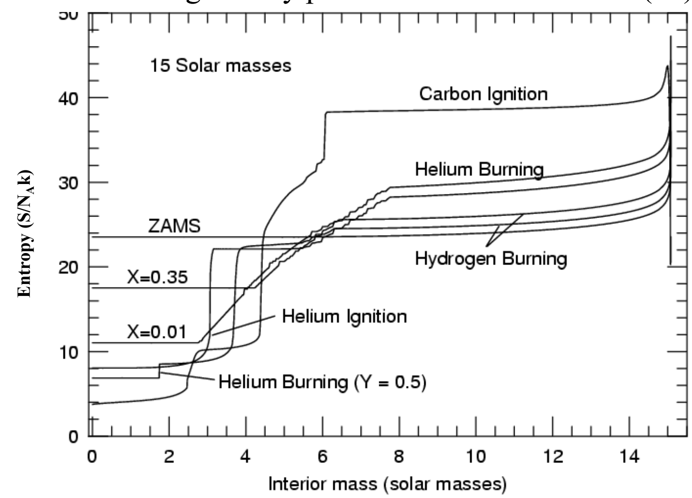
Electron capture

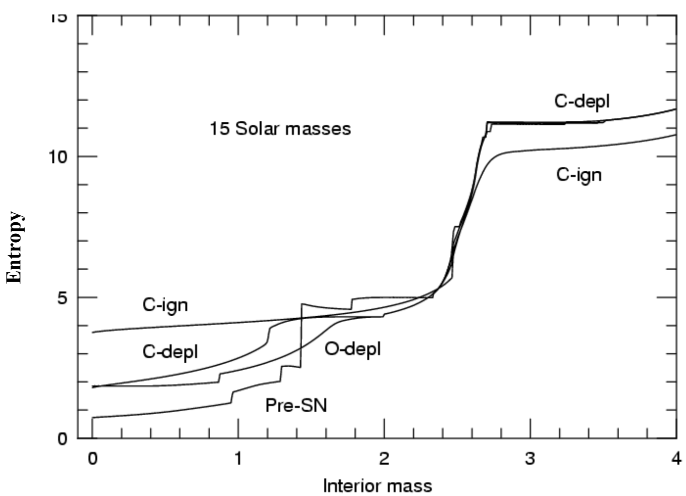
The pressure and entropy come mainly from electrons, but as the density increases, so does the Fermi energy, ε_F. The rise in ε_F means more electrons have enough energy to capture on nuclei turning protons to neutrons inside them. This reduces Y_e which in turn makes the pressure and entropy at a given density smaller.

$$\varepsilon_F = 1.11\,(\rho_7 Y_e)^{1/3} \;\; \text{MeV}$$

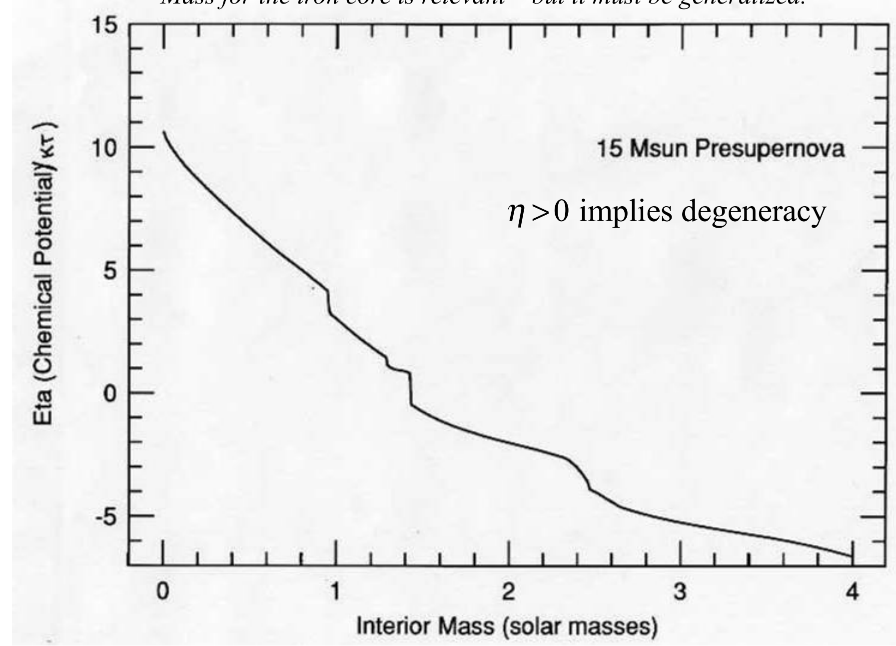
By 2×10^{10} g cm^{-3}, ε_F= 10 MeV which is above the capture threshold for all but the most neutron-rich nuclei. There is also briefly a small abundance of free protons (up to 10^{-3} by mass) which captures electrons.

