University of Canterbury

Stellar Structure

and

Evolution

Astr 323/Phys 323

Plan:

1. Structure equations

2. Microphysics

3. The Sun & the main-sequence evolution

The structure equations

- Stars are *self-gravitating* objects of hot plasma;
- emitting energy in the form of photons from the surface;
- spherical symmetry (absence of rotation and magnetic fields);

→one-dimensional problem with *radius* r being the natural coordinate (Euler description).

Mass and radius

Eulerian description: mass dm in a shell at r and of thickness dr is

$$\mathrm{d}m = 4\pi r^2 \rho \mathrm{d}r - 4\pi r^2 \rho v \mathrm{d}t$$

From the partial derivatives of this equation one can derive:

$$\frac{\partial \rho}{\partial t} = -r^{-2} \frac{\partial(\rho r^2 v)}{\partial r}$$

$$\rightarrow \frac{\partial \rho}{\partial t} = -\nabla(\rho v)$$

(continuity equation in 1-dimensional form and Eulerian description)

Lagrangian description: mass elements m (mass in a concentric shell).

$$\Rightarrow r = r(m,t)$$

Variable change $(r,t) \rightarrow (m,t)$:

$$\frac{\partial}{\partial m} = \frac{\partial}{\partial r}\frac{\partial r}{\partial m}$$

and

$$\left(\frac{\partial}{\partial t}\right)_m = \frac{\partial}{\partial r}\left(\frac{\partial r}{\partial t}\right)_m + \left(\frac{\partial}{\partial t}\right)_r$$

$\Rightarrow$

$$\boxed{\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \qquad (1)}$$

This is the first structure equation (*mass equation* or *mass conservation*).

Contains transformation Euler- $\rightarrow$ Lagrange-description

$$\frac{\partial}{\partial m} = \frac{1}{4\pi r^2 \rho}\frac{\partial}{\partial r}$$

Gravity

Gravitational field

$$\nabla^2 \Phi = 4\pi G \rho$$

($G = 6.673 \cdot 10^{-8}\,\mathrm{dyn\,cm^2\,g^{-2}}$).

In spherical symmetry:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \Phi}{\partial r}\right) = 4\pi G \rho$$

$g = \frac{\partial \Phi}{\partial r} \rightarrow g = \frac{Gm}{r^2}$ is solution of Poisson's equation.

Potential Φ vanishes for $r \rightarrow \infty$.
$-\int_0^\infty \Phi \mathrm{d}r$ is the energy required to disperse the complete star to infinity.

Hydrostatic equilibrium

On layer of thickness dr two forces:

gravity $-g\rho \mathrm{d}r$ and pressure $\triangle P = -\frac{\partial P}{\partial r}\triangle r$.

If shell is at rest (*hydrostatic equilibrium*):

$$\frac{\partial P}{\partial r} = -\frac{Gm}{r^2}\rho$$

in Lagrangian coordinates:

$$\boxed{\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} \qquad (2)}$$

Second structure equation (*hydrostatic equilibrium*).

Estimates for central values of the Sun:

Replace derivatives in the hydrostatic equation by differences between center (P_c) and surface ($P_0 \approx 0$) $\rightarrow$

$$P_c \approx \frac{2GM^2}{\pi R^4}$$

($M/2$ and $R/2$ were used for mean mass and radius)

Sun: $P_c = 7 \cdot 10^{15}$ (cgs units).

With $\rho = \frac{\mu P}{\mathcal{R}T}$ and $\bar{\rho} = (3M)/(4\pi R^3) \Rightarrow$

$$T_c = \frac{8}{3}\frac{\mu}{\mathcal{R}}\frac{GM}{R}\frac{\bar{\rho}}{\rho_c} < 3 \cdot 10^7 \text{ K}$$

Motion:

$$\frac{\partial P}{\partial m} + \frac{Gm}{4\pi r^4} = -\frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2}$$

1. $P = 0 \rightarrow$ free-fall $Gm/r^2 = \ddot{r}$.

2. $\tau_{ff} = \sqrt{R/|\ddot{r}|} \approx \sqrt{R/g}$.

3. $G = 0$, $\tau_{expl} \approx R\sqrt{\rho/P}$ (isothermal sound speed)

4. *hydrostatic timescale* $\tau_{hydro} \approx \frac{1}{2}(G\bar{\rho})^{-1/2}$

5. Examples: $\tau_{hydro} =$
   - 27 minutes for the Sun
   - 18 days for a Red Giant ($R = 100R_\odot$)
   - 4.5 seconds for a White Dwarf ($R = R_\odot/50$)

Conclusion: stars return to hydrostatic equilibrium within an extremely short time.

Energy reservoirs:

1. Thermal (or internal) energy (for an ideal gas)

$$P = \frac{\mathcal{R}}{\mu}\rho T$$

$$\frac{\mathcal{R}}{\mu} = c_P - c_v = \frac{2}{3}c_v$$

The thermal energy per mass unit is $u = c_v T$

and therefore the total energy the integral over mass of u

$$E_{\mathrm{t}} = \int_0^M c_v T \mathrm{d}m = \int_0^M c_v \frac{3\mathcal{R}}{2\mu} T \mathrm{d}m = \frac{3\mathcal{R}}{2\mu}\langle T\rangle M$$

For the Sun, with $\langle T\rangle \approx 10^7$ K $\rightarrow E_{\mathrm{t},\odot} \approx 5 \times 10^{48}$ erg

2. Gravitational energy

$$E_{\mathrm{g}} = -\int_0^M \frac{GM_r}{r}\mathrm{d}M_r \approx -\frac{GM^2}{R}$$

For the Sun, $E_{\mathrm{g},\odot} = -4 \times 10^{48}$ erg

Generally,

$$-E_{\mathrm{g}} \approx E_{\mathrm{t}}$$

Why?

The Kelvin-Helmholtz timescale

$L \approx \left| \frac{\mathrm{d}E_\mathrm{g}}{\mathrm{d}t} \right| \rightarrow \tau_\mathrm{KH} := \frac{|E_\mathrm{g}|}{L} \approx \frac{E_\mathrm{t}}{L}.$

$|E_\mathrm{g}| \approx \frac{GM^2}{2R} \Rightarrow \tau_\mathrm{KH} \approx \frac{GM^2}{2RL}.$

Sun: $\tau_\mathrm{KH} = 1.6 \cdot 10^7$ yrs.

$\Rightarrow$ Sun could shine only for about 10 million years, if gravitational potential or the thermal energy would have been its only energy source!

The Virial Theorem

Integrate eq. (2) after multiplying it by $4\pi r^3$:

$$\int_0^M 4\pi r^3 \frac{\partial P}{\partial m} \mathrm{d}m = -\int_0^M \frac{Gm}{4\pi r^4} 4\pi r^3 \mathrm{d}m$$

Right hand side is obviously total gravitational energy

$$E_\mathrm{g} = -\int_0^M \frac{GM}{r} \mathrm{d}m$$

defined to be 0 at infinity. $-E_\mathrm{g}$ is energy released at assembly of star from parts at infinity.

Left hand side solved by integration by parts:

$$\int_0^M 4\pi r^3 \frac{\partial P}{\partial m} \mathrm{d}m = \left[4\pi r^3 P\right]_0^M - \int_0^M \left(12\pi r^2 \frac{\partial r}{\partial m}\right) P \mathrm{d}m$$

On right hand side, the term in brackets vanishes, if $P(M) \approx 0$, and the integrand becomes [eq. (1)] $3P/\rho$

So finally:

$$3\int_0^M \frac{P}{\rho} \mathrm{d}m = -E_\mathrm{g}$$

Meaning of left hand term:

For ideal gas:

$$\frac{P}{\rho} = \frac{\mathcal{R}}{\mu} T = (c_P - c_v)T = c_v(\gamma - 1)T$$

$\gamma = 5/3$ for mono-atomic gas
(4/3 for photon gas).

With this $\frac{P}{\rho} = \frac{2}{3} c_v T = \frac{2}{3} u$

u: specific internal energy $\Rightarrow E_\mathsf{i} := \int u \mathrm{d}m$ and

$$3 \int_0^M \frac{P}{\rho} \mathrm{d}m = 2E_\mathsf{i}$$

In summary, we obtain the *Virial Theorem*

$$\Rightarrow \qquad E_g = -2E_i \tag{3}$$

For a more general ideal gas, $3(P/\rho) = \zeta u$ ($\zeta = 3(\gamma - 1)$); for monatomic gas $\zeta = 2$ ($\gamma = 5/3$) and for photon gas 1. $\rightarrow$

$$\zeta E_i + E_g = 0.$$

Total energy W (< 0 for bound system):

$W = E_i + E_g = (1 - \zeta)E_i = \frac{\zeta - 1}{\zeta} E_g$

But luminosity L of star must come from this energy reservoir:

$L = -\frac{\mathrm{d}W}{\mathrm{d}t} \quad \Rightarrow \quad L = (\zeta - 1)\frac{\mathrm{d}E_i}{\mathrm{d}t}.$

$$\zeta = 2 \quad \Rightarrow \quad L = -\frac{\dot{E}_g}{2} = \dot{E}_i, \tag{4}$$

Interpretation:

star loses energy $\rightarrow$ decrease in gravitational energy (contraction) $\rightarrow$ but same amount goes into increase in internal energy (heating)

Stars become hotter, because they lose energy!

Note: remember assumptions!
Hydrostatic equilibrium, ideal gas.

White dwarfs lose energy and get cooler! Why?

Energy conservation

$$\mathrm{d}L_r = 4\pi r^2 \rho \epsilon \mathrm{d}m,$$

ϵ (erg/gs): specific *energy generation rate*

Sources for ϵ:

- in a stationary mass shell: $\epsilon = \epsilon_n(\rho, T, \vec{X})$: nuclear energy generation;

- in non-stationary mass shell: interaction with surrounding via $P\mathrm{d}V$

$$\begin{aligned}
\left(\epsilon_n - \frac{\partial L_r}{\partial m}\right)\mathrm{d}t &= \mathrm{d}q = \mathrm{d}u + P\mathrm{d}v \\
\frac{\partial L_r}{\partial m} &= \epsilon_n - \frac{\partial u}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t} \\
&= \epsilon_n - c_P\frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t}
\end{aligned}$$

ϵ_g: *gravothermal energy*

$$\epsilon_g = -c_P\frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t} = -c_P T\left(\frac{1}{T}\frac{\partial T}{\partial t} - \frac{\nabla_{\mathrm{ad}}}{P}\frac{\partial P}{\partial t}\right) \qquad (5)$$

Energy loss due to plasma neutrinos: $-\epsilon_\nu$. *energy conservation equation* reads

$$\boxed{\frac{\partial L_r}{\partial m} = \epsilon_n + \epsilon_g - \epsilon_\nu. \qquad (6)}$$

Global energy conservation

The change in the total energy reservoir equals the loss of energy due to escaping neutrinos and photons:

$$\dot{W} = \frac{\mathrm{d}}{\mathrm{d}t}(E_G + E_i + E_n) = -(L + L_\nu),$$

Integration of the energy equation (6) over m should recover this. Some terms can easily be identified:

$L = \int \frac{\partial L_r}{\partial m}\mathrm{d}m$, $L_\nu = \int \epsilon_\nu \mathrm{d}m$, $\int \epsilon_n \mathrm{d}m = -\frac{\mathrm{d}E_n}{\mathrm{d}t}$.

This leaves the integration of ϵ_g:

We use the formulation

$$\epsilon_g = -\frac{\partial u}{\partial t} + \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}.$$

The first term is easily identified as $-\mathrm{d}E_i/\mathrm{d}t$.

The second term, $\int_0^M \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}\mathrm{d}m$ needs more work:

From the virial theorem derivation we know that (E_G = grav. energy)

$$E_G = -3\int_0^M \frac{P}{\rho}\mathrm{d}m,$$

and compute $\frac{\mathrm{d}E_G}{\mathrm{d}t} = -3\int \frac{\dot{P}}{\rho}\mathrm{d}M_r + 3\int \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}\mathrm{d}M_r$

Taking the deriative w.r.t. t of the hydrostatic equation and integrating it over M_r, we get that

$$-3\int_0^M \frac{1}{\rho}\frac{\partial P}{\partial t}\mathrm{d}M_r = 4\frac{\mathrm{d}E_G}{\mathrm{d}t}$$

using this in the previous equation we arrive at

$$\dot{E}_g = 4\dot{E}_g + 3\int_0^M \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}\mathrm{d}M_r$$

or

$$\int_0^M \frac{P}{\rho^2}\frac{\partial \rho}{\partial t}\mathrm{d}M_r = -\dot{E}_G$$

Therefore

$$\int_0^M \epsilon_g \mathrm{d}M = -\dot{E}_t - \dot{E}_G$$

and all terms in the global energy conservation equation have been recovered!

The nuclear timescale

$\tau_\mathrm{n} := E_n/L$

Nuclear energy reservoir: mass of fuel times the erg/g of fuel.

Sun is in hydrogen burning, which has a heat release of $q = 6.3 \cdot 10^{18}\,\mathrm{erg\,g^{-1}}$

or a total nuclear reservoir of $8.75 \cdot 10^{51}$ erg for the whole Sun.

$$\Rightarrow \tau_\mathrm{n} = 7 \cdot 10^{10}\ \mathrm{yrs}$$

$\tau_\mathrm{n} \gg \tau_\mathrm{KH} \gg \tau_\mathrm{hydr}$

This is the timescale most important for most stars in most evolutionary phases. $\frac{\partial L_r}{\partial m} \approx \epsilon_n$ to high precision is an equivalent statement. It implies that $\epsilon_g \approx 0$ or that the star is said to be in *thermal equilibrium*. Together with the mechanical equilibrium this is also called *complete* equilibrium, because all terms involving $\mathrm{d}t$ are missing. Of course, complete equilibrium cannot be achieved accurately, as will be discussed later.

Energy transport

The energy created in the centre must be transported outwards. This is possible only along a temperature gradient.

T-gradient in Sun: $\triangle T/\triangle r \approx 10^7/10^{11} = 10^{-4}$ (K/cm).

Energy tranport is possible by radiation, convection and conduction, the latter process being usually unimportant except in the case of degenerate electrons.

Formal equation for the temperature gradient:

$$\frac{\partial T}{\partial m} = -\frac{T}{P}\frac{Gm}{4\pi r^4}\nabla$$

Determine ∇ in all cases!

Transport by radiation

Radiation intensity I weakened by intervening matter according to

$$\mathrm{d}I = -I\kappa\rho\mathrm{d}r \Rightarrow -\frac{\mathrm{d}\ln I}{\mathrm{d}r} = \kappa\rho =: \frac{1}{l}$$

Opacity $\kappa(T, \rho, \vec{X})$ (cm^2/g).

Values for solar interior:
$\bar{\rho_\odot} = 1.4\,\mathrm{g/cm^3}$, $\kappa_\odot \approx 1\,\mathrm{cm^2/g} \Rightarrow l_\odot \approx 1$ cm!

Because of this very short mean free path the radiation field is highly isotropic, the stellar interior everywhere in Local Thermal Equilibrium, and the monochromatic intensity well described by Planck-function $B_\nu(T)$.

Anisotropy (of Planck radiation):

$$F \sim T^4 \rightarrow \frac{\triangle F}{F} = 4\frac{\triangle T}{T} = 4\frac{\triangle T}{\triangle r}\frac{l}{T} \approx 4\frac{T}{R}\frac{l}{T} \approx \frac{l}{R}$$

For the Sun, this is of order 10^{-10}! Therefore radiation diffuses outwards only extremely slowly (timescale about $10^6 \cdots 10^7$ years).

Radiation transport by diffusion:

In analogy to particle diffusion equation:
Diffusive flux $\vec{j}$ of particles (per unit area and time) is

$$\vec{j} = -D\vec{\nabla}n = -\frac{1}{3}vl_p\vec{\nabla}n$$

(D is called the diffusion constant; v the diffusion velocity; l_p is the particle free path length and n the particle density).

We now use $U := aT^4$ for the radiation density in place of particle density, $l = 1/(\kappa\rho)$ for the photon mean free path, and c instead of v. In a 1-dimensional problem, we get for $\vec{\nabla}U$

$$\frac{\partial U}{\partial r} = 4aT^3\frac{\partial T}{\partial r}$$

and for the radiation flux F (replacing $\vec{j}$)

$$F = -\frac{4ac}{3}\frac{T^3}{\kappa\rho}\frac{\partial T}{\partial r},$$

or $F = -K_{\text{rad}} \nabla T$.

$K_{\text{rad}} = \frac{4ac}{3}\frac{T^3}{\kappa\rho}$ is the radiative conductivity.

Its value for the solar center is $\approx 3\times10^{15}$ erg K^{-1} s^{-1} cm^{-1}, compared to values of 10^7 for typical metals or 10^4 for concrete.