What about degeneracy? Can the core be supported by electron degeneracy pressure and form a stable (Fe) dwarf?





Because of increasing degeneracy the concept of a Chandrasekhar Mass for the iron core is relevant – but it must be generalized.



The Chandrasekhar Mass

Traditionally, for a fully relativistic, completely degenerate gas:

see lecture 1
$$\frac{P_c^3}{\rho_c^4} = \frac{K_{4/3}^3\,\rho_c^4\,Y_e^4}{\rho_c^4} = \frac{G^3 M^2}{20.745}$$
$$K_{4/3} = 1.2435\times 10^{15}\ \text{dyne cm}^{-2}$$

$$\boxed{M_{Ch} = 5.83\ Y_e^2}$$

$$=1.457\ \mathrm{M}_\odot\ \text{at}\ Y_e = 0.50$$

This is often referred to loosely as 1.4 or even 1.44 solar masses

Relativistic corrections, both special and general, are treated by Shapiro and Teukolsky in *Black Holes, White Dwarfs, and Neutron Stars* pages 156ff. They find a critical density (entropy = 0).

$$\boxed{\rho_c = 2.646\times10^{10}\left(\frac{0.50}{Y_e}\right)^2\ \text{gm cm}^{-3}}$$

Above this density the white dwarf is (general relativistically) unstable to collapse. For $Y_e = 0.50$ this corresponds to a mass

$$\boxed{M_{Ch} = 1.415\ \mathrm{M}_\odot}$$

in general, the relativistic correction to the Newtonian value is

$$\frac{\Delta \mathrm{M}}{\mathrm{M}} = -9\left[4+310\left(\frac{0.50}{Y_e}\right)^{2/3}\right]^{-1}$$
$$=-2.87\%\quad Y_e = 0.50$$
$$=-2.67\%\quad Y_e = 0.45$$

BUT

Effect on M_{Ch}

1) Y_e here is ***not*** 0.50 (Y_e is actually a function of radius or interior mass) ↓
2) The electrons are not fully relativistic in the outer layers (γ is not 4/3 everywhere) ↑
3) General relativity implies that gravity is stronger than classical and an infinite central density is not allowed (there exists a critical ρ for stability) ↓
4) The gas is not ideal. Coulomb interactions reduce the pressure at high density ↓
5) Finite temperature (entropy) corrections ↑
6) Surface boundry pressure (if WD is inside a massive star) ↓
7) Rotation ↑

Coulomb Corrections

Three effects must be summed – electron-electron repulsion, ion-ion repulsion and electron ion attraction. Clayton p. 139 – 153 gives a simplified treatment and finds, over all, a decrement to the pressure (eq. 2-275)

$$\Delta P_{Coul} = -\frac{3}{10}\left(\frac{4\pi}{3}\right) Z^{2/3} e^2\, n_e^{4/3}$$

Fortunately, the dependence of this correction on n_e is the same as relativistic degeneracy pressure. One can then just proceed to use a corrected

$$K_{4/3} = K_{4/3}^0\left[1 - Z^{2/3}\frac{e^2}{\hbar c}\frac{2^{5/3}}{5}\left(\frac{3}{\pi}\right)^{1/3}\right]$$
$$= K_{4/3}^0\left[1-4.56\times10^{-3}\ Z^{2/3}\right]$$

where $K^0_{4/3} = \frac{\hbar c}{4}\left(3\pi^2\right)^{1/3} N_A^{4/3}$

and $\frac{M_{Ch}}{M^0_{Ch}} = \left(\frac{K}{K^0}\right)^{3/2}$

hence

$$M_{Ch} = M^0_{Ch}\left[1-0.0226\left(\frac{Z}{6}\right)^{2/3}\right]$$

Putting the relativistic and Coulomb corrections together with the dependence on Y_e^2 one has

$$M_{Ch} = 1.38\ M_\odot \quad \text{for } ^{12}\text{C} \quad (Y_e = 0.50)$$

$$= 1.15\ M_\odot \quad \text{for } ^{56}\text{Fe} \quad \left(Y_e = \frac{26}{56} = 0.464\right)$$

$$= 1.08\ M_\odot \quad \text{for Fe-core with } \langle Y_e \rangle \approx 0.45$$

So why are iron cores so big at collapse (1.3 - 2.0 $M_\odot$) and why do neutron stars have masses $\approx 1.4\ M_\odot$?

In particular, Baron & Cooperstein (1990) show that

$$P = P_o\left(1+\frac{2}{3}\left(\frac{\pi kT}{\varepsilon_F}\right)^2+...\right)$$

$$\varepsilon_F = p_F c = \left(\frac{3h^3 n_e}{8\pi}\right)^{1/3}$$

$$\varepsilon_F = 1.11(\rho_7\, Y_e)^{1/3}\ \text{MeV}$$

and since $M_{Ch} \propto K_{4/3}^{3/2}$ a first order expansion gives

$$M_{Ch} \approx M^0_{Ch}\left(1+\left(\frac{\pi kT}{\varepsilon_F}\right)^2\right)$$

Finite Entropy Corrections

Chandrasekhar (1938)
Fowler & Hoyle (1960) p 573, eq. (17)
Baron & Cooperstein, ApJ, 353, 597, (1990)

For $n=3,\ \gamma=4/3,$ relativistic degeneracy

$$P_c = K^0_{4/3}\left[1+\frac{1}{x^2}\left(\frac{2\pi^2k^2T^2}{m_e^2c^4}-1\right)+...\right](\rho Y_e)^{4/3}$$

$$x = \frac{h}{mc}\left(\frac{3}{8\pi}n_e\right)^{1/3} = \frac{p_F}{m_e c} = 0.01009\,(\rho Y_e)^{1/3}$$

(Clayton 2-48)

And since, in the appendix to these notes we show

$$s_e = \frac{\pi^2 kTY_e}{\varepsilon_F} \qquad \text{(relativistic degeneracy)}$$

one also has

$$M_{Ch} \approx M^0_{Ch}\left(1+\left(\frac{s_e}{\pi Y_e}\right)^2+...\right)$$

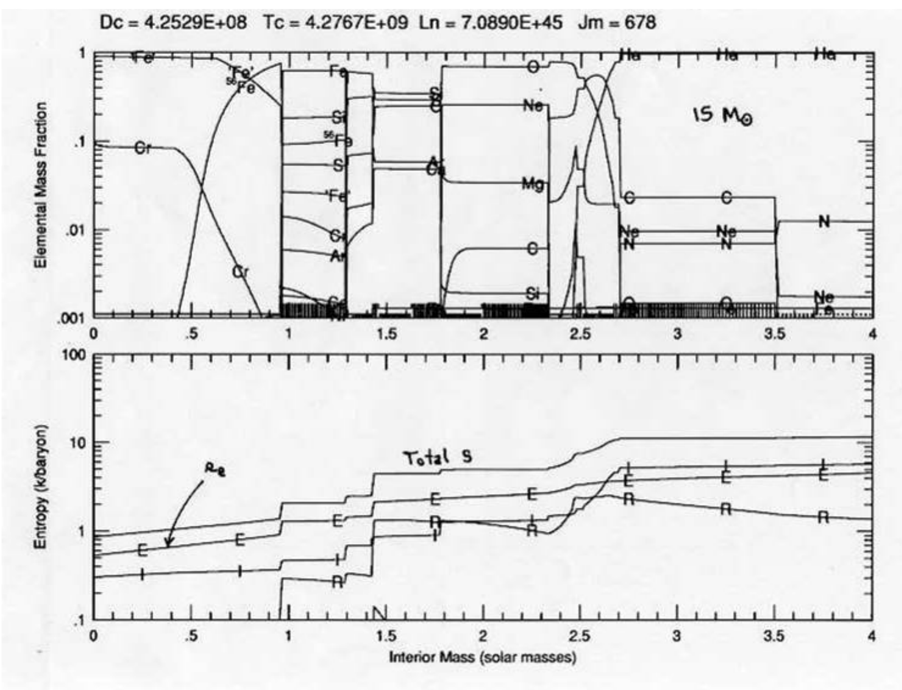
Here M^0_{Ch} incluedes all other non-thermal corrections

The entropy of the radiation and ions also affects M_{Ch}, but much less.

This finite entropy correction is not important for isolated white dwarfs. They' re too cold. But it is very important for understanding the final evolution of massive stars.

Because of its finite entropy (i.e., because it is hot) the iron core develops a mass that, if it were cold, could not be supported by degeneracy pressure.