$\Rightarrow$ Transport by photon gas is extremely efficient; even a small T-gradient allows excellent heat transport [$L_\odot$ transported by $\triangle T/\triangle r \approx 10^7/10^{11} = 10^{-4}$ (K/cm)]

With $L_r = 4\pi r^2 F$, we obtain

$$\frac{\partial T}{\partial r} = -\frac{3}{16\pi ac}\frac{\kappa\rho L_r}{r^2 T^3}$$

or, in Lagrangian coordinates

$$\boxed{\frac{\partial T}{\partial m} = -\frac{3}{64ac\pi^2}\frac{\kappa L_r}{r^4 T^3} \qquad (7)}$$

However, this is – so far – only true for monochromatic radiation. $\kappa = \kappa(\nu)$.

The Rosseland mean opacity

We would like to have

$$\frac{\partial T}{\partial m} = -\frac{3}{64 a c \pi^2} \frac{\bar{\kappa} L_r}{r^4 T^3}$$

with $\bar{\kappa}$ being a suitable mean over frequency of $\kappa(\nu)$.

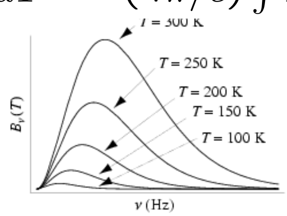
This mean turns out to be

$$\frac{1}{\bar{\kappa}} := \frac{\int_0^\infty \frac{1}{\kappa_\nu} \frac{\partial B_\nu}{\partial T} \mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T} \mathrm{d}\nu}$$

where

$$B_\nu(T) = \frac{2h\nu^3}{c^2} \left(\exp\left(\frac{h\nu}{kT}\right) - 1 \right)^{-1}$$

is the Planck-function for the energy density flux of a black body. ($U = aT^4 = (4\pi/c) \int B_\nu \mathrm{d}\nu$).



$\bar{\kappa}$ is the *Rosseland* mean for opacity, mostly called κ_R or simply κ.

Note that the Rosseland mean is dominated by those frequency intervals, where matter is almost transparent to radiation, i.e. where transport efficiency is highest.

Note: Eq. (7) can also be derived rigidly within radiation theory (see, e.g. Cox & Giuli).

Conduction

In regions of degeneracy, the mean free path of electrons is very large, because the probability for momentum exchange is very small due to the fact that all energy levels are occupied!

The flux of energy is the sum of the radiative and the conductive heat flux:

$$F = F_{\text{rad}} + F_{\text{cond}} = -(K_{\text{rad}} + K_{\text{cond}})\nabla T$$

If we introduce formally κ_{cond}:

$$K_{\text{cond}} = \frac{4ac}{3}\frac{T^3}{\kappa_{\text{cond}}\rho},$$

this allows to replace κ in (7) by

$$\frac{1}{\kappa} = \frac{1}{\kappa_{\text{rad}}} + \frac{1}{\kappa_{\text{cond}}}$$

The mechanism with the smaller *opacity* κ manages the transport!

Effevtive ∇ in transport equation:

$$\nabla = \nabla_{\text{rad}} = \frac{3}{16\pi acG}\frac{\kappa L_r P}{mT^4}$$

Perturbations and stability: convection

For radiative transport, if κ and/or L_r is large, $\frac{\partial T}{\partial r}$ will be large and the question arises whether the stratification is stable against small displacement?

Consider moving mass elements; assume no heat exchange with surrounding (adiabatic movement); Differences between element and surrounding denoted by D, e.g. $DT = T_{\rm e} - T_{\rm s}$

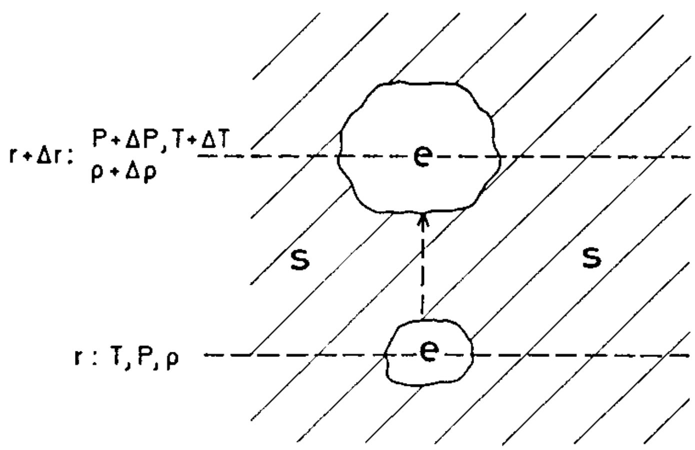


Illustration of blobs moving in unperturbed surrounding

The figure illustrates the picture we have in mind: the temperature excess DT is positive, if the element is hotter than its surrounding. $DP = 0$ due to hydrostatic equilibrium. If $D\rho < 0$ (ideal gas), the element is lighter and will move upwards. Take an element and lift it by $\triangle r$:

$$D\rho = \left[\left(\frac{\partial \rho}{\partial r} \right)_e - \left(\frac{\partial \rho}{\partial r} \right)_s \right] \triangle r$$

The stability condition therefore is

$$\left(\frac{\partial\rho}{\partial r}\right)_e - \left(\frac{\partial\rho}{\partial r}\right)_s > 0.$$

If it is fulfilled, the drop in ρ of the element during the upward movement is smaller than that of the surroundings ($\frac{\partial\rho}{\partial r} < 0!$), so the element will gradually become as dense as the surrounding or denser and will finally experience a downward force and return.

With the EOS $\mathrm{d}\ln\rho = \alpha\mathrm{d}\ln P - \delta\mathrm{d}\ln T - \varphi\mathrm{d}\ln\mu$, the stability condition changes to

$$\left(\frac{\delta}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right)_s - \left(\frac{\delta}{T}\frac{\mathrm{d}T}{\mathrm{d}r}\right)_e - \left(\frac{\varphi}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}r}\right)_s > 0$$

(Here we have used the fact that $DP = 0$.) Multiply this by the pressure scale height

$$H_P := -\frac{\mathrm{d}r}{\mathrm{d}\ln P} = -P\frac{\mathrm{d}r}{\mathrm{d}P} = \frac{P}{\rho g} > 0 \Rightarrow$$

$$\begin{aligned}
\left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_s &< \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln P}\right)_e + \frac{\varphi}{\delta}\left(\frac{\mathrm{d}\ln\mu}{\mathrm{d}\ln P}\right)_s \\
\nabla_s &< \nabla_{\mathrm{ad}} + \frac{\varphi}{\delta}\nabla_\mu \\
\nabla_{\mathrm{rad}} &< \nabla_{\mathrm{ad}} + \frac{\varphi}{\delta}\nabla_\mu
\end{aligned}$$

The last equation holds in general cases and is called the *Ledoux-criterion* for dynamical stability. If $\nabla_\mu = 0$, the *Schwarzschild-criterion* holds. Note that ∇_μ will stabilize.

If the stability criterion is violated, **convective instability** will set in and convection will transport energy.

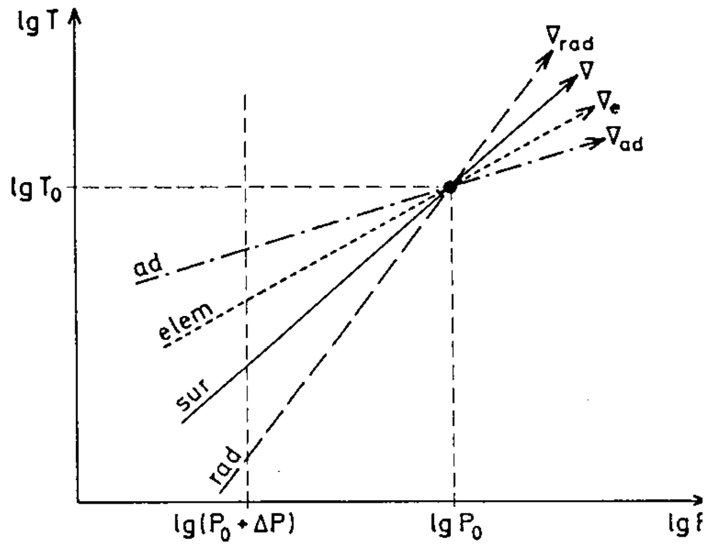


Illustration of the various temperature gradients
in a convectively unstable region

In an unstable layer, the following relations hold:

$$\nabla_{\text{rad}} > \nabla > \nabla_e > \nabla_{\text{ad}} \tag{8}$$

∇ is the actual gradient that will result in the convective layer. The first inequality arises from the fact that only part of the flux can be tranported by radiation, since convection is carrying some in any case. The last is due to the fact that the element will cool more than just adiabatic, because some energy will be lost by other means (radiation, conduction). The middle one is just the stability criterion for the blob not to be pushed back. The task of a convection theory is it to derive ∇!

The chemical composition

The chemical composition of a star enters ρ, κ, ϵ. It can change due to nuclear burning, diffusion, convection and other mixing processes.

Notation:

relative element mass fraction: $X_i := \frac{m_i n_i}{\rho}$, $\sum_i X_i = 1$;
special cases:
hydrogen X, helium Y, "metals" $Z = 1 - X - Y$.
typical values:
$X \approx 0.7 \cdots 0.75$; $Y \approx 0.24 \cdots 0.30$; $Z \approx 0.0001 \cdots 0.04$
$m_i = \mu_i m_u$ and n_i are particle masses and densities.

Changes due to nuclear reactions:

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho}[\sum_j r_{ji} - \sum_k r_{ik}]$$

r_{ji} indicating that species i is created from species j, and r_{ik} that i is destroyed to create k. In most cases more than one input or output particle is involved, e.g. $p +^{12} \mathrm{C} \rightarrow^{13} \mathrm{N}$.

Energy released is $\epsilon_{ij} = \frac{1}{\rho} r_{ij} e_{ij}$. r_{ij} number of reactions per second, and e_{ij} the energy released per reaction, per particle mass it is $q_{ij} = e_{ij}/m_i$.

In the simplest case, for the conversion of hydrogen into helium, we get

$$\frac{\partial X}{\partial t} = -\frac{\epsilon_H}{q_H} = -\frac{\partial Y}{\partial t},$$

$e_{\mathrm{H,He}} \approx 26.7$ MeV/reaction $= 4.72 \cdot 10^{-5}$ erg and $q_{\mathrm{H,He}} = 2.5 \cdot 10^{19}/4 = 6.44 \cdot 10^{-18}$ erg/g.

Changes due to convection:

Usually, the convective timescale $\tau_c \ll \tau_n$, and therefore the mixing can be assumed to be instantaneous. The composition after mixing is simply the mean composition of the convective layer.

If there is nuclear burning happening in the convective layer (e.g. in the convective core of a star), the change in composition is

$$\frac{\partial X_i}{\partial t} = \frac{\int_{\text{c.}z.} \frac{m_i}{\rho} \left(\sum_j r_{ji} - \sum_k r_{ik} \right) \mathrm{d}M_r}{\int_{\text{c.}z.} \mathrm{d}M_r},$$

(c.z. indicates the convection zone) but

$$\frac{\partial X_i}{\partial M_r} = 0$$

.

Complication, if convection zone moves into a chemically different region.

Above equation is sometimes considered the 5$^{\text{th}}$ equation of stellar structure theory.

Changes due to diffusion:

Treated by diffusion equation; the general expression for the diffusive velocity is:

$$\vec{v}_D = -\frac{1}{c} D \left(\vec{\nabla} n + k_T \vec{\nabla} \ln T + k_P \vec{\nabla} \ln P \right)$$

D is the diffusion constant, n the particle density, and k_i suitable scaling constants (for D).

Fick's law is $\vec{j}_D = n\vec{v}_D = -D\vec{\nabla} n$, where $\vec{j}_D$ is the diffusive particle flux.

And the continuity equation states

$$\frac{\partial n}{\partial t} = -\vec{\nabla}\vec{j}_D = \vec{\nabla}(D\vec{\nabla} n) = D\nabla^2 n$$

With corresponding equations for the other two terms, and k_T and k_P describing relative diffusion speeds w.r.t. concentration diffusion, the first equation is obtained.

The three terms correspond to *concentration*, *thermal*, and *pressure* diffusion, the latter better known as *sedimentation*. It often is the dominating effect.

The calculation of the diffusion constants is non-trivial, because it involves interactions on the quantum and electromagnetic level.

Estimate for diffusive timescale in Sun:

Diffusion is *random walk*; therefore distance l_{diff} related to number of scatterings N by

$$l_{\text{diff}} = \lambda\sqrt{N}$$

with λ being the mean free path and $N = \frac{\tau_{\text{diff}}}{\tau_s}$.
τ_s is the mean time between interactions. From this we can write

$$\tau_{\text{diff}} = \tau_s \left(\frac{l_{\text{diff}}}{\lambda}\right)^2$$

λ is related to the particle density n and the interaction cross section σ_s by

$$\lambda = \frac{1}{n\sigma_s}.$$

Assuming elastic scattering by Coulomb-forces, σ_s is of order $10^{-20}\,\text{cm}^2$ for hydrogen in the solar interior and therefore $\lambda \approx 4 \cdot 10^{-7}$ cm.

The thermal velocity of the particles is $\langle v \rangle = \sqrt{3kT/m}$.
We can compute τ_s from

$$\tau_s = \frac{\lambda}{\langle v \rangle} \approx 8 \cdot 10^{-15}\,\text{s}.$$

Sun: $\tau_{\text{diff}} \approx 10^{13}$ yrs for $l_{\text{diff}} = R_\odot$!

However, within the solar age l_{diff} is of order 1% of the solar radius, and therefore small effects are present.

In general, diffusion is unimportant except for (a) high precision models (Sun, high-resolution spectroscopy), and (b) thin surface layers of high temperatures (hot horizontal branch stars, young white dwarfs).

The equations - summary

m is the Lagrangian coordinate;
r, P, T, L_r are the dependent variables;
X_i are the composition variables;
ρ, κ, ϵ, ... are physical functions, all depending on $(P, T, \vec{X})$.

The four structure equations to be solved are:

$$\frac{\partial r}{\partial m} = \frac{1}{4\pi r^2 \rho} \tag{9}$$

$$\frac{\partial P}{\partial m} = -\frac{Gm}{4\pi r^4} - \frac{1}{4\pi r^2}\frac{\partial^2 r}{\partial t^2} \tag{10}$$

$$\frac{\partial L_r}{\partial m} = \epsilon_n - \epsilon_\nu - c_P \frac{\partial T}{\partial t} + \frac{\delta}{\rho}\frac{\partial P}{\partial t} \tag{11}$$

$$\frac{\partial T}{\partial t} = -\frac{GmT}{4\pi r^4 P}\nabla \tag{12}$$

with ∇ depending of the type of energy transport:

$$\nabla = \nabla_{\rm rad} = \frac{3}{16\pi acG}\frac{\kappa L_r P}{mT^4} \tag{13}$$

$$\nabla = \nabla_{\rm con} \, (\approx \nabla_{\rm ad}) \tag{14}$$

Finally, for the composition, we have (schematically)

$$\frac{\partial X_i}{\partial t} = \frac{m_i}{\rho}\left(\sum_j r_{ji} - \sum_k r_{ik}\right) \tag{15}$$

which may include a diffusive term (here: representative for concentration diffusion)

$$\frac{\partial X_i}{\partial t} = \frac{\partial}{\partial m}\left[(4\pi r^2 \rho)^2 D \frac{\partial X_i}{\partial r}\right] \tag{16}$$

Solving the structure equations

- We have I+4 equations for the I species under consideration plus the 4 dependent variables $\rightarrow$ system can be solved!

- The equations are *non-linear partial differential equations* in t and M_r (m). We need additional *boundary conditions* at $M_r = 0$ and M (for the spatial part) and *intial conditions* for $t = 0$ (for the temporal part). Stellar structure is an *initial and boudary value problem*.

- We call a *stellar model* the spatial solution for the structure at a given time t_0 ($r(M_r, t_0)$, $T(M_r, t_0)$, $\ldots X_I(M_r, t_0)$).

- *Initial values:* needed for all variables at $t = 0$. How to obtain them? Possible ways: a previously computed model, a simplified model, a good guess.

- If the initial model is a bad guess, but close enough to the real solution, it will converge over τ_{KH} to this.

- The problem can be separated into a spatial and a temporal part:

Step 1 solve eqs. (9)–(12) for fixed t_1 ($X_i(M_r, t_1)$ given)

Step 2 solve eqa. (15) and (16) for fixed spatial structure from t_1 to $t_2 = t_1 + \triangle t$ using $\epsilon(M_r, t_1)$

Step 3 update composition:
$X_i(t_2) = X_i(t_1) + \left(\frac{\partial X_i}{\partial t}\right)_{t_1} \triangle t$ $\rightarrow$ Step 1

Boundary conditions

at **center**: $M_r = 0 \rightarrow \qquad r(0) = 0, \quad L_r(0) = 0$

at **surface**: different possibilities

(1) "zero" b.c.: for $M_r = M$: $P(M) = 0$, $T(M) = 0$; gives inner parts of stars approximately correct, but outer parts are unrealistic; cannot be compared to observations

(2) **photospheric b.c.**: b.c. taken at photosphere, i.e. at *optical depth* $\tau_{\rm ph} = 2/3$, where $T = T_{\rm eff}$

Stefan-Boltzmann law:

$$L = 4\pi\sigma R^2 T_{\rm eff}^4 \qquad (17)$$

This is the first photospheric boundary condition. It relates three of the four dependent variables.

Temperature relation for gray ($\frac{\partial\kappa}{\partial\nu} = 0$) Eddington-atmosphere:

$$T^4(\tau) = \frac{3}{4} T_{\rm eff}^4 (\tau + 2/3)$$

mass in atmosphere $< 10^{-10}\, M_\odot \rightarrow M(\tau) = M(\tau_{\rm ph})$

We still need a second relation that involves pressure.

The optical depth definition is

$$\mathrm{d}\tau = \kappa\rho\mathrm{d}r \;\rightarrow\; \tau_{\mathrm{ph}} = \int_R^{\infty} \kappa\rho\mathrm{d}r$$

for simplification, replace $\kappa(r)$ by $\bar{\kappa}$, and $\tau_{\mathrm{ph}} = 2/3 \rightarrow$

$$\frac{2}{3} = \bar{\kappa}\int_R^{\infty} \rho\mathrm{d}r$$

From the hydrostatic equation one can integrate the pressure gradient to get

$$P_{\mathrm{ph}} = \int_R^{\infty} g\rho\mathrm{d}r = \frac{GM}{R^2}\int_R^{\infty} \rho\mathrm{d}r$$

which results in the second outer b.c., for P_{ph}:

$$P_{\mathrm{ph}} = \frac{2}{3}\frac{GM}{R^2}\frac{1}{\bar{\kappa}} \tag{18}$$

This is the second relation for the outer b.c., and the system is closed and can be solved.

Eq. (18) can be obtained by more realistic integrations using $\kappa(r) = \kappa(P, T)$ and $g(r)$, or even by using non-gray atmospheres. In general, gray atmospheres are accurate enough.

Note on influence of outer b.c.:

If outer stellar layers are radiative, the solutions will, independent of the outer b.c., quickly converge to the identical inner structure. Hot stars have radiative envelopes.

If they are convective, in particular with $\nabla > \nabla_{\text{ad}}$, they remain different for different outer b.c., and influence the interior. In the extreme case (almost fully convective stars) the central conditions depend on the atmospheric conditions. Cool stars have deep convective envelopes.

Note on numerical methods for solving structure equations:

1. Integrator-type methods (like Runge-Kutta)

Idea: $A(x + \triangle x) = A(x) + \triangle x \frac{\partial A}{\partial x}$

Complication: b.c. not at one boundary, but two of them at each.

Start from $M_r = 0$ and $M_r = M$ using boundary values for two variables and trial values for the other two. Integrate all equations from boundaries to some matching point. If variables at matching point do not agree, go back with new trial values. Iterate until match is good enough. *Fitting method*

Advantage: does not need previous model; works well for homogenous chemical composition; useful for initial main-sequence models and for pre-main sequence models. Accuracy control of solution.

Disadvantage: slow or no convergence if steep gradients are present; convective envelopes require extremely well known outer b.c.

2. Newton-Raphson type methods (like Henyey-method)

Idea: divide star in N shells, and define variables on grid points; replace differential equations by difference equations, e.g.

$$\frac{\partial P}{\partial M_r} \rightarrow \frac{\triangle P}{\triangle M_r} = -\frac{G\bar{M}_r}{4\pi\bar{r}^4}$$

The $\triangle P$ etc. are the differences between values at neighbouring grid points and on the r.h.s. suitable means over the shell between these neighbouring grid points are used. Equations solved by *Newton-Raphson*-type methods.

Advantage: reasonable convergence; can handle all evolutionary phases (depending on grid resolution); changing outer b.c. part of iteration process

Disadvantage: results in system of coupled linear equations, which requires inversion of $4(N-1) \times 4(N-1)$ matrix (for $4(N-1)$ unknowns; N of order 1000); requires good starting values for each model at each grid point for Newton-solver to converge. (What about very first model?)

Henyey-method: clever handling of matrix, which can be split up into many 8x4 matrices. **Method of choice for almost all stellar evolution codes**

Use model at t_1 as first guess for new model at t_2, even if $X_i(M_r)$ have changed. Number of iterations usually < 10, but can go up to ≈ 100.

Simple Stellar Models

Polytropes

Definition: relation between P and ρ of type $P \propto \rho^\nu$

Polytropic relation:

$$P = K\rho^{1+1/n}$$

K: constant of polytrope; n: polytropic index; both constant throughout star.

The polytropic relation is *not* identical to an equation of state of that type! It requires only that T is changing with radius in such a way that the polytropic relation is maintained.

Examples:

1. Isothermal ideal gas: $T = \text{const}_1 \rightarrow n = \infty$

2. Adiabatic stratification:
   $T = \text{const}\, P^{\nabla_{\text{ad}}}$ (in case of convection)

   For ideal gas, it follows that
   $P = \text{const}_2\, \rho^{1/(1-\nabla_{\text{ad}})}$ and $\rightarrow n = \frac{1-\nabla_{\text{ad}}}{\nabla_{\text{ad}}}$.
   For $\nabla_{\text{ad}} = 0.4 \rightarrow n = 3/2$.

A *polytropic equation of state* automatically results in a polytropic stellar interior, e.g. the completely degenerate electron gas:

– non-relat. complt. deg. $\rightarrow P = \text{const}\, \rho^{5/3} \rightarrow n = 3/2$

– relativ. complt. deg. $\rightarrow P = \text{const}\, \rho^{4/3} \rightarrow n = 3$

In this case, the constant K is known (from equation of state); if only stratification is polytropic, K is unknown and depends on T!

Emden's equation

With a polytropic relation the two "mechanical equations" (9) and (10) can be solved independently, by combining them into one.

From (10), written in Euler-coordinates we get an equation for M_r, which, when we differentiate it w.r.t. r gives

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = -\frac{1}{G}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = 4\pi r^2 \rho$$

The left hand side is the Euler version of eq. (9) and we obtain Poisson's equation.

Using the polytropic equation, the derivative of P is

$$\frac{\mathrm{d}P}{\mathrm{d}r} = K(1+\frac{1}{n})\rho^{1/n}\frac{\mathrm{d}\rho}{\mathrm{d}n}$$

and using this in Poisson's equation yields

$$K\left(\frac{n+1}{n}\right)\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\rho^{\frac{1}{n}-1}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -4\pi G\rho$$

With the following substitutions and definition:

$r \rightarrow \alpha x \qquad \rho \rightarrow \lambda y^n \qquad \alpha^2 := \frac{K(n+1)\lambda^{\frac{1}{n}-1}}{4\pi G}$

we get **Emden's equation**

$$\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{2}{x}\frac{\mathrm{d}y}{\mathrm{d}x} = -y^n \qquad (19)$$

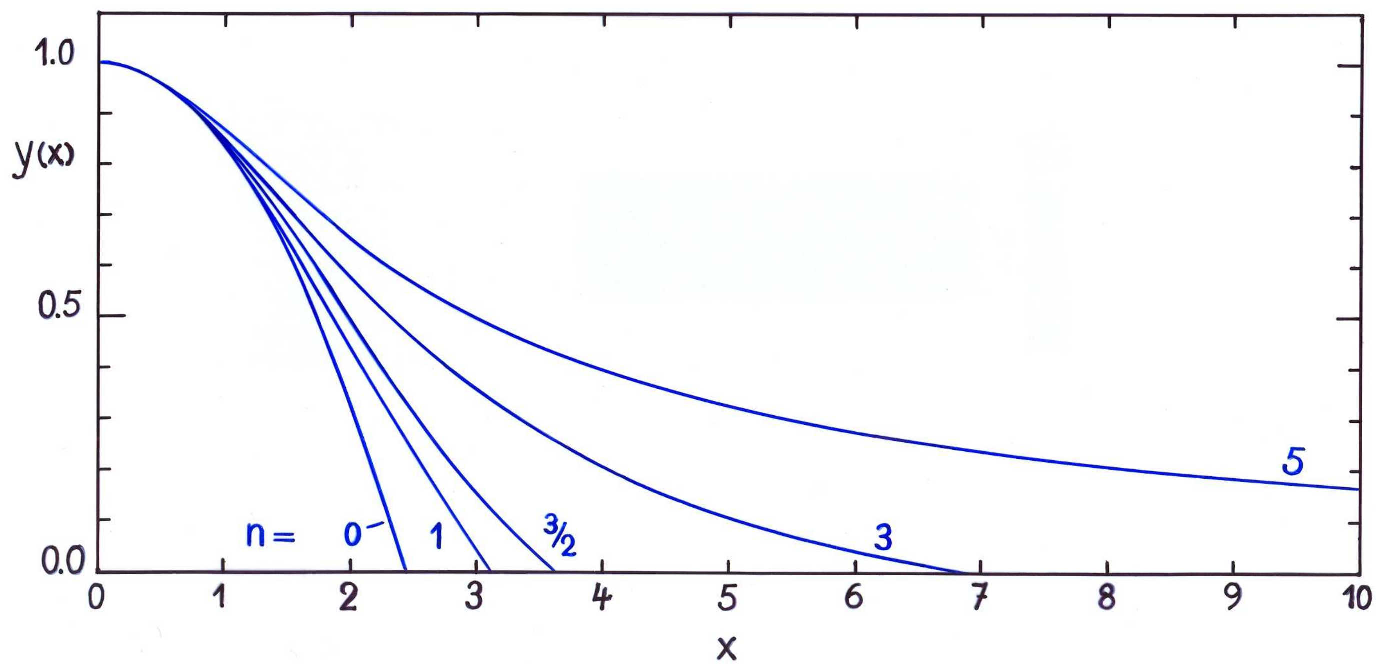
Discussion of solutions of Emden's equation – Lane-Emden-functions

- λ is one free parameter; choose $\lambda = \rho_c$
- at center $x = 0$ and $y(x = 0) = 1$; $\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_0 = 0$
- $R = \alpha x_0$ corresponds to first root of $y(x_0)$
- $M = -4C(K)\rho_c^{\frac{3-n}{2n}} x_0^2 \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x_0}$
- for $n = 3$, M independent of ρ_c (important for White Dwarfs)
- solved either numerically or by power series: $y(x) = 1 - \frac{x^2}{6} + \frac{nx^4}{160} \ldots$
- analytical solutions for $n = 0$ (see above), $n = 1$: $y(x) = \frac{\sin(x)}{x}$, $n = 5$: $y \to 0$ for $x \to \infty$
- for $n < 5$ polytropes have finite radius
- if equation of state is polytropic, K is fixed, and a mass-radius-relation results
- otherwise Lane-Emden-functions scalable with one more free parameter
- from polytropes Chandrasekhar could derive the critical mass (of his name) for completely degenerate stars, above which even electron pressure can no longer balance the gravitational pull:

$$M_{\mathrm{CH}} = \frac{5.76}{\mu_e^2} M_\odot$$

Lane – Emden – Funktionen

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) = -y^n \quad ; \quad y(0)=1 \quad , \quad y'(0)=0$$



Simple stellar models

Homology

for chemically homogeneous (i.e. undeveloped, zero-age, main-sequence) stars.

Basic homology assumption:

Two stars are called *homologous to each other*, if

at $\frac{m_1}{M_1} = \frac{m_0}{M_0} \rightarrow \frac{r_1}{R_1} = \frac{r_0}{R_0}$

this gives scaling functions:

$r\left(\frac{m}{M}\right) = Rf_r\left(\frac{m}{M}\right)$

$L_r\left(\frac{m}{M}\right) = Lf_L\left(\frac{m}{M}\right)$

$P\left(\frac{m}{M}\right) = P_c f_P\left(\frac{m}{M}\right)$

$T\left(\frac{m}{M}\right) = T_c f_T\left(\frac{m}{M}\right)$

where f_i are independent of M, but the constants (R, P_c, etc.) *are* dependent on M and μ.

Assume simple laws for the physics functions:

$$P = \frac{\mathcal{R}}{\mu}\rho T \tag{20}$$

$$\epsilon = \epsilon_0 \rho^\lambda T^\nu \tag{21}$$

$$\kappa = \kappa_0 \rho^n T^{-s} \tag{22}$$

From the (radiative) structure equations we get simple scaling relations, for example for pressure ($x := m/M$):

$$\frac{\mathrm{d}P}{\mathrm{d}m} = P_C \frac{\mathrm{d}f_P}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}m} = \frac{P_c x}{f_P}\frac{\mathrm{d}\ln f_P}{\mathrm{d}\ln x}\frac{1}{M} = \frac{P_c P M}{P_c m M}\frac{\mathrm{d}\ln f_P}{\mathrm{d}\ln x} = \frac{P}{m}\frac{\mathrm{d}\ln f_P}{\mathrm{d}\ln x}$$

and equating this with the structure equation for P we obtain the following relation, and similar ones for the other variables:

$$\frac{\mathrm{d}P}{\mathrm{d}m} = \frac{P}{m}\frac{\mathrm{d}\ln f_P}{\mathrm{d}\ln x} = -\frac{Gm}{4\pi r^4} \quad \rightarrow \quad \frac{P}{m} \sim \frac{m}{r^4} \qquad (23)$$

$$\frac{\mathrm{d}r}{\mathrm{d}m} = \frac{r}{m}\frac{\mathrm{d}\ln f_r}{\mathrm{d}\ln x} = \frac{1}{4\pi r^2 \rho} \quad \rightarrow \quad \frac{r}{m} \sim \frac{1}{r^2\rho} \qquad (24)$$

$$\frac{\mathrm{d}T}{\mathrm{d}m} = \frac{T}{m}\frac{\mathrm{d}\ln f_T}{\mathrm{d}\ln x} = -\frac{3\kappa}{64\pi ac}\frac{L_r}{r^4 T^3} \quad \rightarrow \quad \frac{T}{m} \sim \frac{L_r}{r^4 T^3} \qquad (25)$$

$$\frac{\mathrm{d}L_r}{\mathrm{d}m} = \frac{L_r}{m}\frac{\mathrm{d}\ln f_L}{\mathrm{d}\ln x} = \epsilon \quad \rightarrow \quad \frac{L_r}{m} \sim \epsilon \qquad (26)$$

From (23) and (24) expressions for P and ρ as functions of r and m can obtained; taking the ratio of these and using the same ratio from the ideal gas EOS it follows that

$$\frac{P}{\rho} \sim \frac{m}{r} \sim \frac{T}{\mu}$$

(or $rT = \mu m$) and with equation (25) for L_r
$\Rightarrow L_r \sim \mu^4 m^3$

in particular, for $x = 1$ or $m = M$, we have

$$L \sim \mu^4 M^3,$$

which is the **mass-luminosity relation for main sequence stars**. It does not depend on energy generation, but the proportionality is determined mainly by opacity!

Since also $L_r \sim m\epsilon \sim m\rho^\lambda T^\nu$, we obtain (again for $x = 1$, using $\rho \sim m/r^3$ and $T \sim \mu m/r$)

$$R \sim \mu^{\frac{\nu-4}{\nu+3\lambda}} M^{\frac{\lambda+\nu-2}{\nu+3\lambda}}$$

For $\lambda = 1$ and $\nu \approx 5$ (pp-cylce) $\Rightarrow$

$$R \sim \mu^{0.125} M^{0.5}$$

For $\lambda = 1$ and $\nu \approx 15$ (CNO-cycle) $\Rightarrow$

$$R \sim \mu^{0.61} M^{0.78}$$

These are **mass-radius relations** for the two main nuclear cycles on the main sequence. A representative value for ν is 13, which gives 0.75 for the M-exponent.

This slide is about the main-sequence and lines of constant radius in the HRD, and about MS-lifetimes derived from the relations obtained above.

$\rightarrow$ see 2nd exercise sheet!

Central values on the main sequence:

Set $\lambda = 1$ and μ=const.