Because the core has no choice but to decrease its electronic entropy (by neutrino radiation, electron capture, and photodisintegration), and because its (hot) mass exceeds the Chandrasekhar mass, it must eventually collapse.



E.g., on the following pages are excerpts from the final day in the life of a 15 $M_\odot$ star. During silicon shell burning, the electronic entropy ranges from 0.5 to 0.9 in the Fe core and is about 1.3 in the convective shell.

$$M_{Ch} \approx 1.08\ M_\odot \left(1+\left(\frac{0.7}{\pi(0.45)}\right)^2\right) = 1.34\ M_\odot > \ 0.95\ M_\odot$$

The Fe core plus Si shell is also stable because

$$M_{Ch} \approx \ 1.15\ M_\odot \left(1+\left(\frac{1.0}{\pi(0.47)}\right)^2\right) = 1.67\ M_\odot > 1.3 M_\odot$$

0.45 and 0.47 are average values of Y_e in the region being discussed. For 0.45 we used the smaller value for M_{Ch0} a few pages back. For 0.47 we used the value for ^{56}Fe. These are all crude averages to make a point.

But when Si burning in this shell is complete:

3) The Fe core is now ~1.35 $M_\odot$.

s_e central = 0.4

s_e at edge of Fe core = 1.1

average $\approx$0.7

Y_e ranges from 0.438 (center) to 0.47 (edge).

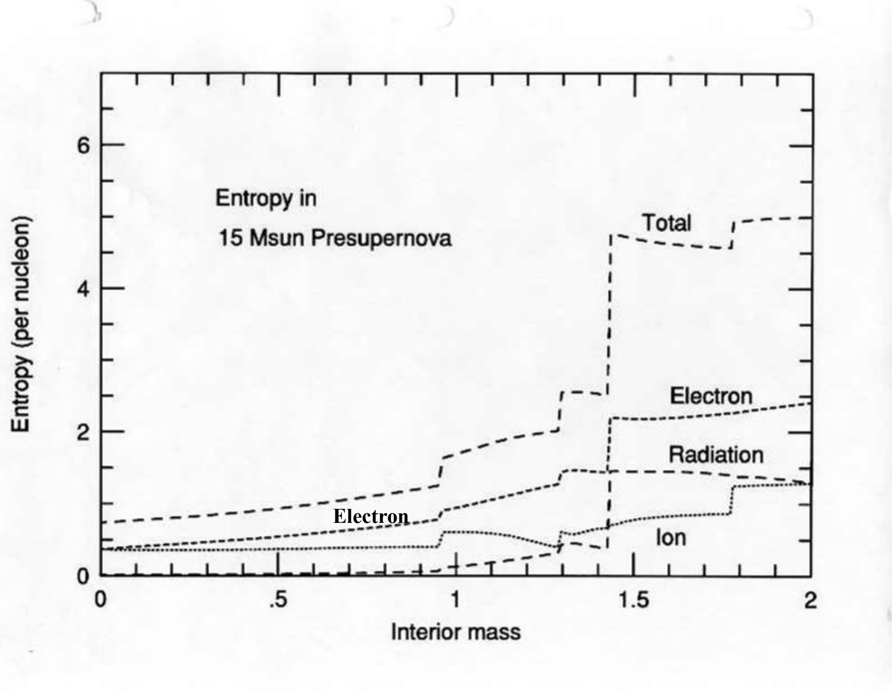
use an averge of 0.45

M_{Ch} now about 1.34 $M_\odot$ (uncertain to at least a few times 0.01 M

Neutrino losses farther reduce s_e. So too do photodisintegration* and electron capture and the boundry pressure of the overlying silicon shell is not entirely negligible.

The core collapses

- Photodisintegration raises the ionic entropy because one nucleus becomes 14. The collapse is approximately adiabatic so total entropy is constant. Thus $s_e = s_{tot} - s_{ion}$ must decrease



Appendix 1: The solution of reaction networks.

Consider the reaction pair:

$$\frac{dY_1}{dt} = -Y_1\lambda_1 + Y_2\lambda_2$$

$$\frac{dY_2}{dt} = Y_1\lambda_1 - Y_2\lambda_2$$

An explicit solution of the linearized equations would be $Y_{new} = Y_{old} + \delta Y$ where

$$\delta Y_1 = (-Y_1\lambda_1 + Y_2\lambda_2)\,\Delta t$$

$$\delta Y_2 = (Y_1\lambda_1 - Y_2\lambda_2)\,\Delta t$$

For large Δt, the answer could oscillate.