$P_c \sim P \sim \frac{P(x)}{f_P(x)}$ and $T_c \sim T \sim \frac{T(x)}{f_T(x)}$

Also $T \sim \frac{M}{R}$, $P \sim \frac{M^2}{R^4}$, $\rho_c \sim \frac{P_c}{T_c}$, and $R \sim M^{\frac{\nu-1}{\nu+3}}$;

$\Rightarrow$

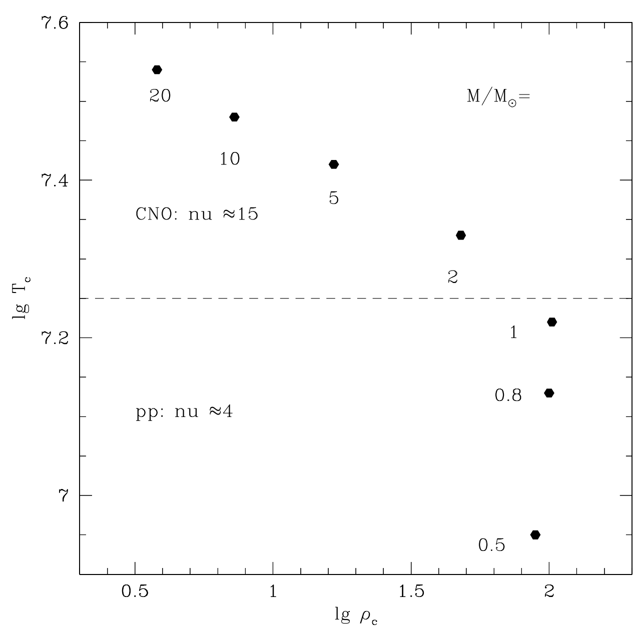
$$T_c \sim M^{\frac{4}{\nu+3}} \quad (27)$$

$$P_c \sim M^{-\frac{2(\nu-5)}{\nu+3}} \quad (28)$$

$$\rho_c \sim M^{-\frac{2(\nu-3)}{\nu+3}} \quad (29)$$

$$T_c \sim \rho_c^{-\frac{2}{\nu-3}} \quad (30)$$

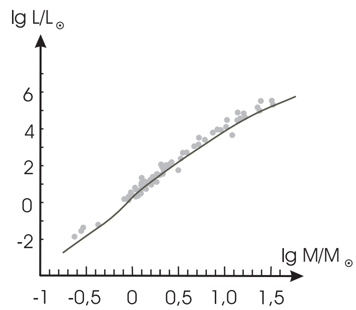
T_c is rising with M; but ρ is *falling* with M for $\nu > 3$ ($M > 0.8\,M_\odot$)!



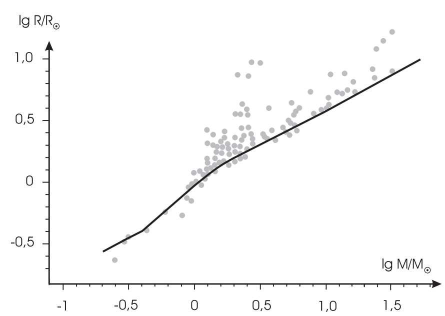
Two different types of stars on the main sequence

	$M \lesssim 1.5\,M_{\odot}$	$M \gtrsim 1.5\,M_{\odot}$
T_{eff}	low	high
core	radiatve	convective
envelope	convective	radiative
H-burning	pp-chain	CNO-cycle
ν	> 10	< 7
gas pressure	small	significant

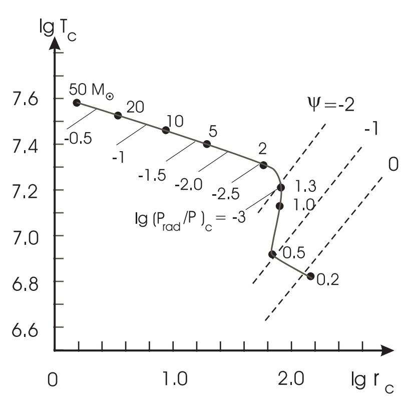
Theoretical and observed M-L-relation:



Theoretical and observed M-R-relation:



Theoretical $\log \rho_c$-$\log T_c$-relation:



The Microphysics

EOS, opacity, energy generation

Equation of State

Ideal gas: (see tutorial handout)

$$P = nk_BT = \frac{\mathcal{R}}{\mu}\rho T$$

with $\rho = n\mu m_u$; μ: molecular weight, mass of particle per m_u.

Several components in gas with relative mass fractions $X_i = \frac{\rho_i}{\rho} \to n_i = \frac{\rho X_i}{m_u \mu_i}$

$$P = P_e + \sum_i P_i = (n_e + \sum_i n_i)k_BT.$$

Completely ionzied atom:

$$P = nk_BT = \mathcal{R}\sum_i \frac{X_i(1+Z_i)}{\mu_i}\rho T = \frac{\mathcal{R}}{\mu}\rho T$$

$\mu := \left(\sum_i \frac{X_i(1+Z_i)}{\mu_i}\right)^{-1}$: *mean molecular weight*

For a neutral gas, $\mu = \left(\sum_i \frac{X_i}{\mu_i}\right)^{-1}$.

[In computation of μ, for the metals with abundance Z one assumes (for complete ionization) that $Z_i/A_i = 1/2$, so their contribution is $Z/2$.]

The mean molecular weight *per free electron* is

$$\mu_e := \left(\sum_i \frac{X_i Z_i}{\mu_i}\right)^{-1} = \frac{2}{(1+X)}$$

Radiation pressure: (see tutorial handout)

$$P_{\rm rad} = \frac{1}{3}U = \frac{a}{3}T^4 \;\left(a = 7.56\cdot 10^{-15}\,\frac{\rm erg}{\rm cm^3K^4}\right)$$

$\beta := \frac{P_{\rm gas}}{P}$.

$\left(\frac{\partial\beta}{\partial T}\right)_P = -\frac{4(1-\beta)}{T}$ and $\left(\frac{\partial\beta}{\partial P}\right)_T = \frac{(1-\beta)}{T}$.

Furthermore

$$\begin{aligned}
\alpha &:= \left(\frac{\partial \ln\rho}{\partial \ln P}\right)_T = \frac{1}{\beta}\\
\delta &:= -\left(\frac{\partial \ln\rho}{\partial \ln T}\right)_P = \frac{4-3\beta}{\beta}\\
\varphi &:= \left(\frac{\partial \ln\rho}{\partial \ln \mu}\right)_{T,P} = 1\\
c_P &:= \frac{\mathcal{R}}{\mu}\left[\frac{3}{2} + \frac{3(4+\beta)(1-\beta)}{\beta^2} + \frac{4-3\beta}{\beta^2}\right]\\
\nabla_{\rm ad} &:= \frac{\mathcal{R}\delta}{\beta\mu c_P}\\
\gamma_{\rm ad} &:= \left(\frac{{\rm d}\ln P}{{\rm d}\ln\rho}\right)_{\rm ad} = \frac{1}{\alpha - \delta\nabla_{\rm ad}}
\end{aligned}$$

For $\beta \to 0$, $c_P \to \infty$, $\nabla_{\mathrm{ad}} \to 1/4$ and $\gamma_{\mathrm{ad}} \to 4/3$.
For $\beta \to 1$, $c_P \to \frac{5\mathcal{R}}{2\mu}$, $\nabla_{\mathrm{ad}} \to 2/5$, and $\gamma_{\mathrm{ad}} \to 5/3$.

Note: there are further thermodynamic derivatives in use in the literature, the so-called gammas. Here is a list of relations:

$$\begin{aligned}
\gamma_{\mathrm{ad}} &=: \Gamma_1 \\
\nabla_{\mathrm{ad}} &=: \frac{\Gamma_2 - 1}{\Gamma_2} \\
\Gamma_3 &:= \left(\frac{\mathrm{d}\ln T}{\mathrm{d}\ln \rho}\right)_{\mathrm{ad}} + 1 \\
\frac{\Gamma_1}{\Gamma_3 - 1} &= \frac{\Gamma_2}{\Gamma_2 - 1}
\end{aligned}$$

Ionization

Boltzmann-equation: occupation numbers of different energy states in thermal equilibrium. Applied to atoms being ionized, taking into account the distribution of electrons in phase space ⇒ *Saha-equation*

$$\frac{n_{r+1}}{n_r} P_e = \frac{u_{r+1}}{u_r} 2 \frac{(2\pi m_e)^{3/2}}{h^3} (kT)^{5/2} \exp(-\chi_r / kT),$$

with n_r: number density of atoms in ionization state r; χ_r ionization energy; u_r partition function; $P_e = n_e kT$ the electron pressure ($k = k_B$)

Application: hydrogen ionziation in Sun

$n = n_0 + n_1$; $n_e = n_1$; $x := \frac{n_1}{n_0 + n_1}$

$P_e = P_{\text{gas}} \frac{n_e}{n_e + n} = P_{\text{gas}} \frac{x}{x+1}$

$$\Rightarrow \frac{x^2}{1 - x^2} = \frac{u_1}{u_0} \frac{2}{P_{\text{gas}}} \frac{(2\pi m_e)^{3/2}}{h^3} (kT)^{5/2} \mathrm{e}^{-\chi_1/kT}$$

$u_0 = 2$, $u_1 = 1$ are ground-state statistical weights; $\chi_1 = 13.6$ eV.

Solar surface, $T = 5700$ K, $P_{\text{gas}} = 6.8 \cdot 10^4$, $\rightarrow x \approx 10^{-4}$; at $P_{\text{gas}} = 10^{12}$, $T = 7 \cdot 10^5$, $\rightarrow x \approx 0.99$.

The mean molecular weight for a partially ionized gas $\mu = \mu_0/(E + 1)$, where μ_0 is the molecular weight of the unionized gas, and E the number of free electrons per all atoms. With μ again the ideal gas equation can be used.

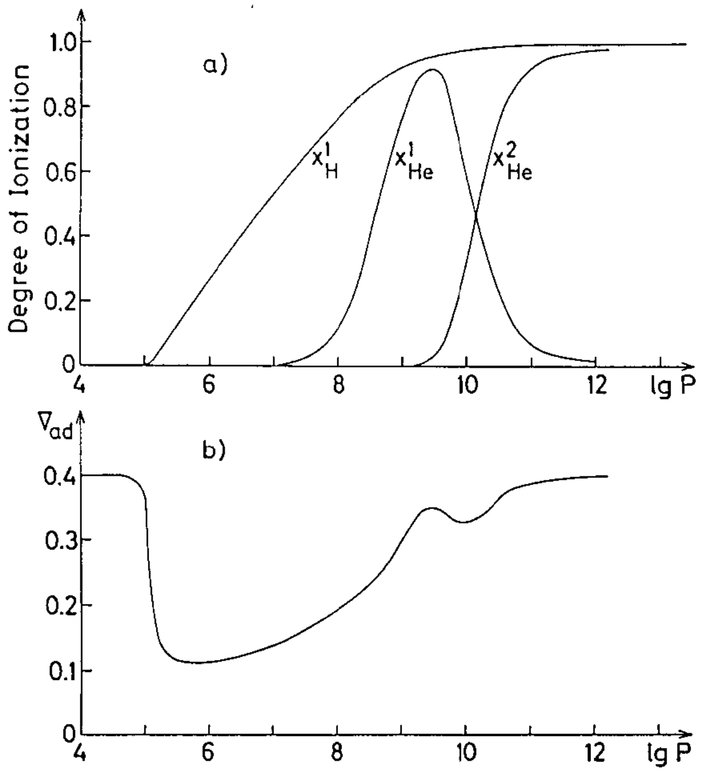


Illustration of ionization of hydrogen and helium within a stellar envelope. In panel (b) the corresponding run of $\nabla_{\rm ad}$ is shown. The depression is due to the increase in c_P due to ionziation. Since $\nabla_{\rm ad}$ is getting smaller, convection will set in.

Note the following defect of the Saha-equation: the ionization increases with T and decreases with P. When $T \approx$ const., as in stellar cores, the ionization degree should decrease, which is unphysical. The explanation lies in the fact that the ionization potential is suppressed, if the atoms approach each other, and individual potentials overlap. This is called *pressure ionziation*, and is treated in practice by "complete ionziation"-conditions or a change in the χ_i.

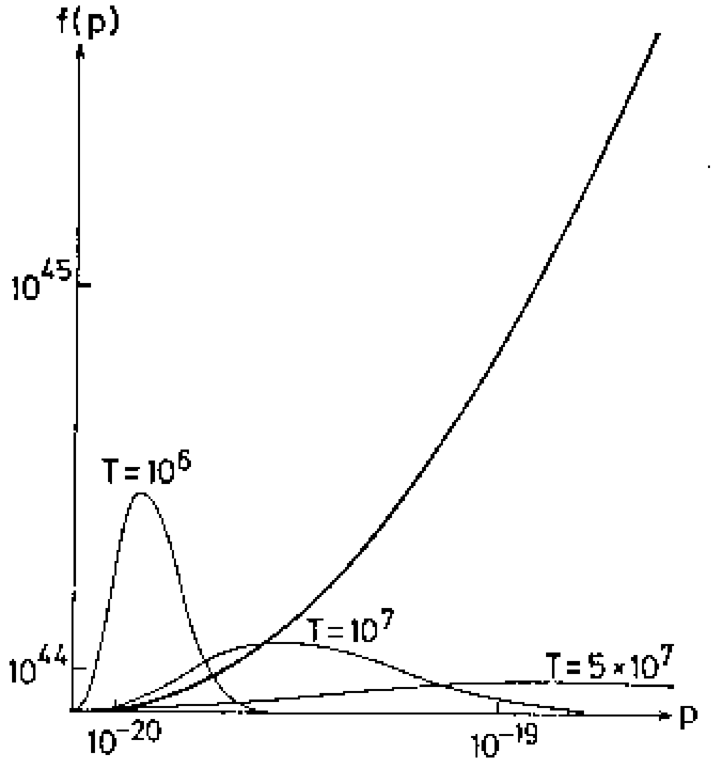
Electron degeneracy

The distribution of electrons in momentum space (Boltzmann equation; p is momentum):

$$f(p)\mathrm{d}p\mathrm{d}V = n_e \frac{4\pi p^2}{(2\pi m_e kT)^{3/2}} \exp\left(-\frac{p^2}{2m_e kT}\right) \mathrm{d}p\mathrm{d}V$$

Pauli-principle:

$$f(p)\mathrm{d}p\mathrm{d}V \leq \frac{8\pi p^2}{h^3}\mathrm{d}p\mathrm{d}V$$



Completely degenerate gas:

$$
\begin{aligned}
f(p) &= \frac{8\pi p^2}{h^3} \text{ for } p \le p_F \propto n_e^{1/3} \\
&= 0 \text{ for } p > p_F
\end{aligned}
$$

$E_F = \frac{p_F^2}{2m_e} \propto n_e^{2/3}$ is the Fermi-energy; for $E_F \approx m_e c^2$, $v_e \approx c \rightarrow$ *relativistic complete degeneracy*.

1. $p_F \ll m_e c$ (non-relativistic)

$$P_e = 1.0036 \cdot 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} \qquad P_e = \frac{2}{3} U_e$$

2. $p_F \gg m_e c$ (relativistic)

$$P_e = 1.2435 \cdot 10^{15} \left(\frac{\rho}{\mu_e}\right)^{4/3} \qquad P_e = \frac{1}{3} U_e$$

$P_i \ll P_e$

Partial degeneracy:

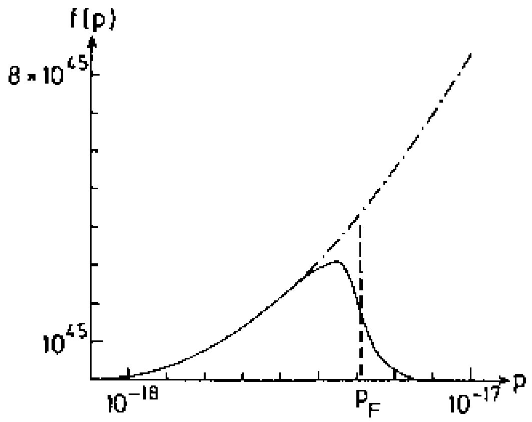
Finite T $\rightarrow$ *Fermi-Dirac* statistics:

$$f(p)\mathrm{d}p\mathrm{d}V = \frac{8\pi p^2}{h^3} \frac{1}{1 + \exp\left(\frac{E}{kT} - \Psi\right)} \mathrm{d}p\mathrm{d}V$$

$\Psi = \frac{n_e}{T^{3/2}}$ is the *degeneracy parameter*.
At constant Ψ, $T \propto \rho^{2/3}$ for the non-relativistic case, and $\propto \rho^{1/3}$ in the relativistic one.

For Fermions, the following relations are valid:

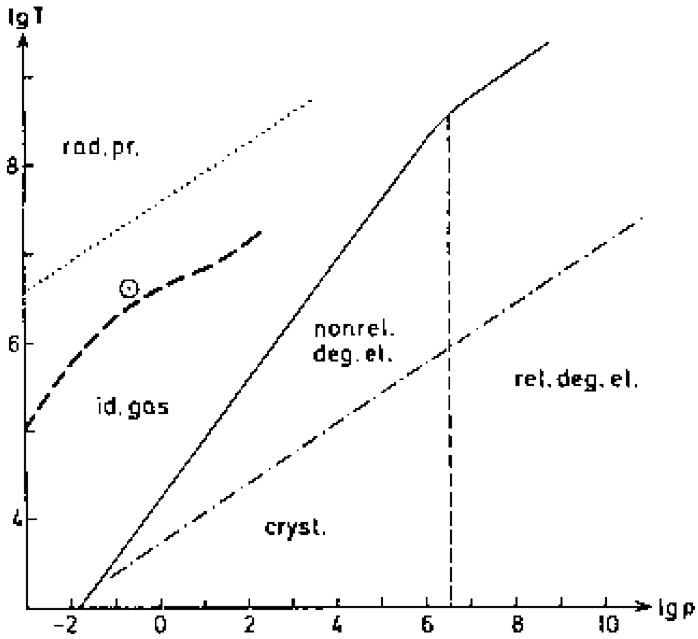
$$\begin{aligned} n_e &= \frac{8\pi}{h^3}\int_0^\infty \frac{p^2 \mathrm{d}p}{1+\exp\left(\frac{E}{kT}-\Psi\right)} \\ P_e &= \frac{8\pi}{3h^3}\int_0^\infty \frac{p^3 v(p) \mathrm{d}p}{1+\exp\left(\frac{E}{kT}-\Psi\right)} \\ U_e &= \frac{8\pi}{h^3}\int_0^\infty \frac{E p^2 \mathrm{d}p}{1+\exp\left(\frac{E}{kT}-\Psi\right)} \end{aligned}$$



$f(p)$ for partially degenerate gas with $n_e = 10^{28}$ cm^{-3} and $T = 1.9 \cdot 10^7$ K corresponding to $\Psi = 10$.