In the usual case that the two species were not in equlibrium $Y_1\lambda_1 \neq Y_2\lambda_2$, a large time step, $\Delta t \rightarrow \infty$, would lead to a divergent value for the change in Y, including negative values.

On the other hand, forward or "implicit" differencing would give

$$\frac{\delta Y_1}{\Delta t} = -(Y_1 + \delta Y_1)\lambda_1 + (Y_2 + \delta Y_2)\lambda_2$$

$$\frac{\delta Y_2}{\Delta t} = -(Y_2 + \delta Y_2)\lambda_2 + (Y_1 + \delta Y_1)\lambda_1$$

$$\delta Y_1\left(\frac{1}{\Delta t} + \lambda_1\right) + \delta Y_2(-\lambda_2) = -Y_1\lambda_1 + Y_2\lambda_2$$

$$\delta Y_1(-\lambda_1) + \delta Y_2\left(\frac{1}{\Delta t} + \lambda_2\right) = Y_1\lambda_1 - Y_2\lambda_2$$

In general an n x n matrix

n = 2 here

Add equations $\Rightarrow \delta Y_1 = -\delta Y_2$; substituting, one also has:

$$\delta Y_1 = \left(\frac{Y_2\lambda_2 - Y_1\lambda_1}{1/\Delta t + \lambda_1 + \lambda_2}\right) \qquad \text{(if } 1/\Delta t >> \lambda\text{, same as the explicit solution)}$$

Even if $\Delta t \rightarrow \infty$ the change in Y is finite and tends to the equilibrium value $Y_1\lambda_1 = Y_2\lambda_2$

Appendix 2: Energy generation during silicon burning

<u>Energy Generation Rate</u>

Crudely $2(^{28}\mathrm{Si}) \rightarrow {}^{56}\mathrm{Ni}$

$$\epsilon_{nuc}^{Si} = (9.65 \times 10^{17})(5.46)\,\frac{dY(^{28}\mathrm{Si})}{dt}$$

$$= 5.3 \times 10^{18}\,\frac{dY(^{28}\mathrm{Si})}{dt}$$

$$\frac{dY(^{28}\mathrm{Si})}{dt} = -2\,Y(^{24}\mathrm{Mg})\,\lambda_{\gamma\alpha}(^{24}\mathrm{Mg})$$

one photodisintegrates
one goes to Ni

This is very like neon burning except that 7 alpha-particles are involved instead of one.

<u>Reaction rates governing the rate at which silicon burns:</u>

Generally speaking, the most critical reactions will be those connecting equilibrated nuclei with A > 24 (magnesium) with alpha-particles. The answer depends on temperature and neutron excess:

Most frequently, for η small, the critical slow link is $^{24}\mathrm{Mg}(\gamma,\alpha)^{20}\mathrm{Ne}$

The reaction $^{20}\mathrm{Ne}(\gamma,\alpha)^{16}\mathrm{O}$ has been in equilibrium with $^{16}\mathrm{O}(\alpha,\gamma)^{20}\mathrm{Ne}$ ever since neon burning. At high temperatures and low Si-mass fractions, $^{20}\mathrm{Ne}(\alpha,\gamma)^{24}\mathrm{Mg}$ equilibrates with $^{24}\mathrm{Mg}(\gamma,\alpha)^{20}\mathrm{Ne}$ and $^{16}\mathrm{O}(\gamma,\alpha)^{12}\mathrm{C}$ becomes the critical link.

However for the values of η actually appropriate to silicon burning in a massive stellar *core*, the critical rate is $^{26}\mathrm{Mg}(p,\alpha)^{23}\mathrm{Na}(p,\alpha)^{20}\mathrm{Ne}$

$$Y(^{24}\mathrm{Mg}) = Y(^{28}\mathrm{Si})\,\frac{\lambda_{\gamma\alpha}(^{28}\mathrm{Si})}{\rho Y_\alpha \lambda_{\alpha\gamma}(^{24}\mathrm{Mg})}$$

To get $^{24}\mathrm{Mg}$

$$= 9.87 \times 10^{9}\, T_9^{3/2} \left(\frac{(24)(4)}{28}\right)^{3/2} e^{-11.605\, Q_{\alpha\gamma}(^{24}\mathrm{Mg})/T_9} \cdot \frac{Y(^{28}\mathrm{Si})}{\rho Y_\alpha}$$

$$= 6.3 \times 10^{10}\, e^{-115.83}\, \frac{T_9^{3/2}}{\rho Y_\alpha}\, Y(^{28}\mathrm{Si})$$

what is ρY_α ?