The equation of state for normal stellar matter:

$$\begin{aligned} P &= P_{\text{ion}} + P_e + P_{\text{rad}} \\ &= \frac{\mathcal{R}}{\mu_0}\rho T + \frac{8\pi}{3h^3}\int_0^\infty \frac{p^3 v(p) \mathrm{d}p}{1+\exp\left(\frac{E}{kT}-\Psi\right)} + \frac{a}{3}T^4 \\ \rho &= \frac{4\pi}{h^3}(2m_e)^{3/2} m_u \mu_e \int_0^\infty \frac{E^{1/2}\mathrm{d}E}{1+\exp\left(\frac{E}{kT}-\Psi\right)} \end{aligned}$$



Further effects:

1. Non-ideal effects (Coulomb screening in Debye-Hückel theory; van-der-Waals forces)

2. Collective effects like *crystallization* (White Dwarfs)

3. At nuclear matter densities, neutronisation (neutron stars)

In practice:

use of precompiled tables ("OPAL", "Mihalas-Hummer-Dappen") for different mixtures, or modified in-line EOS ("Eggleton-Faulker-Flannery"), which mimic non-ideal effext.

Opacity

Physical effects determining κ:

1. **Electron scattering:** (Thomson-scattering)
   $\kappa_{\rm sc} = \frac{8\pi}{3}\frac{r_e^2}{m_e m_u} = 0.20(1+X)$ cm^{2}g^{-1}

2. **Compton-scattering:**
   $T > 10^8$: momentum exchange $\rightarrow \kappa < \kappa_{\rm sc}$

3. **free-free transitions:**
   $\kappa_{\rm ff} \propto \rho T^{-7/2}$ (*Kramers* formula)

4. **bound-free transitions:**
   $\kappa_{\rm bf} \propto Z(1+X)\rho T^{-7/2}$

5. **b-f for H^--ion** below 6000 K (major source);

   electrons for H^--ion from metals with low ionization potentials

   $\kappa_{H^-} =$ const $q n_e \rho T^{\nu-7/2}$, $\nu 7 \cdots 10$

6. **bound-bound transitions:** below 10^6 K. No simple formula.

7. e^-**-conduction:** $\kappa_{\rm c} \propto \rho^{-2} T^2$

8. **molecular absorption** for $T < 10^4$ K

9. **dust absorption** for $T < 3000$ K

Opacity tables used in calculations

- Cox & coworkers (Los Alamos) started in late 60's with calculations of tables for Rosseland mean opacity

- Calculations include: detailed EOS and atomic physics

- Opacity Project (Seaton et al. 1994 . . . 2006): updated opacities for solar exterior conditions and variable compositions

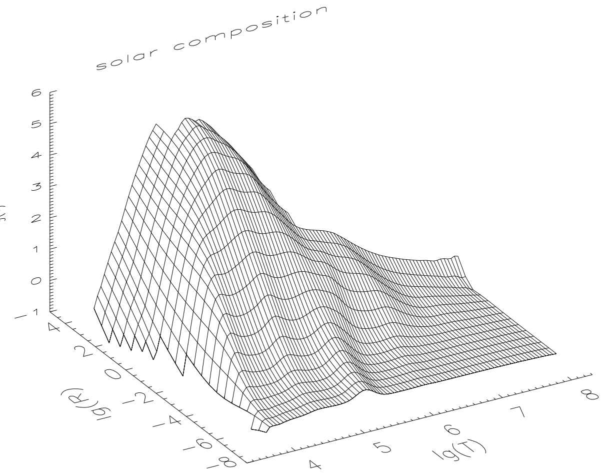
- OPAL: Rogers & Iglesias (LLNL, 1992–1996): updated opacities for extended range of stellar conditions and fixed set of compositions; $T >$ 6000 K

- OPAL & OP agree very well in common (T, R)-range ($R = \rho/T_6^3$)

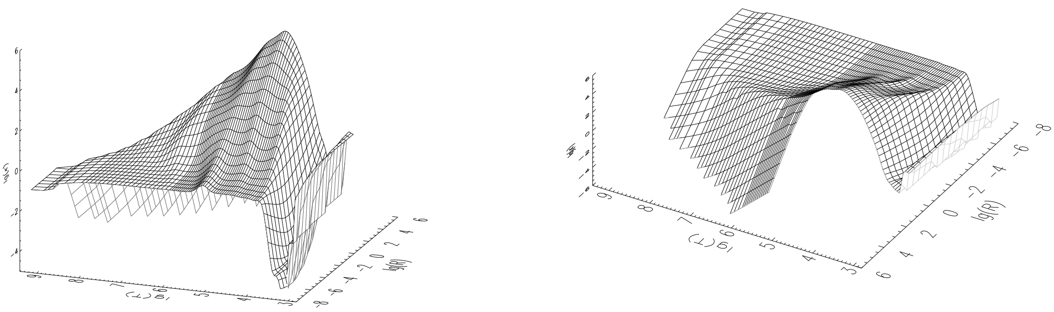
- Alexander & Ferguson (1998, 2006): updated molecular opacities ($T > 10^3$ K) for fixed set of compositions

- Itoh et al.; Potehkin: fitting formulae for electron conduction opacities

Sample table of Rosseland-opacity for "solar" composition ($X = 0.70$, $Y = 0.28$, $Z = 0.02$); only atomic absorption; source: OPAL



$R = \frac{\rho}{T_6^3}$, $\qquad T_6 = T/10^6\,\mathrm{K}$ (because massive stars are approximately $n = 3$ polytropes)



Two tables, combined from atomic, molecular, and "conductive" opacities for the solar (left) and a metal-poor (right) composition.

Note: lower metallicity → lower opacity → lower ∇_{rad} → higher surface temperature. *Pop. II stars are generally hotter than Pop. I!*

Nuclear Energy Production

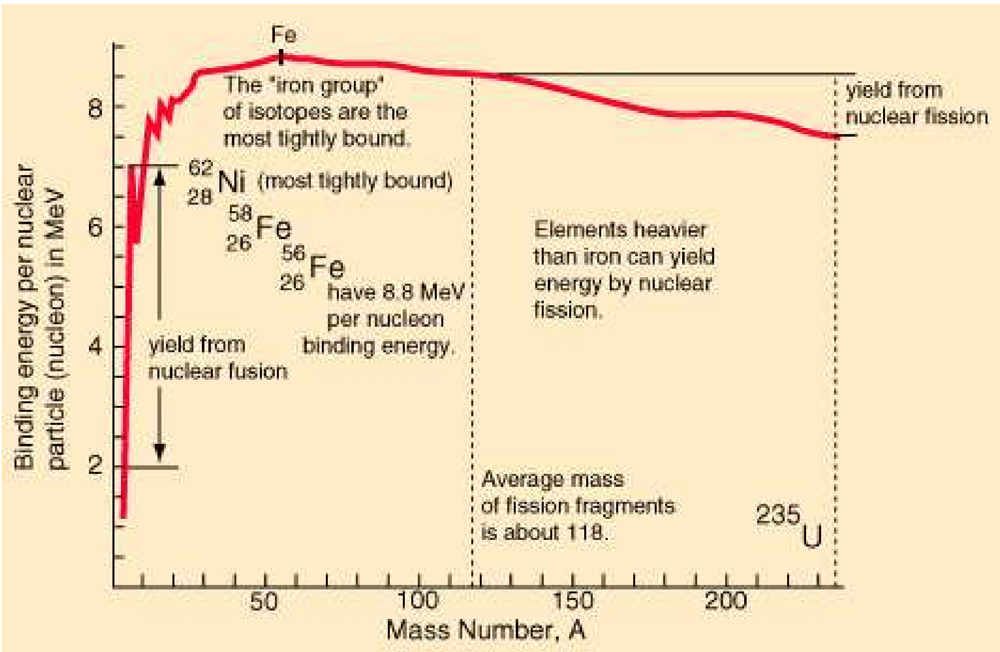
Mass defect $\rightarrow$ energy

Example:
4 ^{1}H (protons) : $4 \cdot 1.0081 m_u$ ^{4}He: $4.0089 m_u$.
Difference (0.7%) : 26.5 MeV $\Rightarrow$
the Sun can shine for 10^{11} yrs.

Binding energy:

$$E_B := [(A - Z)m_n + Zm_p - M_{\text{nuc}}]c^2$$

B.E. per nucleon $f := E_B/A$ (of order 8 MeV). Its maximum (8.4 MeV) is reached for ^{56}Fe.



Nuclear fusion:

Non-resonant reactions:

Coulomb barrier $V = \frac{1.44 Z_1 Z_2}{r_0[\text{fm}]}$ MeV cannot be overcome by particles with average thermal velocity at $10^7 \cdots 10^8$ K (energy $E_{\text{th}} \approx 10^4$ eV).

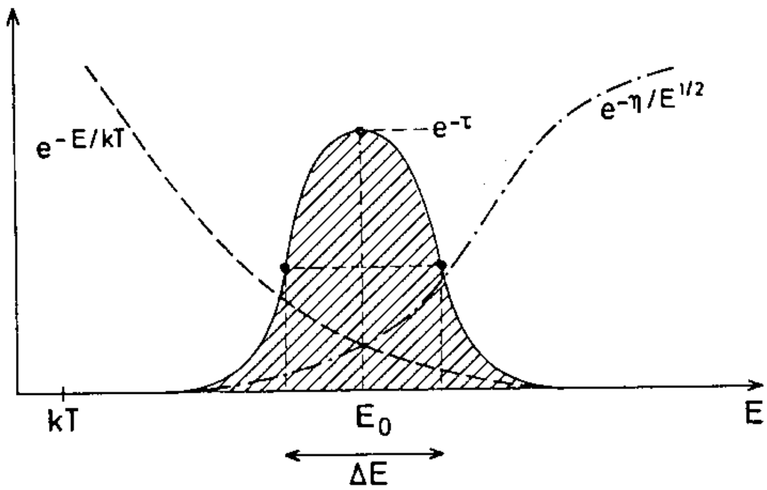
r_0 is radius off effective core (Yukawa-)potential:

$$r_0 \approx A^{1/3} 1.44 \cdot 10^{-13} \text{ cm}.$$

In high-energy tail of Maxwell-distribution only 10^{-43} fraction of particels.

Solution: quantum-mechanical tunneling effect (Gamow), which yields a maximum cross section at E_0 which is around stellar temperatures.

Maxwell-Boltzmann-distribution of temperature T and tunnelling probability:



The Gamow peak (strongly magnified); the dashed line is the Maxwell-distribution, the dot-dashed one the tunnelling probability

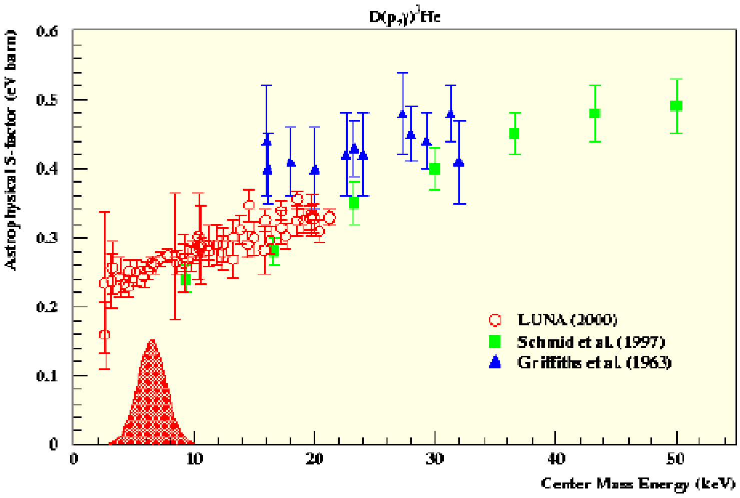
Nuclear cross sections

Non-resonant reactions; general form:

$$\sigma(E) = S(E)E^{-1}e^{-\pi\eta} \quad \eta = \sqrt{(m/2)}\frac{2\pi Z_1 Z_2 e^2}{hE^{1/2}}$$

with $m := \frac{m_1 m_2}{m_1+m_2}$

$S(E)$ is the *astrophysical cross section* and has the advantage of being a smooth function of T. Laboratory measurements are usually at energies above stellar energies, but can be extrapolated in $S(E)$.



Thermonuclear reaction rate

$$r_{jk} = \frac{n_j n_k}{1+\delta_{jk}}\langle\sigma v\rangle$$

(no. of reactions per unit volume and time);
$\langle\sigma v\rangle$: reaction probability per pair of reacting nuclei and second, averaged over the Maxwellian velocity distribution:

$$f(E)\mathrm{d}E = \frac{2}{\sqrt{\pi}}\frac{\sqrt{E}}{(kT)^{3/2}}e^{-E/kT}\mathrm{d}E$$

$$\langle\sigma v\rangle = \int_0^\infty \sigma(E) v f(E)\mathrm{d}E$$

The energy released by the reaction is then

$$\epsilon_{jk} = \frac{1}{1+\delta_{jk}} \frac{q_{jk}}{m_j m_k} \rho X_i X_k \langle \sigma v \rangle$$

with q_{jk} being the energy released per reaction.

Approximation: $\epsilon_{jk} \approx \epsilon_{jk,0} \left(\frac{T}{T_0}\right)^{\nu}$.

H-burning: $\nu = 5 \dots 15$; He-burning $\nu = 40$.

Non-resonant reactions:

If nucleus has energy level close to incoming particle's energy, reaction rate (dramatically) increased. No simple formula; depends on theoretical models for nucleus structure and/or experimental measurements of energy levels and their width.

Electron shielding

clouds of negatively charged electrons reduce repulsive Coulomb-potential of bare nuclei and can increases r_{jk} by about 10%.

Approximative formula by Salpeter for the *Weak limit:*

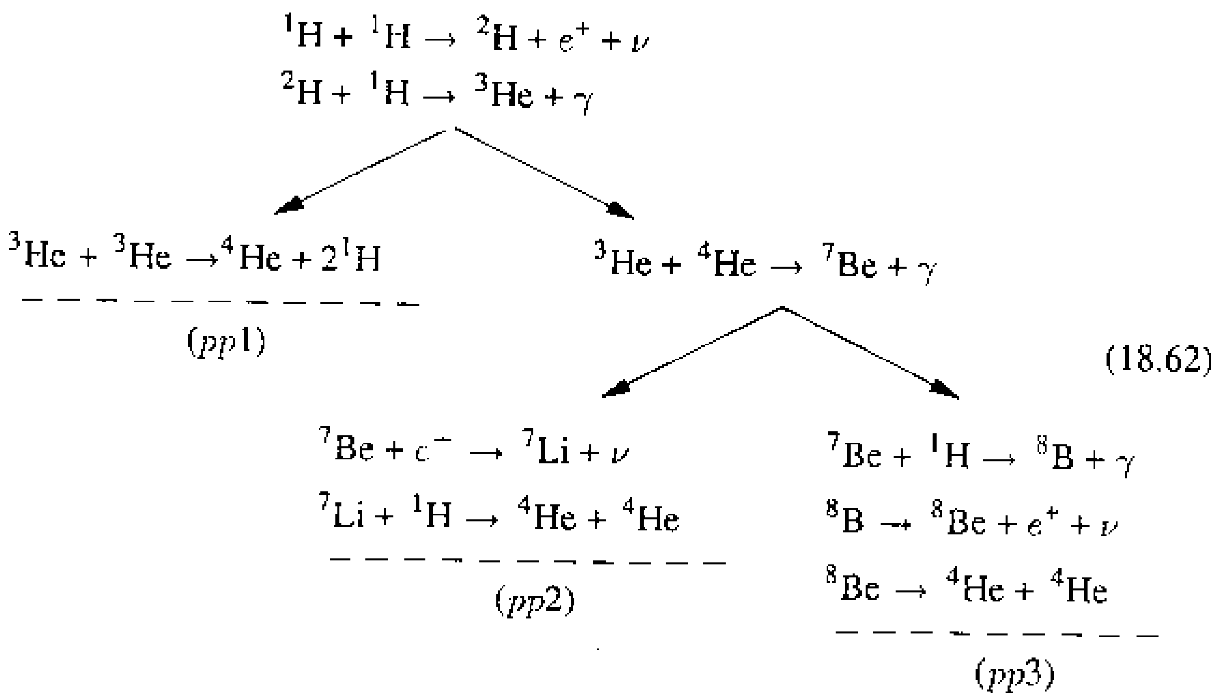
$E_D = \frac{Z_1 Z_2 e^2}{r_D} \ll kT$
$r_D = \sqrt{\frac{kT}{4\pi e^2 \chi n}}$ is the Debye-Hückel length and χ an average particle density)

$$\tilde{r}_{ij} = r_{ij} \left(1 + 0.188 Z_1 Z_2 \sqrt{\frac{\rho \chi}{T_6^3}}\right)$$

Major burning stages in stars

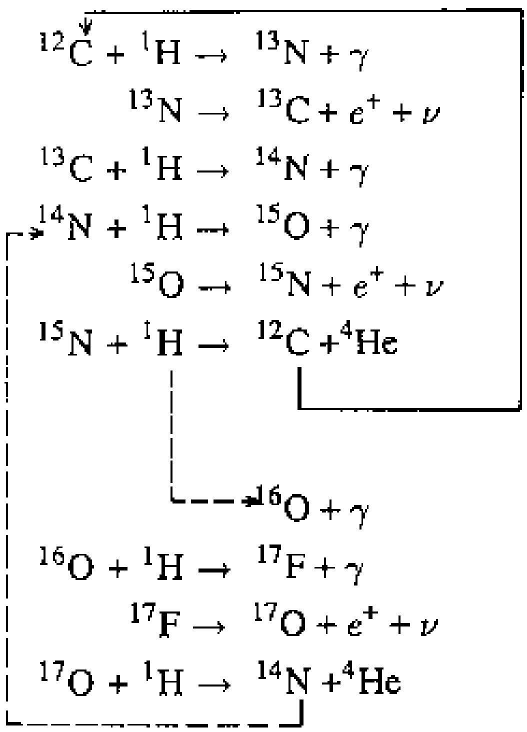
Sequence of phases: hydrogen → helium → carbon/oxygen → neon → silicon → iron.