Assume $X(^{28}\mathrm{Si}) + X(^{56}\mathrm{Ni}) = 1$ let $X(^{28}\mathrm{Si}) = f$

To get α

$$\frac{Y(^{56}\mathrm{Ni})}{Y(^{28}\mathrm{Si})} = \frac{1-f}{2f} = C(^{56}\mathrm{Ni}, \rho, T_9)\, Y_\alpha^{7}$$

$$C(^{56}\mathrm{Ni}, \rho, T_9) = (\rho N_A)^7 (5.94 \times 10^{33})^{-7}\, T_9^{-21/2} \left(\frac{56}{28}\right)^{3/2} \left(\frac{1}{4}\right)^{21/2} \exp\left[BE(^{56}\mathrm{Ni}) - BE(^{28}\mathrm{Si}) - 7\,BE(\alpha)\right]/kT$$

$$\log_{10} C_{56} = -242.288 - \frac{21}{2}\log_{10} T_9 + 248.936/T_9 + 7\log_{10}(\rho N_A)$$

a) Solve for ρY_α in terms of f

b) substitute $Y(^{24}Mg)$ as f^n of $Y(^{28}Si)$

c) express $\lambda_{\gamma\alpha}(^{24}Mg)$ " " " $\lambda_{\alpha\gamma}(^{20}Ne)$

get

$$\frac{dY(^{28}Si)}{dt} = -Y(^{28}Si)\,\mathcal{R}(^{28}Si)$$

$$\mathcal{R}(^{28}Si) = 9.55\times10^{10}\,T_9^{3/2}\underbrace{\left(\frac{2f}{1-f}\right)^{1/7}}_{\approx 1} e^{-142.06/T_9}\,\lambda_{\alpha\gamma}(^{20}Ne)$$

independent of ρ.

$$\boxed{\epsilon_{nuc}^{Si} \approx 1.8\times10^{28}\,T_9^{3/2}\,X(^{28}Si)\,e^{-142.06/T_9}\,\lambda_{\alpha\gamma}(^{20}Ne)}$$

$$\lambda_{\alpha\gamma}(^{20}Ne) \approx 124\left(\frac{T_9}{3.5}\right)^5$$

$$\epsilon_{nuc}^{Si} \text{ @ } T_9 = 3.5 \quad \propto T^{47} \qquad 47 = 5 + 1.5 + \frac{142.06}{3.5}$$

Balanced Power

$$\epsilon_{nuc} \propto \rho^0 T^{47} \qquad \epsilon_\nu \propto \rho^{-1} T^9$$

$$\frac{\langle\epsilon_{nuc}\rangle}{\langle\epsilon_\nu\rangle} = 1 \Rightarrow \quad \overset{\circ}{\epsilon}_{nuc} = 13.1\,\overset{\circ}{\epsilon}_\nu \qquad \text{take } X(^{28}Si) = 0.5$$

T_9	ϵ_{nuc}	$\epsilon_\nu(\rho=10^7)$	$\epsilon_\nu(\rho=10^8)$
3.0	7.0(9)	4.1(12)	1.1(11)
3.5	1.7(13)	2.0(13)	7.8(11)
4.0	6.2(15)	7.3(13)	3.7(12)

So burn Si at about $T_9 \simeq 3.5$

Lifetime

$$\frac{1}{\tau_{28}} = \frac{1}{Y_{28}}\frac{dY_{28}}{dt}$$

$$= 9.5\times10^{10}\,(3.5)^{3/2}\,(124)\,e^{-142.06/3.5}$$

$$= 1.8\times10^{-4} \Rightarrow \tau = 1.5 \text{ hr.}$$

lengthened by convection to ~ 1 day

3. Appendix on Entropy

Zentralfriedhof
Vienna, Austria

S = k log W

As discussed previously

For a mixture of ions and radiation,

$$\epsilon = \frac{aT^4}{\rho} + \frac{3}{2}\frac{N_A \rho k T}{\mu \rho}, \qquad P = \frac{1}{3} a T^4 + \frac{\rho N_A k T}{\mu}$$

it follows (Clayton, page 120-121) that

$$\boxed{S = const + \frac{N_A k}{\mu} ln(\frac{T^{3/2}}{\rho}) + \frac{4aT^3}{3\ \rho}.}$$

for ideal gas plus radiation

Note that this implies, if T^3/ρ is a constant, that S will *decrease* with increasing T. The constant is both complicated and arbitrary. By convention

$(T^{3/2}/\rho)/(T^3/\rho) = T^{3/2}$

$$S_{\rm rad} = \frac{4aT^3}{3\ \rho}$$

It is more common to normalize this to the gas constant, $N_A k$, to obtain the *entropy per baryon*

$$\boxed{s_{\rm rad} = \frac{4}{3}\frac{aT^3}{N_A k \rho}}$$

dividing by k makes s dimensionless

The ions are rarely relativistic or degenerate. Any good thermodynamics text (e.g.,

Eg., just the radiation part

$$V \equiv 1/\rho$$

$$T\,dS = d\varepsilon + P\,dV$$

$$T\,dS = \frac{4aT^3}{\rho}dT + aT^4\left(-\frac{1}{\rho^2}d\rho\right) + \frac{1}{3}aT^4\,dV$$

$$= \frac{4aT^3}{\rho}dT + aT^4\,dV + \frac{1}{3}aT^4\,dV$$

$$dS = 4aT^2V\,dT + \frac{4}{3}aT^3 dV$$

$$= d\left(\frac{4}{3}aT^3 V\right)$$

So $S_{\rm rad} = \left(\frac{4}{3}\frac{aT^3}{\rho}\right)$

Reif p. 362) gives

$$\boxed{s_{\rm ion} = \frac{1}{\bar{A}}\left[ln\left[\frac{g_0}{n}\left(\frac{2\pi M kT}{h^2}\right)^{3/2}\right] + \frac{5}{2}\right]}$$

Reif
Fundamentals of Statistical and Thermal Physics
McGraw Hill

where g_0 is the partition function (usually taken as 1), $n = \rho N_A/\bar{A}$, $\bar{A} = (\Sigma Y_i)^{-1}$, $M = \bar{A}/N_A$.

A similar expression holds for non-degenerate, non-relativistic electrons. The more general expression for electrons is more complicated (Cox and Guili 1966).

Cox and Guili
Principles of Stellar Structure
Second edition
A. Weiss et al
Cambridge Scientific Publishers

$$s_e = \frac{1}{\rho N_A}\left(\frac{P_e + E_e}{kT} - \eta n_e\right) = \frac{P_e + E_e}{\rho N_A kT} - \eta Y_e$$

Cox and Guili (24.76b)

where η is the electron chemical potential/kT

$$\eta = \frac{\mu}{kT}$$

where μ, the chemical potential is defined by

$$TS = E + PV - \mu N \qquad (CG\,10.20)$$

hence

$$S_e / V = (\varepsilon_e \rho + P_e - \mu n_e)/T$$

$$S_e = \left(\varepsilon_e + \frac{P_e}{\rho} - \frac{\mu n_e}{\rho} \right) / T$$

$$Y_e = \frac{n_e}{\rho N_A}$$

$$s_e = S_e / N_A k = \left(\frac{\varepsilon_e \rho + P_e}{\rho N_A kT} \right) - \eta Y_e$$

$$= \left(\frac{5}{2} - \eta \right) Y_e \qquad \eta \ll 0$$

For an ideal gas

$$\varepsilon \rho + P = \left(\frac{3}{2} + 1 \right) \rho N_A kT$$

For η>>1 (great degeneracy)

$$n_e \approx \frac{8\pi}{c^3 h^3} (kT)^3 \frac{1}{3} \eta^3 \left[1 + \frac{\pi^2}{\eta^2} \right]$$

$$P_e \approx \frac{8\pi}{3c^3 h^3} (kT)^4 \frac{1}{4} \eta^4 \left[1 + \frac{2\pi^2}{\eta^2} + \frac{7\pi^4}{15\eta^4} \right] \qquad \left(P_e \propto n_e^{4/3} \right)$$

$$\frac{P_e}{n_e kT} \approx \frac{1}{4} \eta \left[\frac{1 + \frac{2\pi^2}{\eta^2}}{1 + \frac{\pi^2}{\eta^2}} \right] >> 1$$

$\eta \gg \pi$ and keeping only terms in η^{-2}

$$\frac{P_e}{n_e kT} \approx \frac{\eta}{4} \left[\left(1 + \frac{2\pi^2}{\eta^2} \right) \left(1 - \frac{\pi^2}{\eta^2} \right) \right]$$

$$\approx \frac{\eta}{4} \left(1 + \frac{2\pi^2}{\eta^2} - \frac{\pi^2}{\eta^2} \right) = \frac{\eta}{4} + \frac{\pi^2}{4\eta}$$