Hydrogen-burning



The pp-chain for the fusion of hydrogen to helium

Energy per completion: 26.20 (ppI), 25.67 (ppII), 19.20 MeV (ppIII).



The CNO-cycle

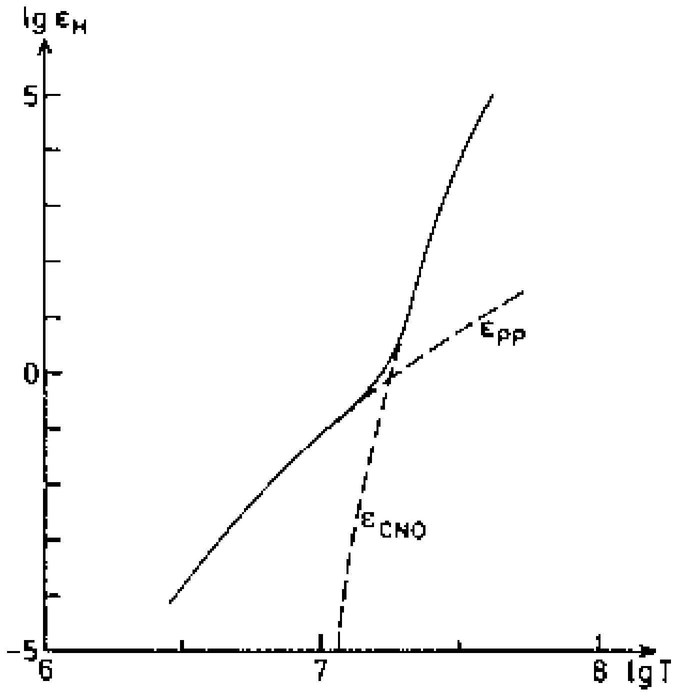
- The e^{+} reactions happen instantaneously.
- The complete cycle is dominated by the slowest reaction, which is $^{14}N(p,\gamma)^{15}O$.
- $q_{\text{CNO}} \approx 25 MeV$
- In equilibrium, $\approx$ C & O $\Rightarrow$ ^{14}N.

- $^{12}C/^{13}C \approx 3 \ldots 6$ (solar: 40)

- At low T (solar center), the cycle is too slow to be important, but the $C \rightarrow N$ transformation is working.

- Equivalent cycles involving Na and Mg exist and operate partially at higher temperature ($\approx 5 \cdot 10^7$ K).

- With increasing temperature, the CNO-cycle becomes more dominant.



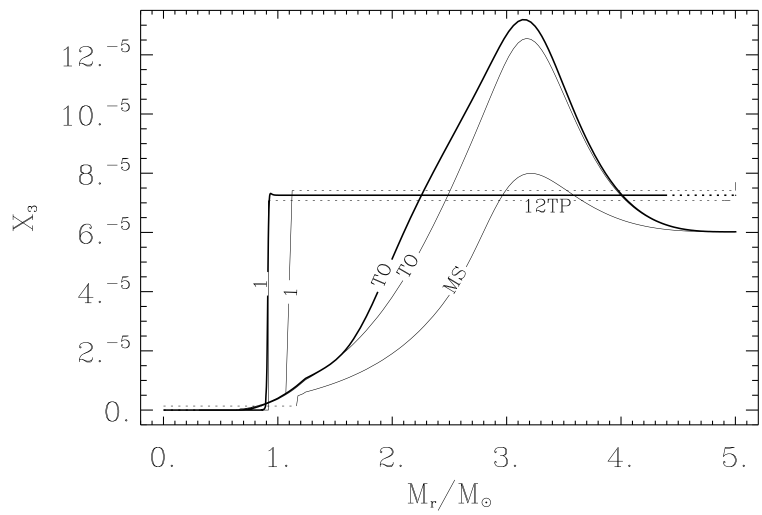
^{3}He production in stars

- ^{3}He can achieve an equilibrium abundance in the pp-chain
- this abundance is higher for lower T
- the time to reach it is also larger for lower T

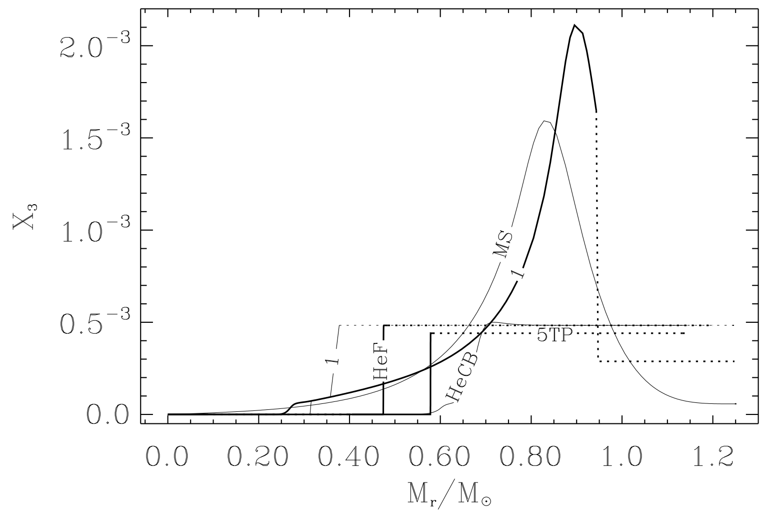
$\rightarrow$ more massive stars quickly reach (low) equilibrium abundances of ^{3}He, which are lower than the primordial value ($\approx 10^{-5}$)

$\rightarrow$ low-mass stars can *produce* ^{3}He, if they live long enough

- BUT: observations indicate NO ^{3}He-production in the galaxy!



^{3}He-abundance in a $5\,M_\odot$ star during its evolution



^{3}He-abundance in a $1.25\,M_\odot$ star during its evolution

Helium-burning

burning temperature: $\geq 10^8$ K; reactions:

1. $3-\alpha$-process: $2\alpha(\alpha,\gamma)^{12}C$; actually two steps: $\alpha(\alpha,\gamma)^8Be$ and $^8Be(\alpha,\gamma)^{12}C$; $q=7.27$ MeV.

2. $^{12}C(\alpha,\gamma)^{16}O$: reaction rate uncertain by a factor of 2! $q=7.6 MeV$

3. $^{16}O(\alpha,\gamma)^{20}Ne$: important only during the end of helium burning; $q=4.77$ MeV

4. final core composition: C/O = 50/50 ... 20/80

The core He-flash in low-mass stars

- for $M \lesssim 2.2 M_\odot$ H-exhausted He-core becomes highly degenerate
- maximum T not in center, but slightly below H-shell (ν-cooling!)
- at $T \approx 10^8$ K, 3α-reactions set in
- due to degeneracy, released energy cannot be used for expansion, but for further heating

$\rightarrow$ thermal runaway

- ends only, when raised T lifts degeneracy
- L_{He} can reach $10^6 L_\odot$ (for some days); is used for expansion of the core

Burning times of burning phases:

$$
\begin{array}{rcl}
\mathrm{H} & : & 10^{10}\ \mathrm{(yrs)} \\
\mathrm{He} & : & 10^{8} \\
\mathrm{C} & : & 10^{4} \\
\vdots & : & \vdots \\
\mathrm{Si} & : & \mathrm{hrs}
\end{array}
$$

Plasma neutrino emission

Stellar plasma emits neutrinos, which leave star without interaction and lead to energy loss L_ν.

Processes are:

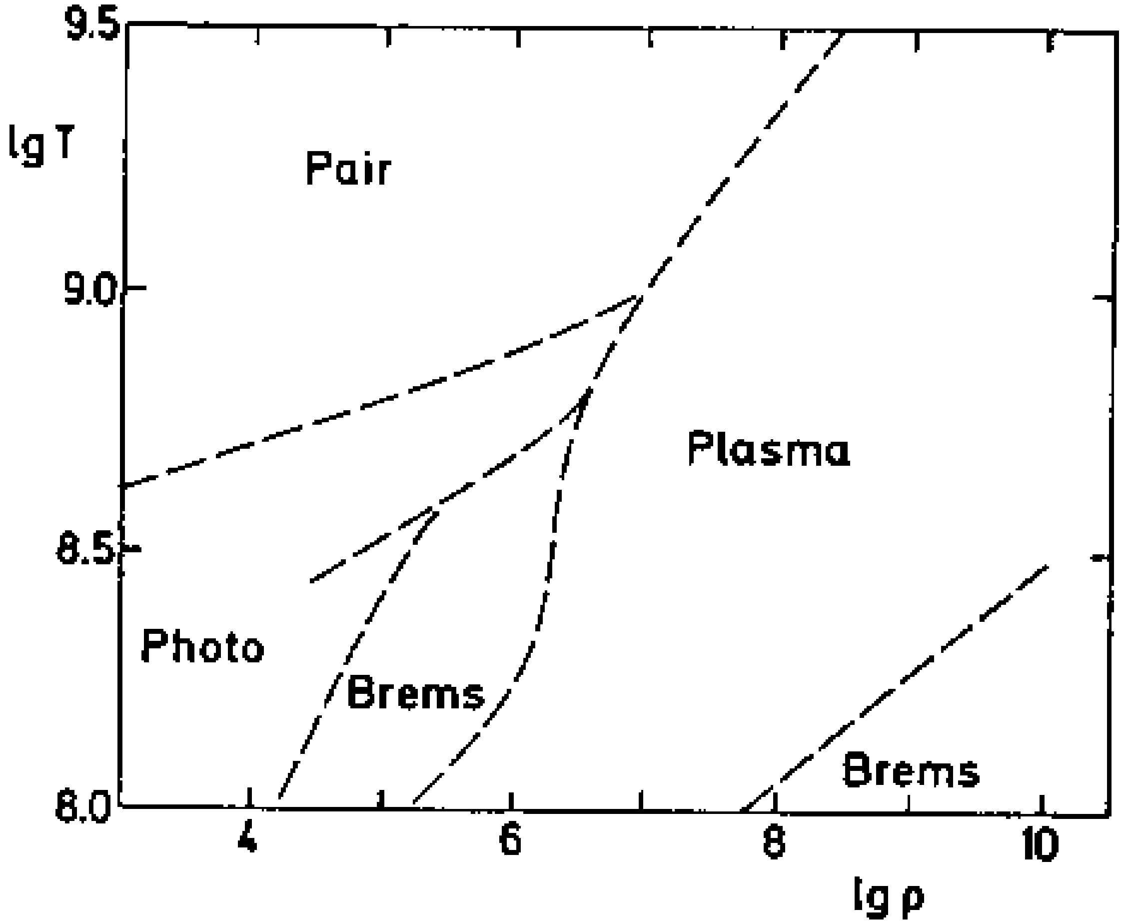
1. Pair annihilation: $e^- + e^+ \rightarrow \nu + \bar{\nu}$ at $T > 10^9$ K.

2. Photoneutrinos: $\gamma + e^- \rightarrow e^- + \nu + \bar{\nu}$ (as Compton scattering, but with ν-pair instead of γ).

3. Plasmaneutrinos: $\gamma_{\text{pl}} \rightarrow \nu + \bar{\nu}$; decay of a plasma state γ_{pl}.

4. Bremsstrahlung: inelastic nucleus–e^- scattering, but emitted photon replaced by a ν-pair.

5. Synchroton neutrinos: as synchroton radiation, but again a photon replaced by a ν-pair.



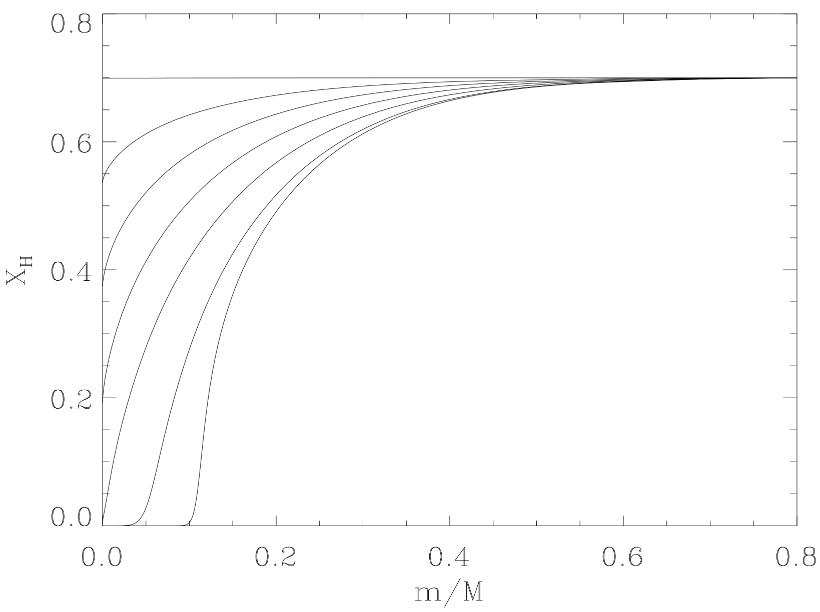
The regions in the ρ–T plane, where the different plasma-neutrino processes are dominant

Evolution of Low-Mass Stars

$0.1 \lesssim M/M_\odot \lesssim 2.5$; mostly $0.8 \lesssim M/M_\odot \lesssim 1.5$

On the main-sequence

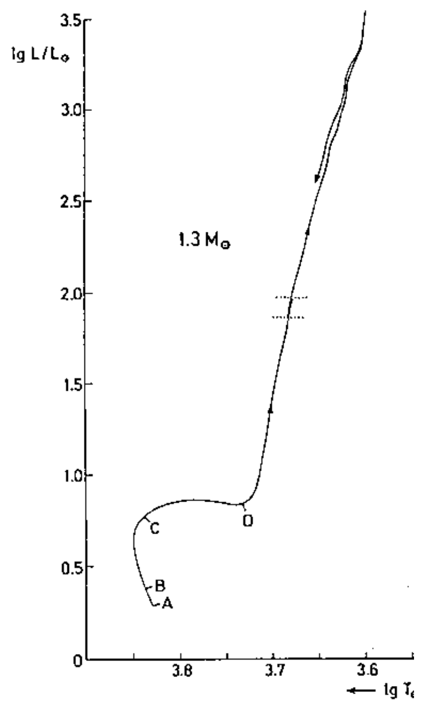
- long lifetime (billion years)
- core hydrogen burning via pp-chains ($M \lesssim 1.3 M_\odot$) and CNO-cycle (in Sun only 1.5%)
- radiative core, convective envelope ($M \lesssim 0.2 M_\odot$: convective envelope extends to center; fully convective star)
- gradual consumption of H; faster towards centre (higher T)



H-profile developing in $1 M_\odot$ star on main sequence

Evolutionary tracks and shell burning

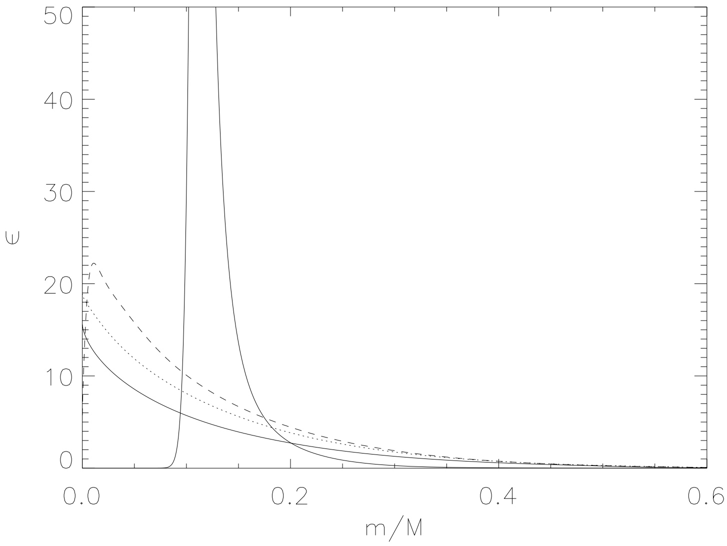
- stars initially evolve almost parallel to zero-age main sequence
- later turn to cooler temperatures (core contracts, envelope expands)
- evolve on nuclear timescale at almost constant L (subgiant branch) to bottom of red giant branch



Evolution of a low-mass star in the HRD

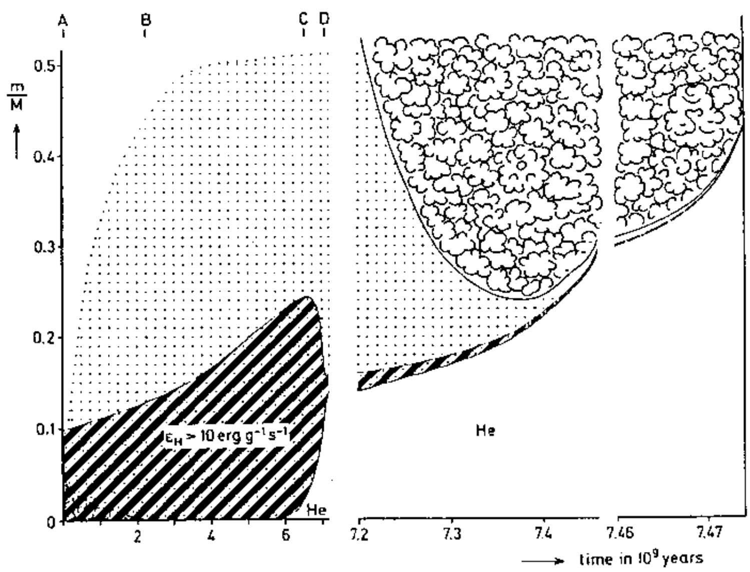
- and then up the Red Giant Branch (this is an evolution along the *Hayashi-line*, the location of the coolest stars with fully convective envelopes in thermal and hydrostatic equilibrium)

- at end of main-sequence *hydrogen-burning shell* around core established



energy production in $1M_\odot$ during main sequence evolution

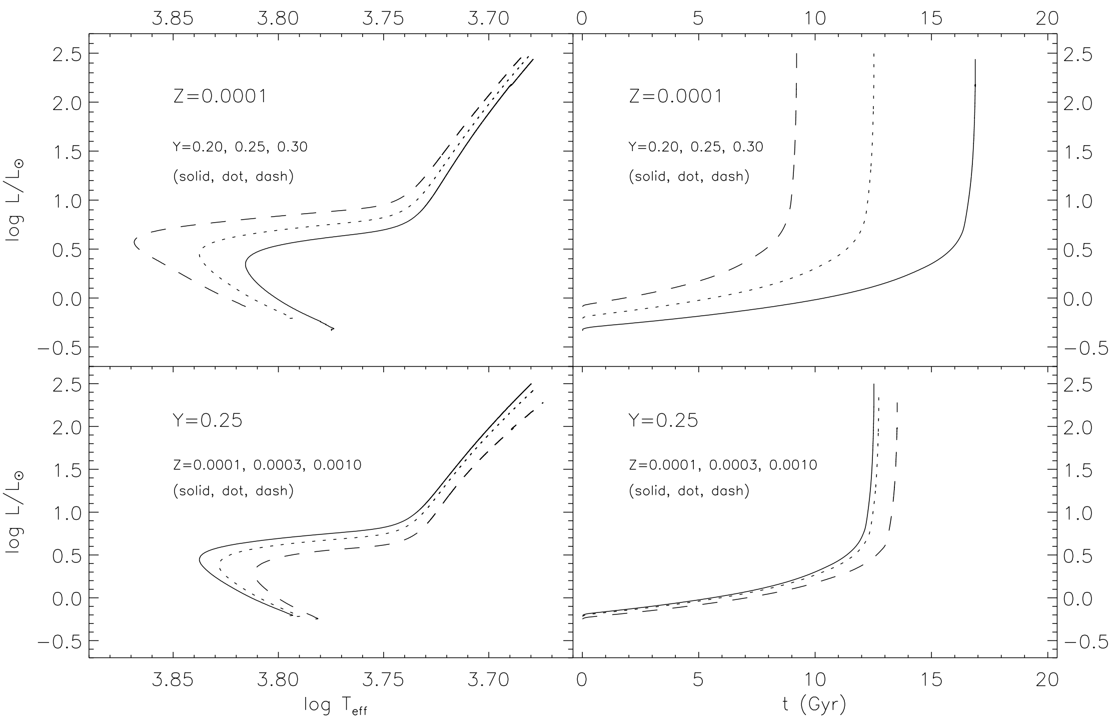
- Internal structure from main sequence to *Red Giant Branch Tip*



(hatched: energy production, clouds: convection; dots: composition change)

Influence of composition

- lower Z: stars are hotter and brighter (lower κ)
- live shorter (higher L)
- lower X (higher Y): hotter and brighter (lower κ)
- live shorter (higher L and less H)



On the RGB and He ignition

- evolution determined by He-core mass: $L \sim M_c^7$ (from homology for thin shells)

- core isothermal + contraction energy (negative T-gradient) + cooling by plasma neutrinos (positive T-gradient) $\rightarrow$ T-maximum below H-shell!

- core increasingly degenerate; $\rho_c \rightarrow 10^5 \cdots 10^6\,\mathrm{g/cm^3}$

- T_{max} also function of M_c

- He ignites off-centre at $T_{\mathrm{max}} \approx 10^8$ K

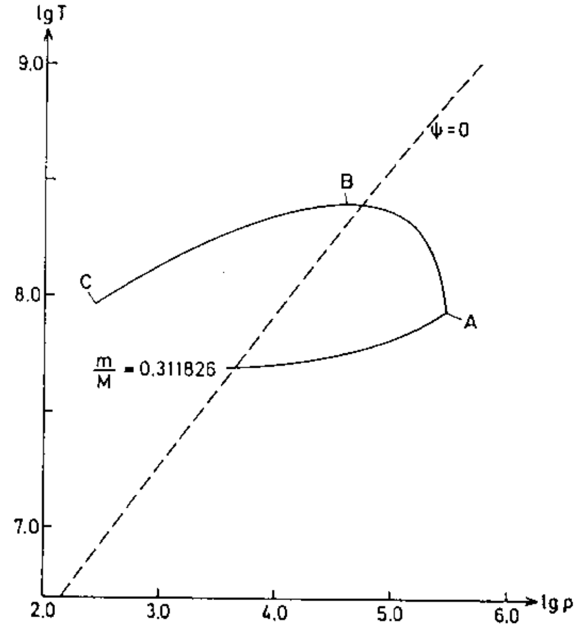
- **at $\approx$ the same $M_c \approx 0.48\,M_\odot$ and L for all stars!**

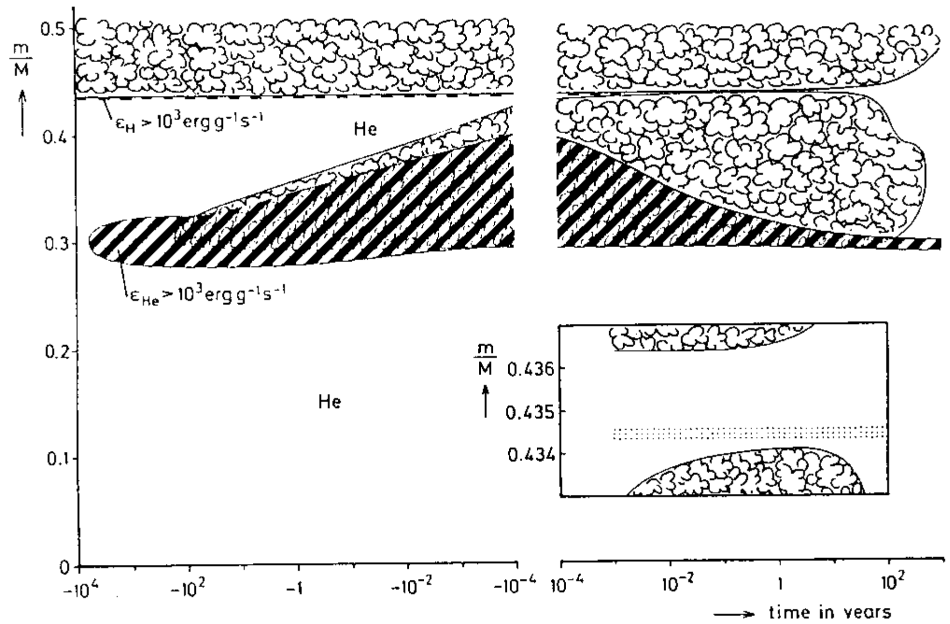
$\Rightarrow$ tip of RGB used as *standard candle* and distance indicator

- He-ignition very violent because of high degeneracy; $L_{\mathrm{He}} > 10^6\,L_\odot$ for short time

- degenerate gas only heating due to energy input from He-burning $\rightarrow$ thermonuclear runaway

- only later energy used to expand matter, decrease degeneracy





Evolution of the helium-shell during the flash

After the RGB:

- on the RGB mass loss small (a few 0.1 $M_\odot$) following *Reimers mass loss formula*:

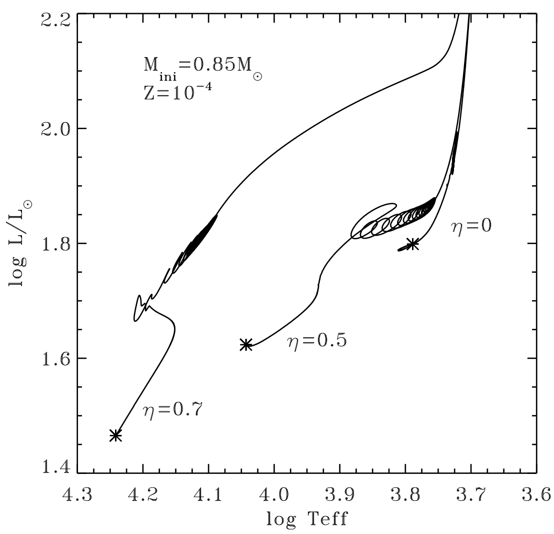
$$\dot{M} = -4 \times 10^{-13} \eta \frac{LM}{R} \frac{R_\odot}{L_\odot M_\odot} M_\odot/\text{yr}$$

- expansion of core to $\rho \approx 10^2\,\text{g/cm}^3$ and quiet helium burning in (convective) core

- location: higher T_{eff}, lower L ($\approx$ same for all M)

→ *Horizontal Branch*; T_{eff} depends on envelope mass left (less = hotter)

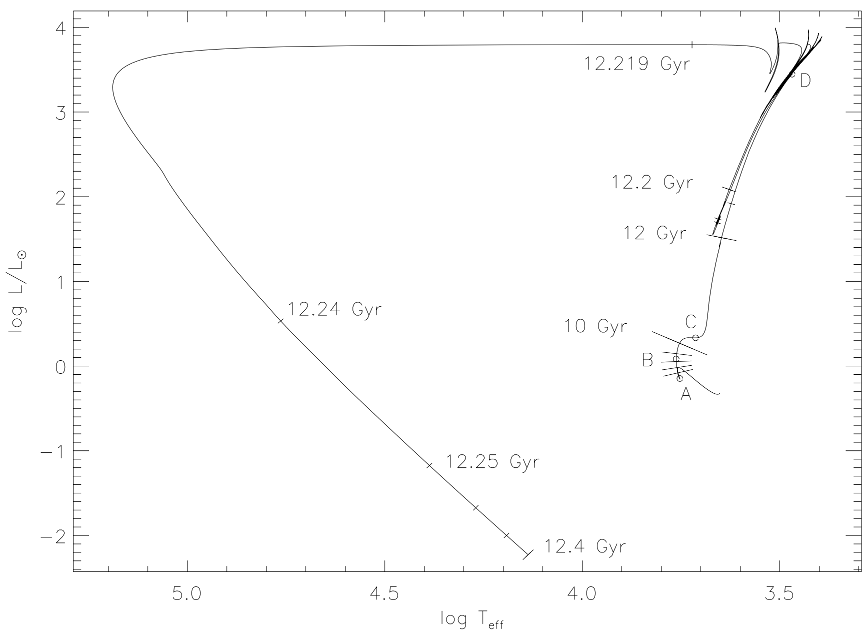
- evolution from RGB-tip to HB in few million years



Evolution to horizontal branch for models evolved from the main sequence but with different Reimers mass loss parameter η as indicated. ZAHB models are indicated with a star.

After the Horizontal Branch:

- end of core He burning
- He-shell around C/O core + H-shell as before
- evolution to giants → *Asymptotic Giant Branch* (see intermediate mass stars)
- depending on remaining envelope mass departure from AGB, crossing of HRD, cooling to White Dwarf



Evolution of model for Sun from ZAMS to White Dwarf

Evolution of Intermediate-Mass Stars

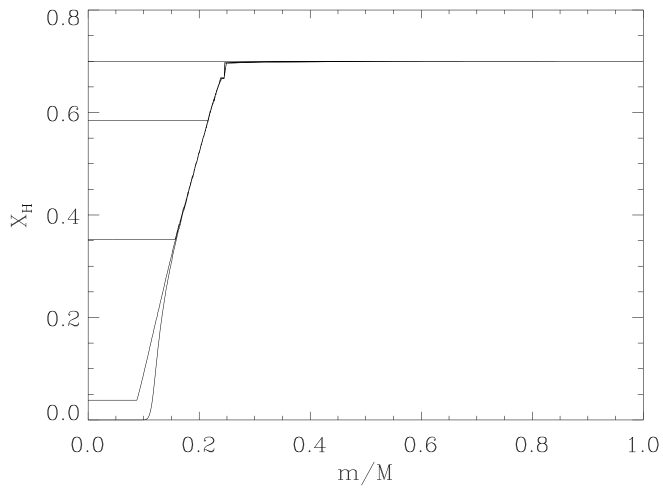
General properties:

- mass range: $2.5 \lesssim M/M_\odot \lesssim 8$
- early evolution differs from $M/M_\odot \lesssim 1.3$ stars; for $1.3 \lesssim M/M_\odot \lesssim 2.5$ properties of both mass ranges
- MS: convective core and radiative envelope
- CNO-cycle; $\epsilon_{\rm nuc} \propto \rho X Z_{\rm CNO} T^{18}$
- rapid transition from MS to RGB (Hertzsprung-gap)
- Helium-core non-degenerate $\rightarrow$ quiet He-ignition
- after core helium burning: degenerate C/O-core and two nuclear shells (H and He)

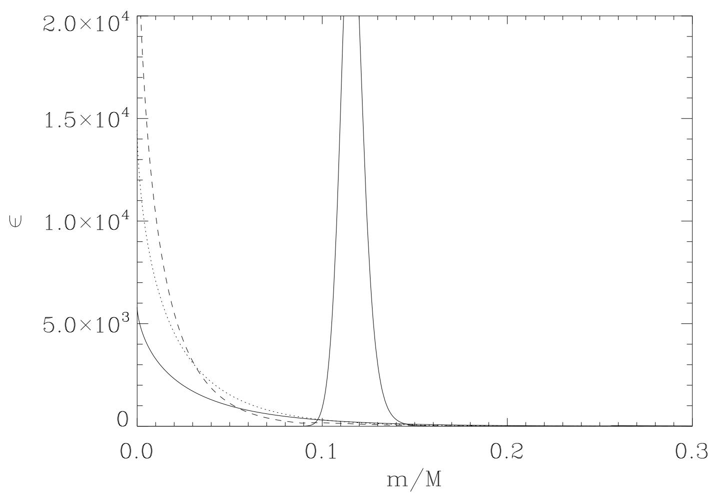
Main-sequence phase

- convective core burning: mixing and burning mainly at central temperature

→ gradual exhaustion of convective core // (Fig: $5M_\odot$ star)



- development of off-center H-burning shell (within former H-burning core!)