For a non-relativistic, non-degenerate electron gas, Clayton 2-63 and 2-57 imply (for η<< 0)

$$n_e = \frac{2(2\pi m_e kT)^{3/2}}{h^3} \left(e^{\eta} - \frac{e^{2\eta}}{2^{3/2}} + \ldots \right)$$

which implies

$$-\eta \approx \ln \left[\frac{2(2\pi m_e kT)^{3/2}}{n_e h^3} \right]$$

which gives the ideal gas limit for electron entropy (similar to ions but has Y_e and m_e)

$$u_e = 3P_e \qquad \eta = \frac{\mu}{kT} \qquad \mu \approx \varepsilon_F$$

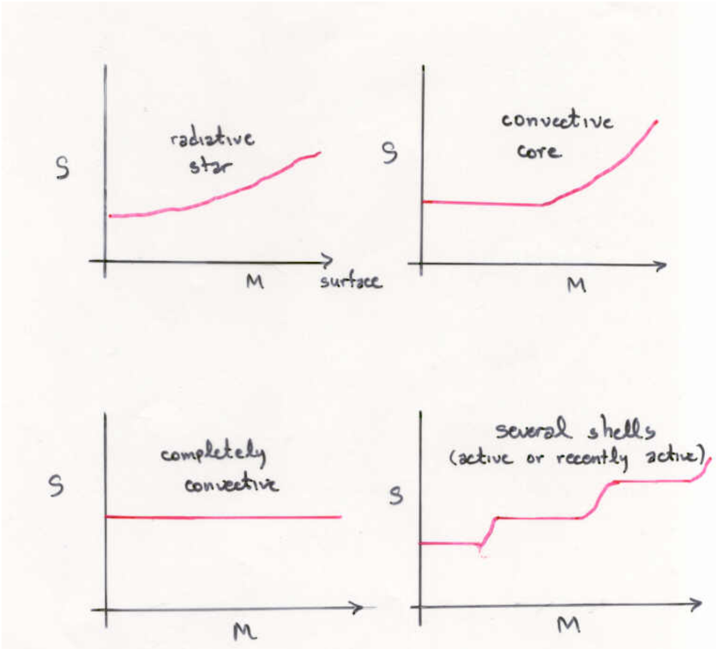
$$\frac{S_e}{V} = (u_e + P_e - \mu n_e)/T \approx \frac{(4P_e - \mu n_e)}{T}$$

$$\approx \frac{n_e kT \left(\eta + \frac{\pi^2}{\eta} \right) - \eta n_e kT}{T} = \frac{\pi^2 n_e k}{\eta}$$

$$= \frac{\pi^2 \rho N_A Y_e k}{\eta}$$

$$s_e = \frac{S_e}{\rho N_A k} \approx \frac{\pi^2 Y_e}{\eta}$$

$$\boxed{s_e \approx \frac{\pi^2 kTY_e}{\varepsilon_F}}$$



3. Completely convective stars with negligible degeneracy will be polytropes of index n = 3/2. ln $(T^{3/2}/\rho)$ = constant implies $T \propto \rho^{2/3}$ and $P \propto \rho^{5/3}$.

and negligible radiation pressure (entropy)

4. Stars in which radiation completely dominates the entropy which are completely convective will be n = 3 polytropes. This is the case for extremely massive stars. T^3/ρ = constant and $P \propto T^4$ implies $P \propto \rho^{4/3}$

5. Stars have negative heat capacity.

$$C = \frac{dQ}{dT} = T\frac{dS}{dT} = \frac{dS}{dlnT}$$

For an ideal gas with negligible radiation,

$$S \sim \left(ln\frac{T^{3/2}}{\rho}\right)\frac{N_A k}{\mu}$$

$$\rho_c \sim \frac{T_c^3}{M^2}$$

$$S \sim \left(ln\frac{M^2}{T_c^{3/2}}\right)\frac{N_A k}{\mu}$$

$$= \left(2\,ln\,M - \frac{3}{2}ln\,T_c\right)\frac{N_A k}{\mu}$$

$$C \approx -\frac{3}{2}\frac{N_A k}{\mu}$$

Thus the well known property of stars to get hotter as they radiate. It can be shown that the presence of radiation acts to inhibit this negative heat capacity

6. Stars of greater mass have higher entropy. Again $S \sim ln\,T^{3/2}/\rho$, but T_c^3/ρ_c increases as M^2. Thus as M rises so does S. This implies that more massive stars have higher entropy and a less centrally condensed

structure - less core convergence. This will be quite important.