Schönberg-Chandrasekhar mass:

- with increasing mass of exhausted core the ideal gas pressure alone can no longer sustain pressure of (massive) envelope

- critical core mass is $(M_c/M)_{\mathsf{SC}} = 0.37(\mu_e/\mu_c)^2$

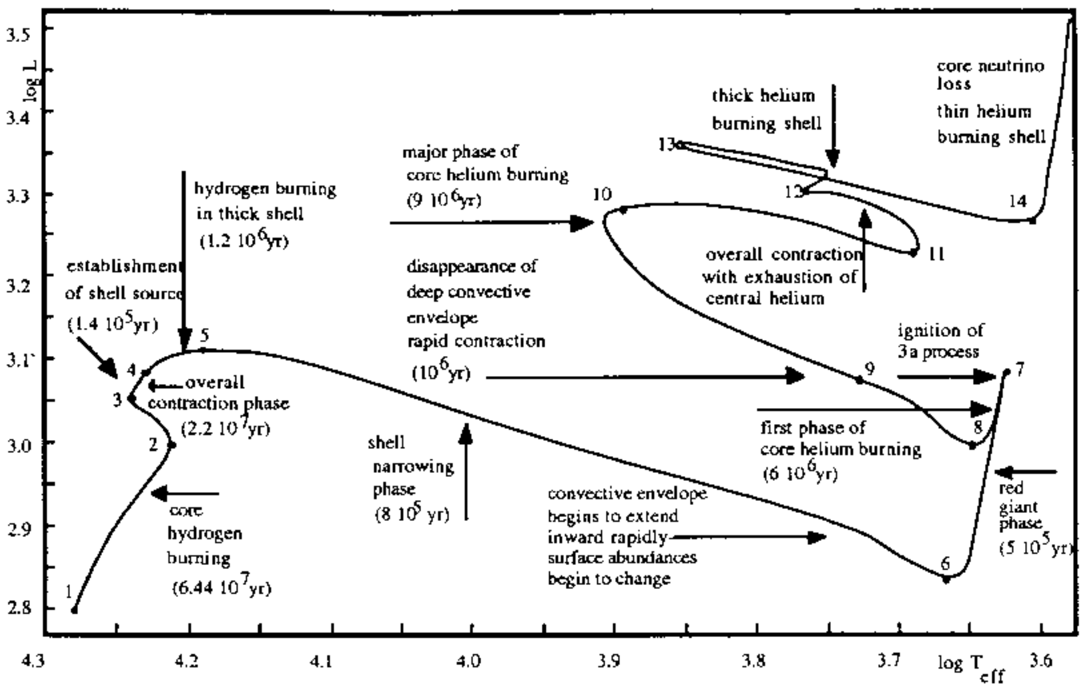
- core contracts on *thermal timescale* (fast), while envelope expands $\rightarrow$ fast crossing of HRD to giants $\Rightarrow$ low probability to observe such star $\rightarrow$ *Hertzsprung-gap*

- at RGB further expansion at constant T_{eff}

- evolution stopped when core becomes degenerate or helium ignites

- core is not isothermal (due to contraction energy)

Evolution of $5\,M_\odot$-star



Core helium burning:

- non-degenerate ignition; longer lifetime → *clump* of stars next to RGB
- loops through Cepheid-instability strip, details depending on internal chemical structure
- at exhaustion of He-core → He-shell develops around degenerate C/O-core
- asymptotic return to RGB: *Asymptotic Giant Branch* evolution

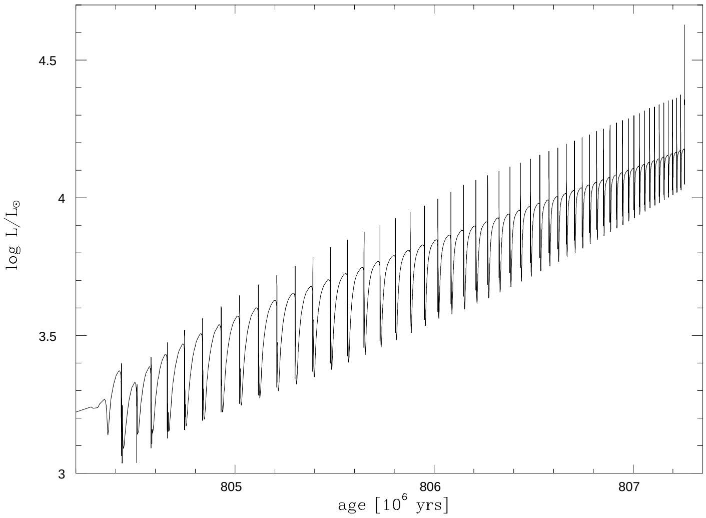
The AGB-Phase

- double-shell burning phase, most important for intermediate mass stars

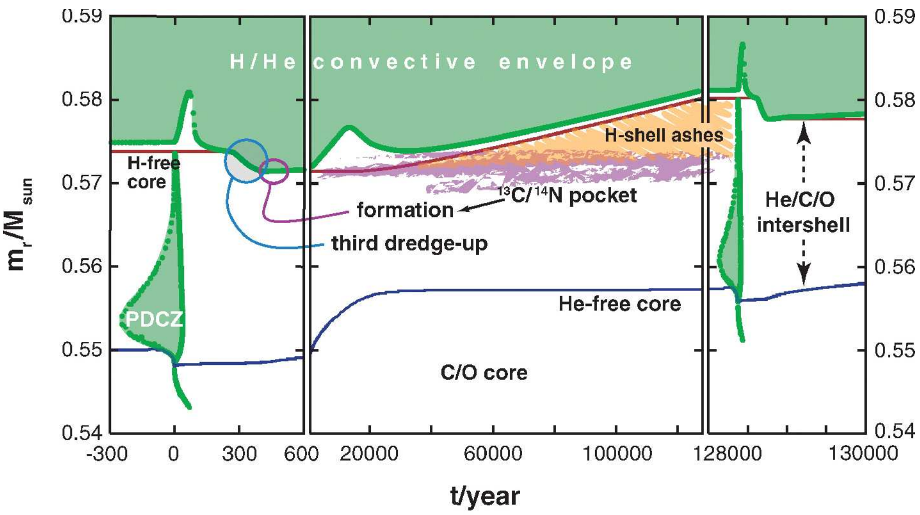
- special features: thermal pulses (TP), nucleosynthesis of rare-earth elements (Ba, Sr, . . .) through slow neutron capture chains (*s-process*); strong mass loss removing up to 90% of stellar mass within 10^5 years, returning synthesized elements to interstellar medium

- Thermal pulses:
  - runaway events in helium shell
  - duration: few hundred years
  - interpulse time: few thousand years
  - strong luminosity variations in shell
  - variable convective zones
  - mixing between He- and H-burning regions and envelope (3rd dredge-up)

Luminosity variations during TPs in a $2.5M_\odot$ star:

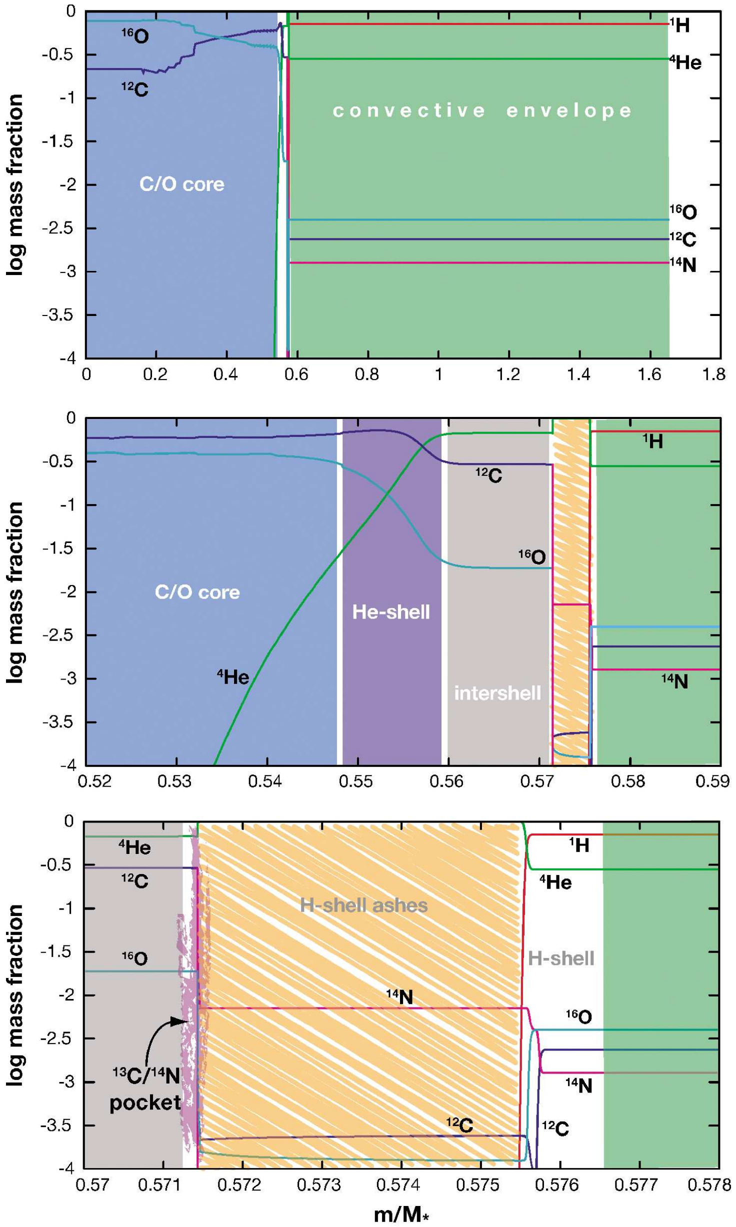


Interior structure changes in a $2\,M_\odot$ star:



Herwig, F. 2005
Annu. Rev. Astron. Astrophys. 43: 435–79

Chemical profile in the $2\,M_\odot$ star:



Herwig, F. 2005
Annu. Rev. Astron. Astrophys. 43: 435–79

Nucleosynthesis and mixing due to convection:

the convective mixing between He-shell and envelope leads to

- enrichment of envelope with carbon $\rightarrow$ formation of carbon stars

- entrainment of protons into hot C-rich regions, which results in *production of neutrons:*

  $$^{12}\mathrm{C}(p,\gamma)^{13}\mathrm{N}(\beta^{+}\nu)^{13}\mathrm{C} \rightarrow (p,\gamma)^{14}\mathrm{N} \text{ or } (\alpha,n)^{16}\mathrm{O}$$

- alternatively, C and O converted in H-shell to ^{14}N and then

  $$^{14}\mathrm{N}(\alpha,\gamma)^{18}\mathrm{F}(\beta^{+}\nu)^{18}\mathrm{O}(\alpha,\gamma)^{22}\mathrm{Ne}(\alpha,n)^{25}\mathrm{Mg}$$

  (C-13 and Ne-22 neutron sources)

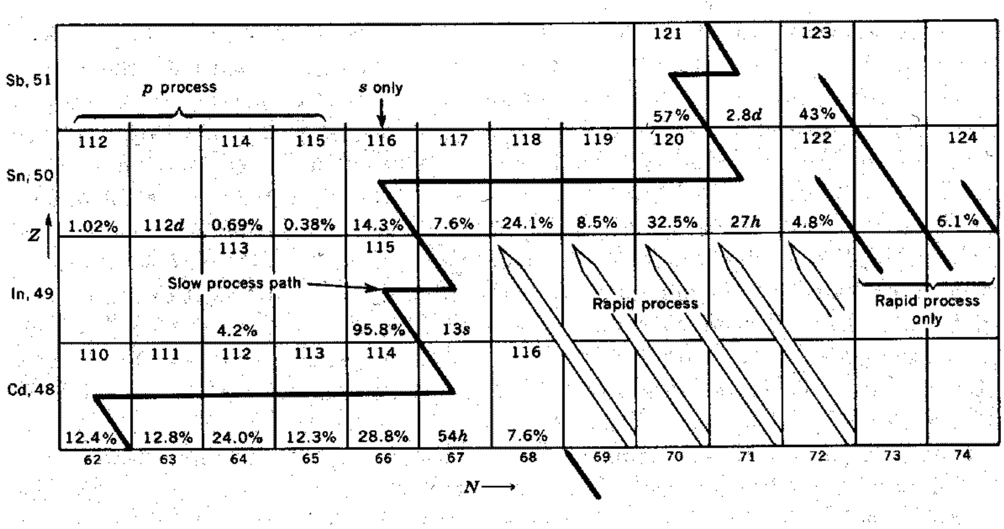
- neutrons start neutron-capture nucleosynthesis

- more likely for: later pulses, lower metallicity, lower stellar mass, more mixing efficiency

s-process neutron capture nucleosynthesis

- The *s-process* is a sequence of *neutron captures* and β^-- (electron-) decays, in which the former happen *s*lower than the latter ones (there is also a *r*(apid)-process in the opposite case).

- It begins on *seed* nuclei at the Fe-group elements and creates rare-earth elements



Path of the s-process through cadmium (Cd), indium (In), antimony (Sb); for stable isotopes the isotop-fraction is given, for those that decay by β-decay the half-life time

- Other s-process elements are Ba, Bi (last β-stable element), Pb, Ag, In, . . .

- the abundances of such elements in meteorites and on Earth can partially be explained with this stellar source

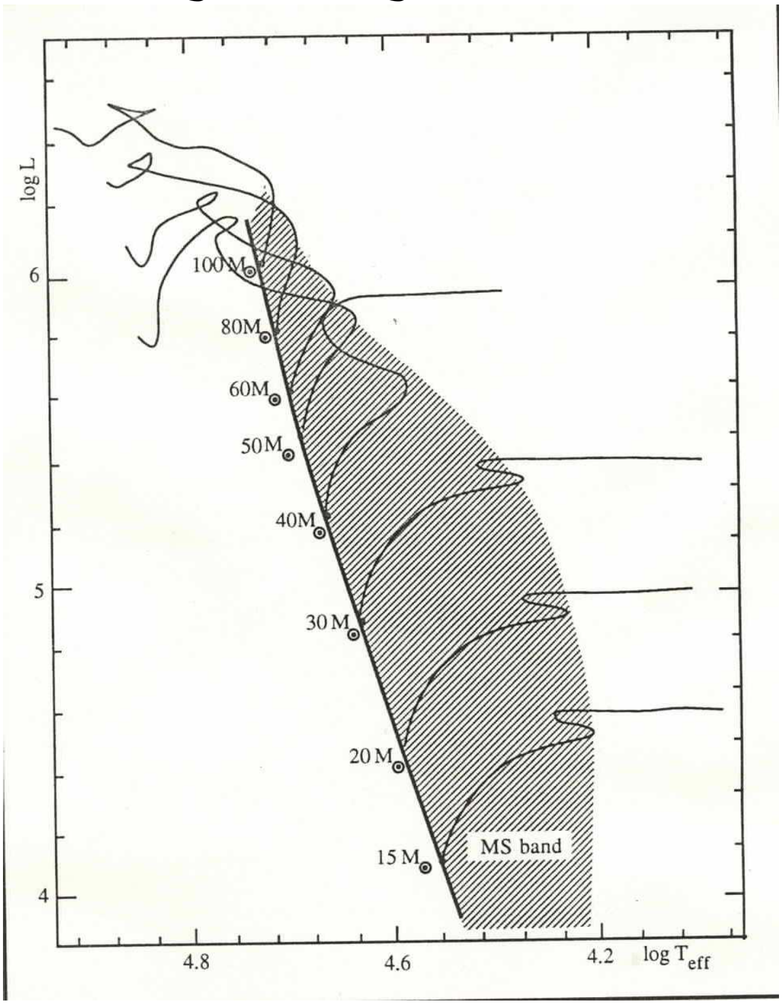
After the AGB:

- envelope lost quickly by *superwind* with $\dot{M} \approx 10^{-5}\,M_\odot$/yr (radiation pressure on dust grain)
- if almost all envelope is lost $\rightarrow$ star crosses HRD to hot $T_{\rm eff}$
- may illuminate former envelope, now expanding shell $\rightarrow$ *Planetary Nebula*
- then former stellar He-core cools to White Dwarf

Evolution of Massive Stars

- $M \gtrsim 8 M_\odot$; upper mass limit (stability limit) around $100 M_\odot$
- large convective cores on main-sequence (up to 80% or more of M)
- effect of *convective overshooting* very important:
  - larger convective cores
  - extends lifetime on main-sequence
  - leads to higher luminosity and broader MS
- radiation pressure and electron scattering dominant
- strong mass loss on main-sequence due to *radiation-driven* winds from hot and luminous surface
- both convection and mass loss decisive for evolution
- e.g. can uncover H-burning or He-burning regions (Wolf-Rayet stars)
- central evolution up to Fe
- then Supernova of type II because of collapse of core and formation of central neutron star

Sample evolution up to end of H-burning including some overshooting and large mass loss



Note: evolution quite uncertain because of lack of physical knowledge about mass loss and convection (overshooting and other effects), the influence of rotation (an active area of research), instabilities in the atmospheres and more as well as due to the low number of objects that can be observed (initial final mass function and short lifetimes).